1. (a) Show that a graph $G$ is bipartite if, and only if, all the circuits of $G$ are of even length.

(b) The line graph $L(G)$ of a simple graph $G(V, E)$ is the graph whose vertices are in 1-1 correspondence with the edges of $G$, two vertices of $L(G)$ being adjacent if and only if the corresponding edges of $G$ are adjacent.

Show that the line graphs of the two graphs below are isomorphic:

Show that the line graph of $G$ has $\varepsilon$ vertices and $\sum_{v \in V} \left(\frac{d(v)}{2}\right)$ edges (where $\varepsilon$ is the number of edges of $G$, and $d(v)$ is the degree of $v$ in $G$).

Hence, or otherwise, show that the second of the above graphs cannot be the line graph of any graph $G$. 

continued ...
2. (a) Let the line graph $L(G)$ of a simple graph $G(V,E)$ be defined as in Question 1(b).

Show that:

(i) If $G$ is Eulerian, then $L(G)$ is Hamiltonian,

(ii) If $G$ is Eulerian, then $L(G)$ is Eulerian,

(iii) If $G$ is Hamiltonian, then $L(G)$ is Hamiltonian.

Give an example of a Hamiltonian line graph $L(G)$ for which $G$ is neither Eulerian nor Hamiltonian.

(b) Say, giving reasons, for each of the three graphs below, whether it is (i) Eulerian, (ii) Hamiltonian.
3. (a) Let $G$ be a graph. Define: $G$ is planar.

Show that $K_{3,3}$ is not planar, but that it can be embedded on the surface of a Möbius band.

How would your answer change if $K_{3,3}$ were replaced with $K_{3,4}$?

(b) Wagner’s Theorem says that a graph $G$ is planar if, and only if, it contains no subgraph contractible to $K_5$ or $K_{3,3}$.

Use Wagner’s Theorem to show that the Petersen graph is non planar:

Further, show that the Thomassen graph below is non planar:
(c) Let $G$ be a plane connected graph with $\varphi$ faces, $\varepsilon$ edges, $\nu \geq 3$ vertices, in which no face is bounded by $< k$ edges. Show that $k\varphi \leq 2\varepsilon$.

Further, assuming Euler’s formula:

$$\nu - \varepsilon + \varphi = 2,$$

deduce that

$$\varepsilon \leq \frac{k(\nu - 2)}{(k - 2)}.$$

Hence show that the Folkman graph (below) is non-planar:

4. (a) A bipartite tournament is a digraph obtained from a complete bipartite graph by giving a direction to each edge.

(i) Are bipartite tournaments semi-Hamiltonian?

(ii) Are strongly connected bipartite tournaments Hamiltonian?

Give a proof or counterexample in each case.

(b) Prove Moon’s theorem:

If $D$ is a strongly connected tournament with $\nu \geq 3$, then $D$ contains a dicircuit of length $k$ for each $k$ with $3 \leq k \leq \nu$.

Deduce that every strongly connected tournament is Hamiltonian.

continued …
5. \(\textbf{(a)}\) Show that every planar graph is 6-colourable.

[You may assume that any planar graph has a vertex of degree \(\leq 5\).]

Show that every Hamiltonian plane graph is 4-face colourable.

\(\textbf{(b)}\) Find an embedding of the Petersen graph (below) on the projective plane.

\[\text{Deduce that the chromatic number } \chi(P) \text{ of a projective plane } P \text{ is at least 6.}
\]
Use Heawood’s inequality

\[\chi(S) \leq \left\lceil \frac{1}{2}(7 + \sqrt{49 - 24n}) \right\rceil\]

for a surface \(S\) of Euler characteristic \(n < 2\) to show that \(\chi(\text{projective plane}) = 6\).