MATH-303201

This question paper consists of 4 printed pages, each of which is identified by the reference MATH 303201

<u>No calculators allowed</u>

Examination for the Module MATH 3032

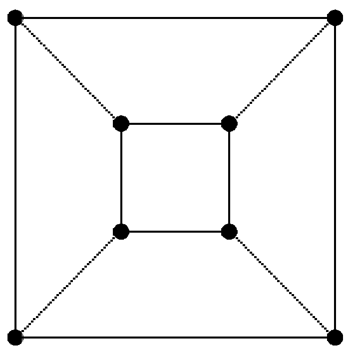
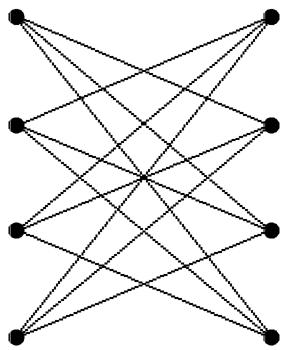
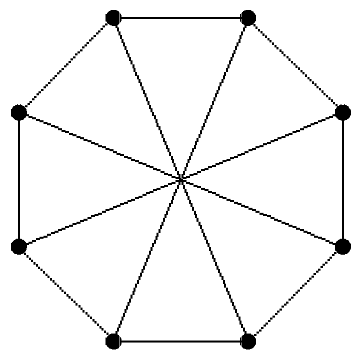
(January 2007)

GRAPH THEORY

Time allowed : 2 hours

Do not answer more than *FOUR* questions.
All questions carry equal marks.

1. (**a**) Among the graphs below, which pairs are isomorphic?

(Give reasons for your answer.)

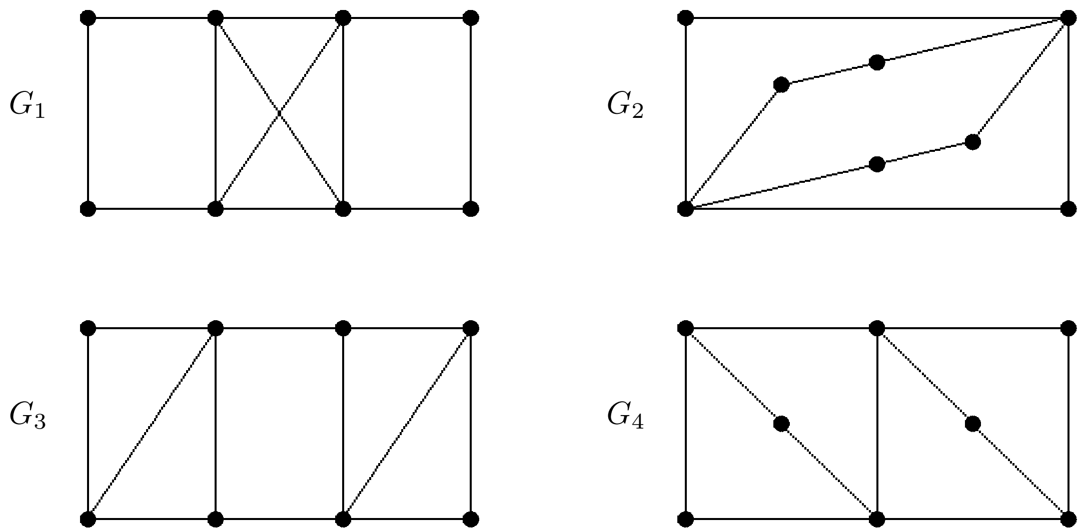
(**b**) If G has vertices $v_1, v_2, \ldots, v_\nu$, the sequence $(d(v_1), d(v_2), \ldots, d(v_\nu))$ is called a *degree sequence* of G. Show that the sequence $(5, 5, 5, 5, 3, 3)$ is the degree sequence of some graph.

A sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ is *graphic* if there is a *simple* graph with degree sequence $\mathbf{d}$. Say, for each of the sequences $(7, 6, 5, 4, 3, 3, 2)$ and $(6, 6, 5, 4, 3, 3, 1)$, whether or not they are graphic.

(**c**) Define: G is *k-regular*, and G^c is the *complement* of G.

How many distinct (up to isomorphism) 4-regular graphs are there on 7 vertices? And how many non-isomorphic 5-regular graphs are there on 7 vertices?

2. (**a**) For each of the four graphs below, say, giving reasons, whether or not it is (i) Eulerian, or (ii) Hamiltonian.



(**b**) Prove Dirac's sufficient condition for G to be Hamiltonian:

If G is simple, and $\nu \geq 3$, and $d(u) \geq \frac{\nu}{2}$ for every vertex u of G, then G is Hamiltonian.

Show that a party of three or more girls can be seated around a table in such a way that everyone has two of her friends at her side provided that each person has as friends at least half of the total number of people in the party.

(**c**) G is *Hamilton-connected* if every two vertices of G are connected by a Hamilton path.

Show that if G is Hamilton-connected with $\nu \geq 4$ vertices, then each vertex of G has degree ≥ 3.

Deduce that G has $\varepsilon \geq [\frac{1}{2}(3\nu + 1)]$ edges (where $[x]$ denotes the greatest integer $\leq x$).

3. (**a**) Let G be a connected plane graph with ν vertices, ε edges and ϕ faces. Assuming Euler's formula:

$$\nu - \varepsilon + \phi = 2,$$

deduce that if G is simple, with $\nu \geq 3$, then $\varepsilon \leq 3\nu - 6$.

Show that if G is a planar graph for which $\nu \geq 4$ and $\varepsilon = 3\nu - 6$, then G has no vertices of degree ≤ 2.

(**b**) Let P be a regular polyhedron in which each face has p edges and in which q faces meet at each vertex.

Show that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{\varepsilon},$$

where ε is the number of edges of P.

Deduce that there are at most five regular polyhedra.

(**c**) Define the *dual graph* G^* of a plane graph G.

Question 3 continues . . .

Show that if F is the set of faces of the plane graph G, then

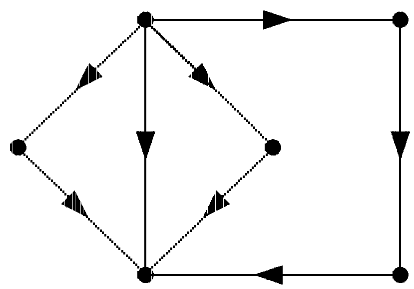
$$\sum_{f \in F} d(f) = 2\varepsilon\,.$$

Show that if G is a connected plane graph for which every vertex is of degree at least three, then G has a face of degree less than six.

4. (**a**) Define the terms *strongly connected* and *dicomponent* for a digraph D.

Find the dicomponents for the digraph:

D_1

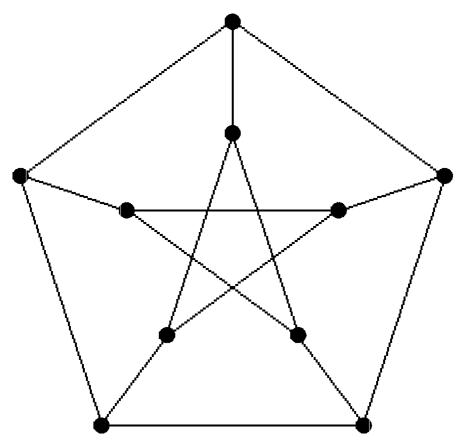


(**b**) Define: D is an *Eulerian* digraph.

Show that a connected digraph D is Eulerian if, and only if, $d^{\rightarrow}(v) = d^{\leftarrow}(v)$ for every vertex v in D.

(**c**) Prove *Robbins' Theorem:* If every edge of a connected graph G is a circuit edge, then G has a strongly connected orientation.

Find a strongly connected orientation of the Petersen graph (below).



continued ...

5. (**a**) Define: G is a *k-critical* graph.

List all 3-critical graphs, and give an example of a 4-critical graph on 4 vertices.

For all $n \geq 6$, construct a 4-critical graph on n vertices.

(**b**) Let G be a Hamiltonian cubic graph. Show that G has a Tait colouring (that is, a 3-edge colouring in which no two adjacent edges take the same colour).

Give an example of a cubic graph with no Tait colouring, proving that your graph has no Tait colouring.

(**c**) State the *Map Colour Theorem*.

Find the Euler characteristic and the chromatic number for the double torus.

END