1. (a) Among the graphs below, which pairs are isomorphic?

(Give reasons for your answer.)

(b) If $G$ has vertices $v_1, v_2, \ldots, v_\nu$, the sequence $(d(v_1), d(v_2), \ldots, d(v_\nu))$ is called a degree sequence of $G$. Show that the sequence $(5, 5, 5, 5, 3, 3)$ is the degree sequence of some graph.

A sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ is graphic if there is a simple graph with degree sequence $\mathbf{d}$. Say, for each of the sequences $(7, 6, 5, 4, 3, 2)$ and $(6, 6, 5, 4, 3, 3, 1)$, whether or not they are graphic.

(c) Define: $G$ is $k$-regular, and $G^c$ is the complement of $G$.

How many distinct (up to isomorphism) 4-regular graphs are there on 7 vertices? And how many non-isomorphic 5-regular graphs are there on 7 vertices?
2. (a) For each of the four graphs below, say, giving reasons, whether or not it is (i) Eulerian, or (ii) Hamiltonian.

![Graphs G1, G2, G3, G4](image)

(b) Prove Dirac’s sufficient condition for $G$ to be Hamiltonian:

*If $G$ is simple, and $\nu \geq 3$, and $d(u) \geq \frac{\nu}{2}$ for every vertex $u$ of $G$, then $G$ is Hamiltonian.*

Show that a party of three or more girls can be seated around a table in such a way that everyone has two of her friends at her side provided that each person has as friends at least half of the total number of people in the party.

(c) $G$ is Hamilton-connected if every two vertices of $G$ are connected by a Hamilton path.

Show that if $G$ is Hamilton-connected with $\nu \geq 4$ vertices, then each vertex of $G$ has degree $\geq 3$.

Deduce that $G$ has $\varepsilon \geq [\frac{1}{2}(3\nu + 1)]$ edges (where $[x]$ denotes the greatest integer $\leq x$).

3. (a) Let $G$ be a connected plane graph with $\nu$ vertices, $\varepsilon$ edges and $\phi$ faces. Assuming Euler’s formula:

$$\nu - \varepsilon + \phi = 2,$$

deduce that if $G$ is simple, with $\nu \geq 3$, then $\varepsilon \leq 3\nu - 6$.

Show that if $G$ is a planar graph for which $\nu \geq 4$ and $\varepsilon = 3\nu - 6$, then $G$ has no vertices of degree $\leq 2$.

(b) Let $P$ be a regular polyhedron in which each face has $p$ edges and in which $q$ faces meet at each vertex.

Show that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{\varepsilon},$$

where $\varepsilon$ is the number of edges of $P$.

Deduce that there are at most five regular polyhedra.

(c) Define the dual graph $G^*$ of a plane graph $G$.  

2 Question 3 continues ...
Show that if $F$ is the set of faces of the plane graph $G$, then

$$\sum_{f \in F} d(f) = 2\varepsilon.$$

Show that if $G$ is a connected plane graph for which every vertex is of degree at least three, then $G$ has a face of degree less than six.

4. (a) Define the terms strongly connected and dicomponent for a digraph $D$.

Find the dicomponents for the digraph:

![Diagram](image)

(b) Define: $D$ is an Eulerian digraph.

Show that a connected digraph $D$ is Eulerian if, and only if, $d^→(v) = d^←(v)$ for every vertex $v$ in $D$.

(c) Prove Robbins’ Theorem: If every edge of a connected graph $G$ is a circuit edge, then $G$ has a strongly connected orientation.

Find a strongly connected orientation of the Petersen graph (below).

![Diagram](image)
5. \textbf{(a)} Define: $G$ is a $k$-critical graph.

List all 3-critical graphs, and give an example of a 4-critical graph on 4 vertices.

For all $n \geq 6$, construct a 4-critical graph on $n$ vertices.

\textbf{(b)} Let $G$ be a Hamiltonian cubic graph. Show that $G$ has a Tait colouring (that is, a 3-edge colouring in which no two adjacent edges take the same colour).

Give an example of a cubic graph with no Tait colouring, proving that your graph has no Tait colouring.

\textbf{(c)} State the \textit{Map Colour Theorem}.

Find the Euler characteristic and the chromatic number for the double torus.

END