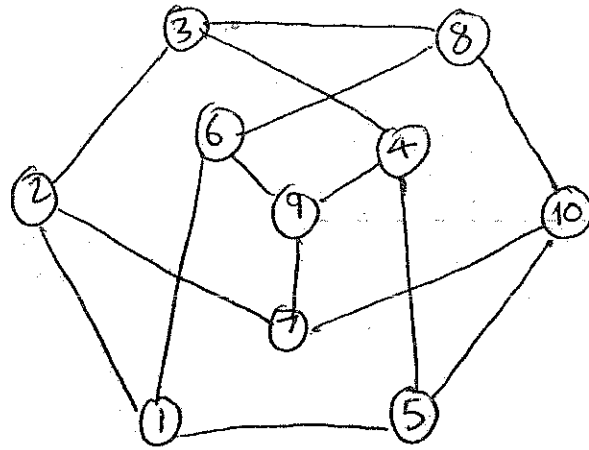
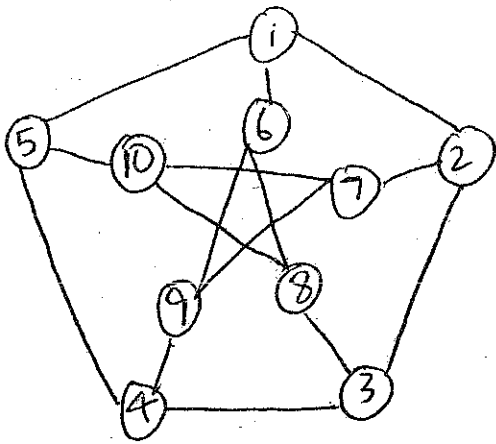


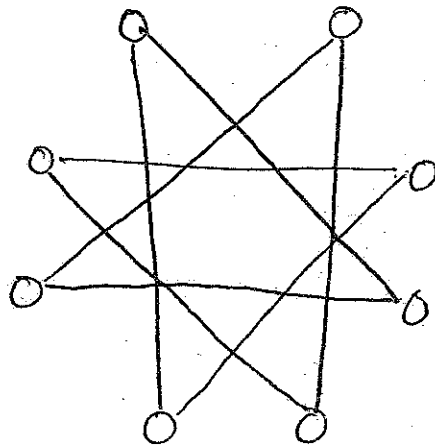
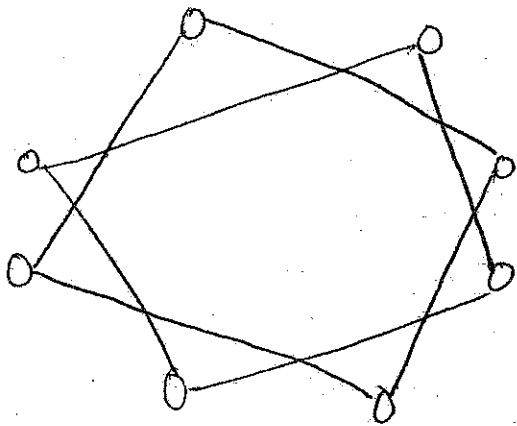
1. (a) Isomorphism is given by:



the vertices

4

(b) Complements of the given graphs are:



3

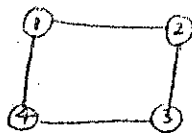
new

Since graphs are isomorphic iff their complements are — and the first is disconnected, and the second is not — the original graphs are not isomorphic.

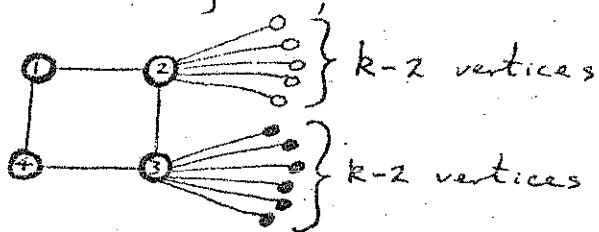
2

2

(c) Let G be k -regular of girth 4. G must contain a circuit of length 4:



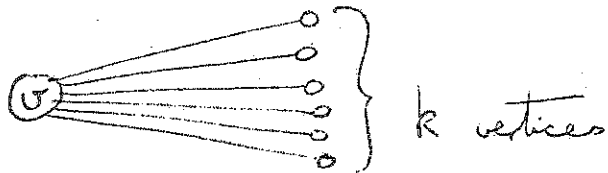
Since G is k -regular, we can extend G :



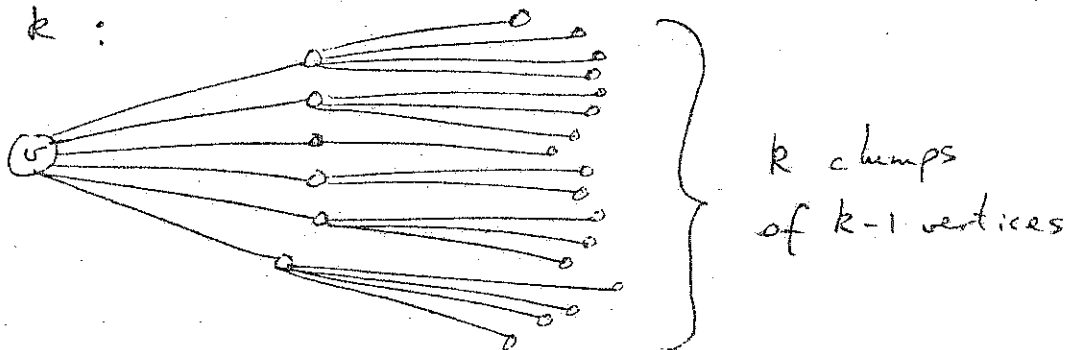
Since G has girth 4, all these vertices are distinct. So G has at least $4 + 2(k-2) = 2k$ vertices. (5)

If G has exactly $2k$ vertices, can take $G = K_{k,k}$. (2)

Let G be k -regular of girth 5. Let v be a vertex of G . Since G is k -regular, v must have degree k :



Again, each of the vertices adjacent to v must have degree k :



Since G has girth 5, all of these vertices must be distinct. So G has at least $1 + k + k(k-1) = k^2 + 1$ vertices. (4)

similar problem sheet

new

similar problem sheet

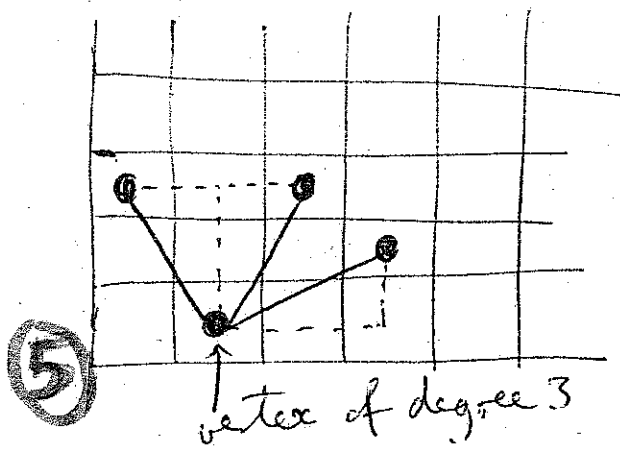
2) (a) An Euler tour for G is a closed chain which includes all edges of G . G is Eulerian iff it has an Euler tour.

① G is Eulerian iff G is connected and all vertices have even degree.

Take a graph G with vertices corresponding to the 64 squares of the chessboard, in which 2 vertices are adjacent iff a knight can move legitimately between the 2 corresponding squares.

Then the knight can travel round the board performing each move just once iff G is semi-Eulerian, i.e., iff G has at most 2 vertices of odd degree.

But it can be seen from the diagram that there are at least 8 vertices of odd degree - so answer is NO.

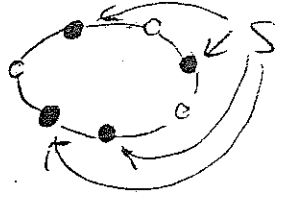


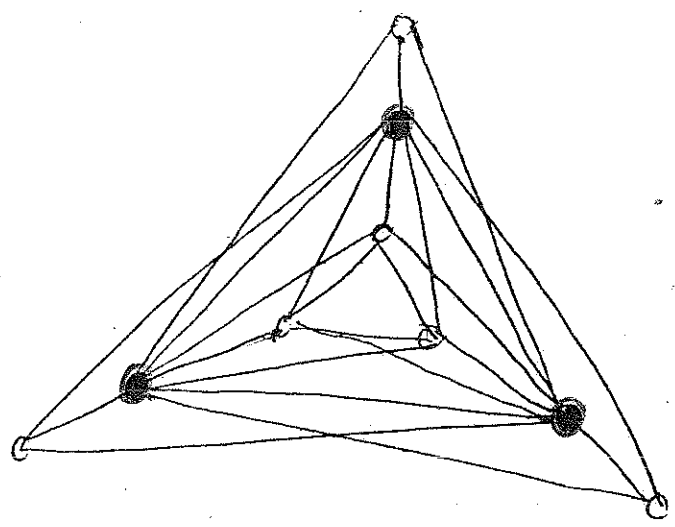
(b) Let C be a Hamilton circuit for G .

Then $c(C-S) \leq |S|$

But $c(G-S) \leq c(C-S)$

so result follows.

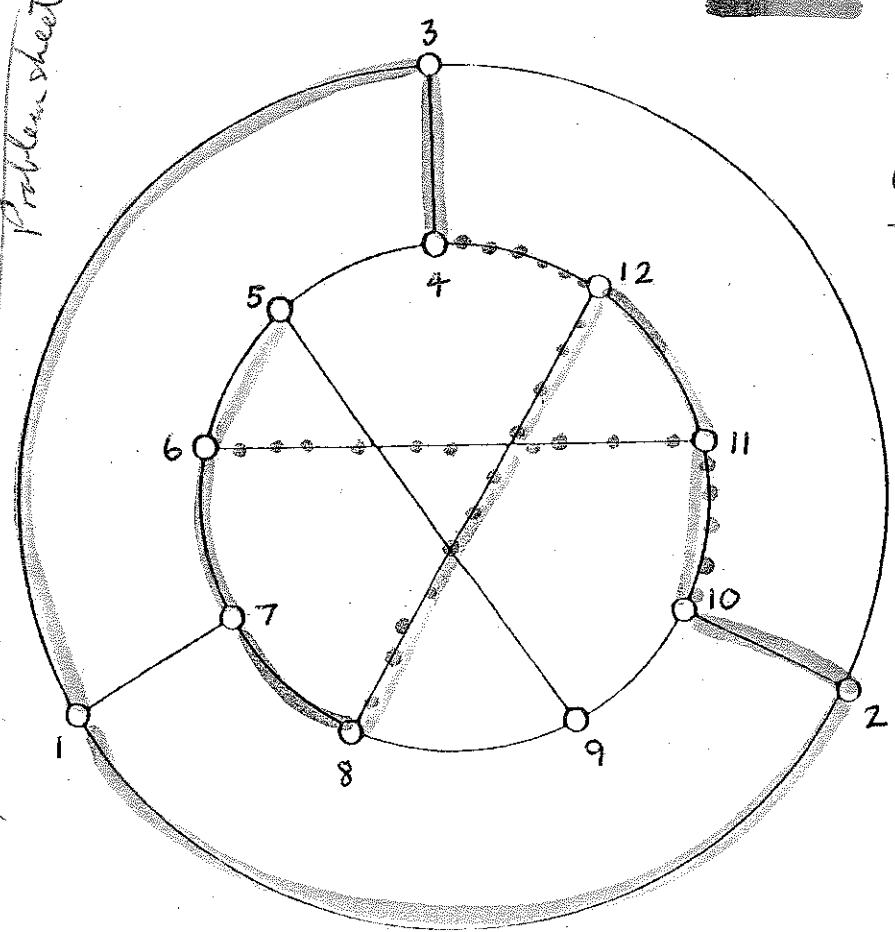




Take S to be the red vertices.
 Then $c(G-S) = 4 \neq |S| = 3$.

⊕ So G is not Hamiltonian.

new, but similar Petersen graph on problem sheet (c)



— = initial form of Hamilton circuit

Case 1. Miss out edge $11 \rightarrow 12$ - then must have edges - impossible

Case 2. Include $11 \rightarrow 12$ (so $11 \rightarrow 10$ or $12 \rightarrow$ Get edges —

Then both $5 \rightarrow 9$ and $5 \rightarrow 4$ are impossible

3) (a) Proof by induction on no. ϵ of edges of G .

IH : Fmla. holds for any connected plane graph with $< \epsilon$ edges

Assume G a connected plane graph with ϵ edges.

Case 1 : G has no circuits. Then G is a tree, and $\epsilon = v - 1$ and $\phi = 1$.

So $\epsilon = v + \phi - 2$, i.e., $v - \epsilon + \phi = 2$.

Case 2 : G has at least one circuit, and ϵ is an edge in this circuit. Then $G - \epsilon$ is a connected plane graph with v vertices, $\epsilon - 1$ edges, and $\phi - 1$ faces.

So by the IH, $v - (\epsilon - 1) + (\phi - 1) = 2$,

or $v - \epsilon + \phi = 2$, and the result follows. (6)

(b) $v(G^*) = \phi(G)$, $\epsilon(G^*) = \epsilon(G)$, $d_{G^*}(f^*) = d_G(f)$

follow straight from the defn. of dual graph. (2)

(i) Let G have dual G^* .

Then $\epsilon = \epsilon^*$, $\phi = v^*$, $v = \phi^*$.

So if $G \cong G^*$, $v = v^* = \phi$

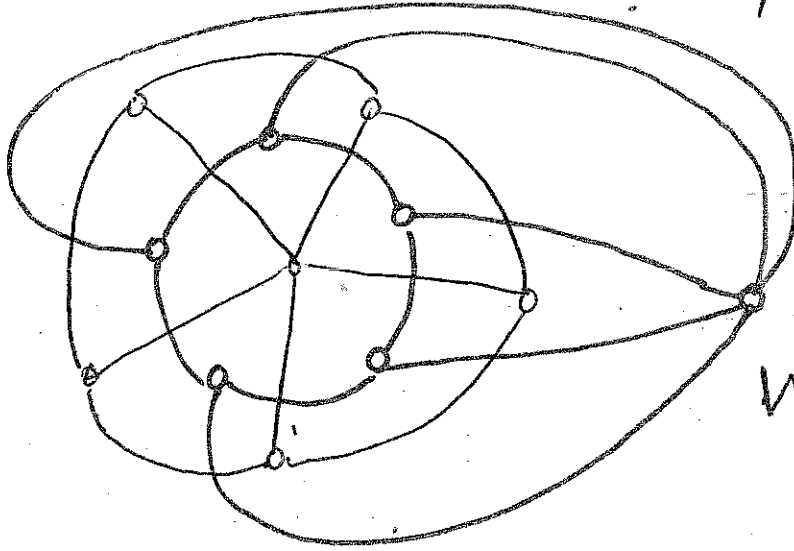
From Euler's Fmla., $v - \epsilon + \phi = v - \epsilon + v = 2$

i.e., $\epsilon = 2v - 2$. (4)

in notes

similar problem sheet

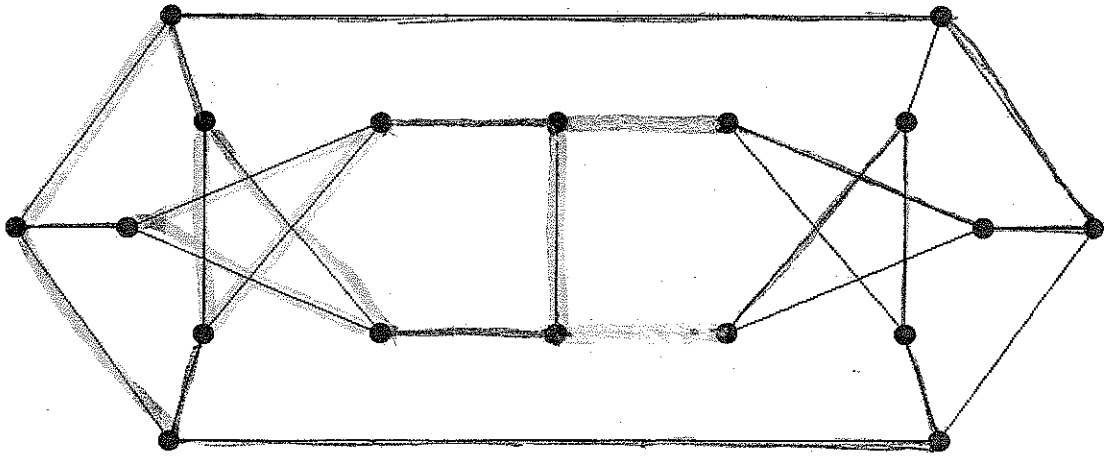
For each $n \geq 4$, the wheel W_{n-1} with $n-1$ spokes is self-dual. For example:



$$W_5^* \cong W_5$$

2

(c)



Take the red and green subgraph shown — and then contract all the red edges to get K_5 .

similar problem sheet

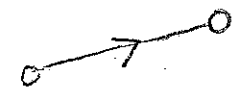
was - did Potomac graph in lectures

4) (a) D is a tournament if it is an orientation of a complete simple graph.

Clearly, any vertex adjacent to a sink cannot be a sink. Similarly for sources.

(b) D is semi-Hamiltonian iff \exists a spanning dipath for D .

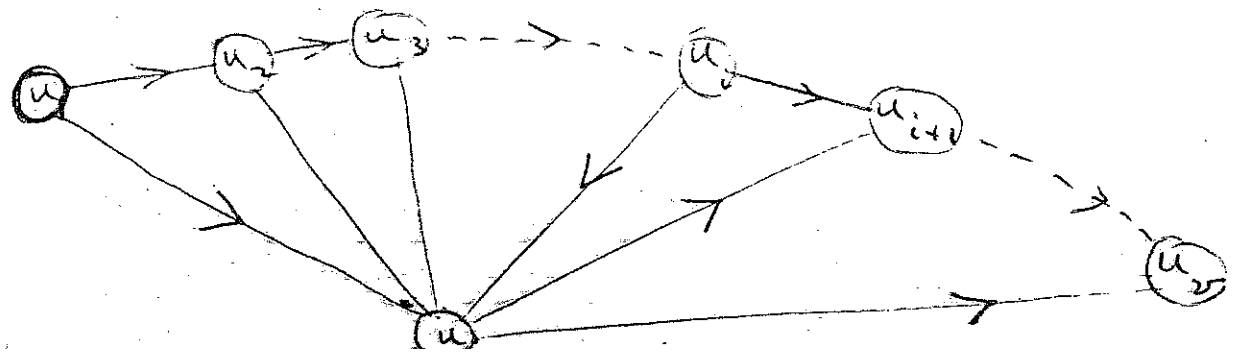
Prove every tournament semi-Hamiltonian by induction on v .

① For $v = 2$, theorem is true! 

② INDUCTIVE HYPOTHESIS. Every tournament on v vertices is semi-Hamiltonian.

Let T be a tournament on $v+1$ vertices.

Let u be a vertex of T . Then $T-u$ is a tournament on v vertices, and by the inductive hypothesis, $T-u$ has a Hamilton dipath $u_1, u_2, u_2, u_3, \dots, u_{v-1}, u_v$.



CASE I, uu_1 is an arc of T .

Then $uu_1, u_1u_2, \dots, u_{v-1}u_v$ is the required Hamilton dipath.

CASE II, u_vu is an arc of T .

Then $u_1u_2, \dots, u_{v-1}u_v, u_vu$ is the Hamilton dipath.

CASE III. Otherwise.

Let i be the least no. for which u_iu and uu_{i+1} are in T .

Then $u_1u_2, \dots, u_iu, uu_{i+1}, \dots, u_{v-1}u_v$ is the required dipath ✓

This completes the induction. \square

II

(c) Let D be a tournament and let v be a vertex of maximum outdegree in D .

Let u be any other vertex of D .

Then either (1) u is an out-neighbour of v , or (2) u is an out neighbour of an out neighbour w of v ,

(3) or (3) v and every out neighbour of v is an out neighbour of u - But (3) would give $d^{\rightarrow}(u) \geq d^{\rightarrow}(v) + 1$, contradicting the choice of v .

5) (a) Let G be an Eulerian map with dual graph G^* . So $d_G(v)$ is even each $v \in V$.

Let C^* be any circuit of G^* , and let V' be the set of vertices inside C^* .

Then length $C^* = \sum_{v \in V'} d_G(v) - \sum_{v \in V'} d_{G[V']}(v)$

\uparrow even since each $d_G(v)$ even \uparrow even since $= 2|E_{G[V']}|$

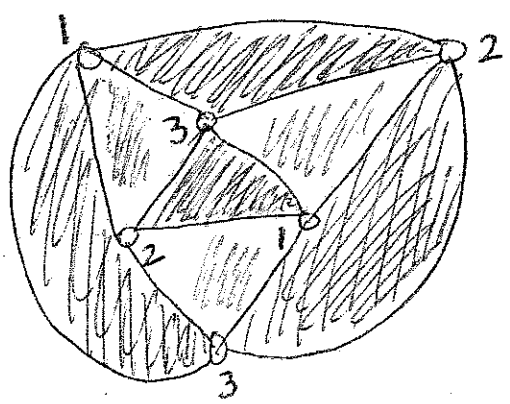
So length C^* is even, so G^* is bipartite.

Hence G is 2-face-colourable.

6

(b) Let G be plane, Eulerian, all faces bounded by triangles. So by part (a), G is 2 face colourable — say the faces of G are coloured red and green.

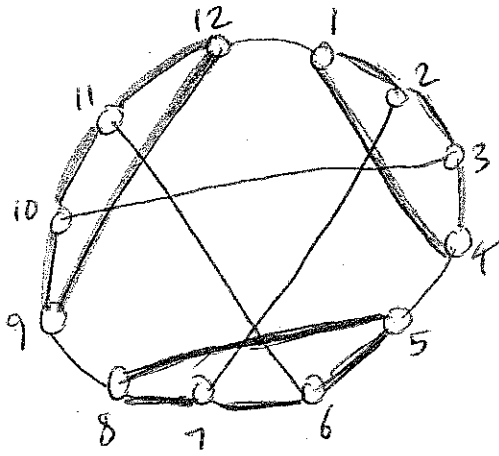
We can now 3-colour the vertices of G as follows: First colour the vertices of a red triangle 1, 2, 3 clockwise. Then move outwards, colouring the vertices of the other red triangles 1, 2, 3 clockwise:



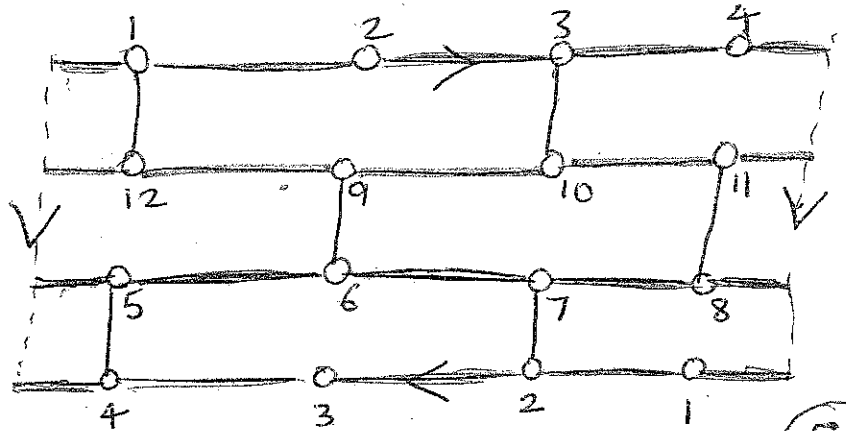
4

in notes

(c)



Franklin graph



Embedding on Klein bottle

5

Since all 6 faces of the embedding are mutually adjacent, need 6 colours, giving $\chi(\text{Klein bottle}) \geq 6$.

2

Can use the above embedding to calculate the Euler characteristic:

2

$$n(\text{Klein bottle}) = v - e + f = 12 - 18 + 6 = 0$$

So Heawood's inequality gives

$$\chi(\text{Klein bottle}) \leq \left[\frac{1}{2} (7 + \sqrt{49 - 24 \times 0}) \right] = 7$$

1