

MATH-303201

This question paper consists of 5 printed pages, each of which is identified by the reference MATH 303201

No calculators allowed

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Examination for the Module MATH 3032

(January 2002)

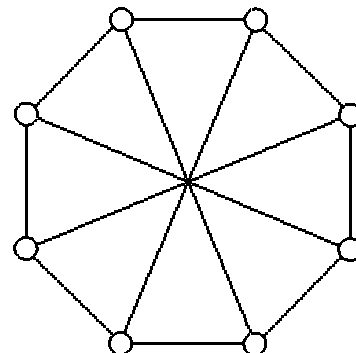
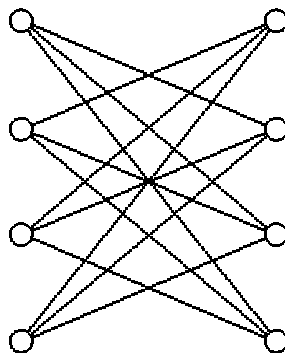
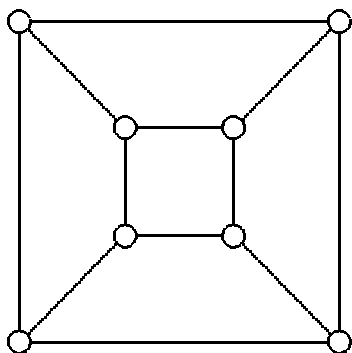
GRAPH THEORY

Time allowed : 2 hours

Do not answer more than *FOUR* questions.

All questions carry equal marks.

1. (a) Show that a simple bipartite graph G with ν vertices has at most $\frac{\nu^2}{4}$ edges.
 (b) Among the graphs below, which pairs are isomorphic?



(Give reasons for your answer.)

- (c) Define: G is *connected*, H is a *component* of G , for G a graph.

Show that if G is a simple graph on ν vertices with ε edges and c components, then G satisfies

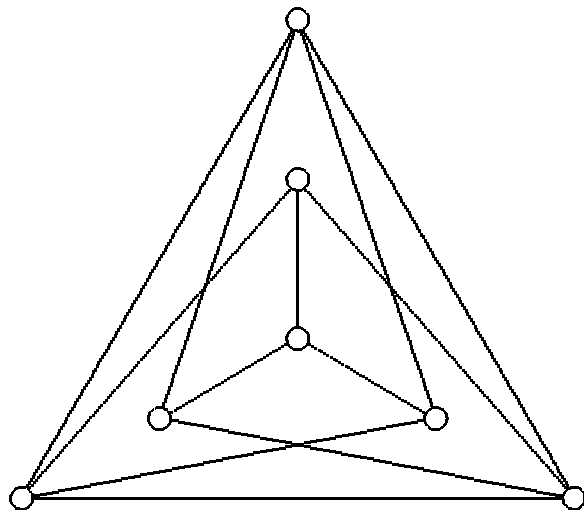
$$\varepsilon \leq \frac{1}{2}(\nu - c)(\nu - c + 1).$$

Say, with reasons, whether there exists a disconnected simple graph with 6 vertices and 11 edges.

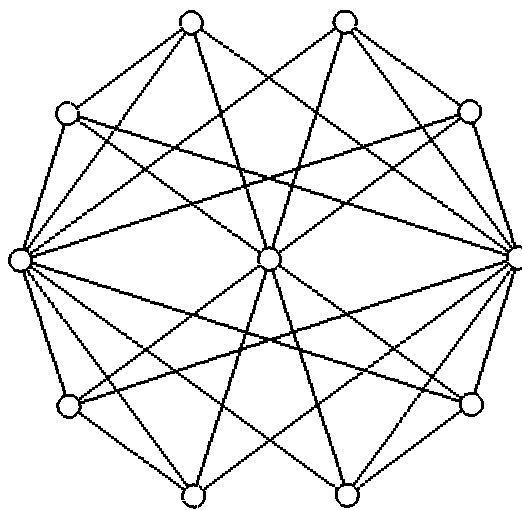
2. (a) Let G be a graph. Define: G is *Eulerian*, G is *Hamiltonian*.

Show that a connected graph G is Eulerian if and only if the degree of each vertex of G is even. [You may assume that every non-empty graph containing no vertices of degree < 2 contains a closed chain.]

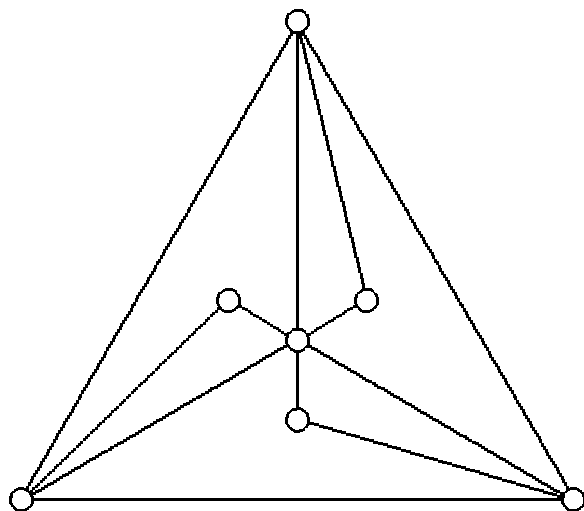
- (b) Say, giving reasons, for each of the four graphs below, whether it is (i) Eulerian, (ii) Hamiltonian.



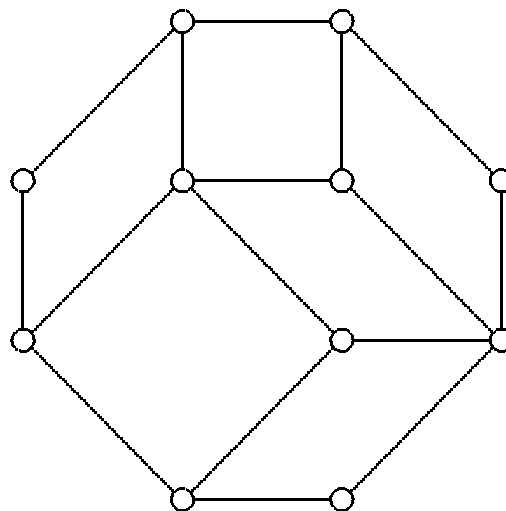
G_1



G_2



G_3



G_4

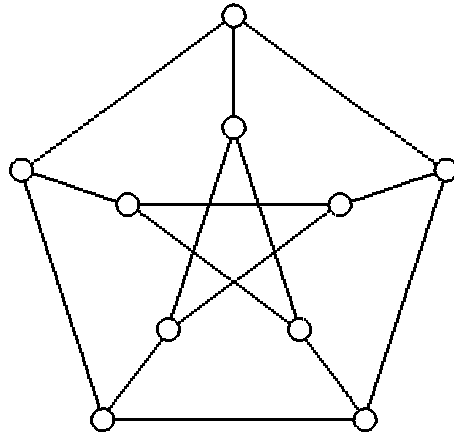
3. (a) Let G be a graph. Define: G is *planar*.

Show that $K_{3,3}$ is not planar, but that it can be embedded on the surface of a Möbius band.

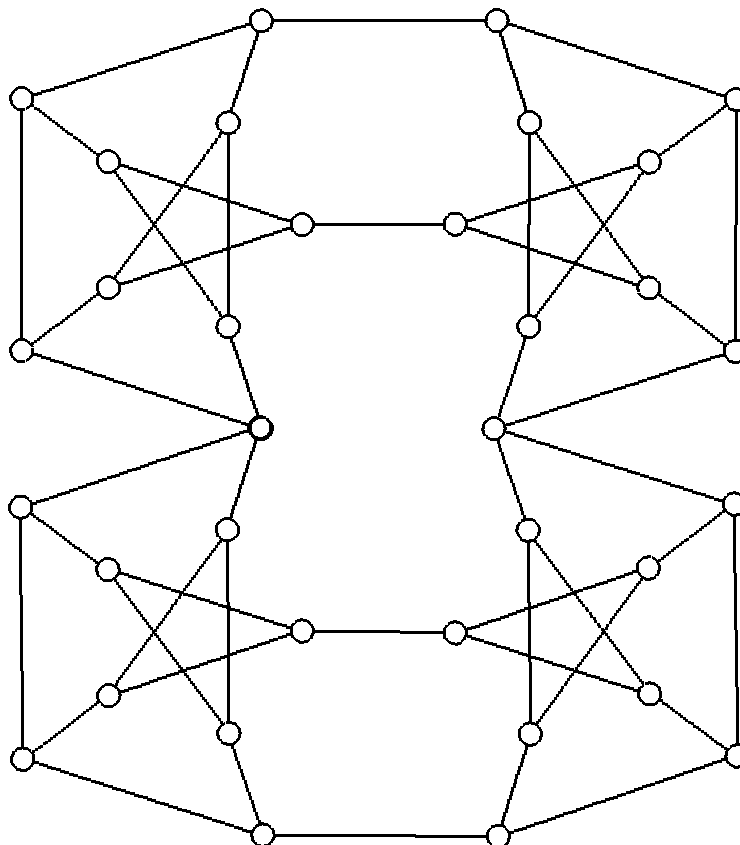
How would your answer change if $K_{3,3}$ were replaced with $K_{3,4}$?

(b) Assume *Wagner's Theorem*, which says that a graph G is planar if, and only if, it contains no subgraph contractible to K_5 or $K_{3,3}$.

Show that the Petersen graph below is non planar.



Further, show that the *Thomassen graph* below is non planar.



(c) If G is a plane connected graph with φ faces, ε edges, $\nu \geq \frac{k}{2} + 1$ vertices, and no circuits of length $< k$, show that $k\varphi \leq 2\varepsilon$.

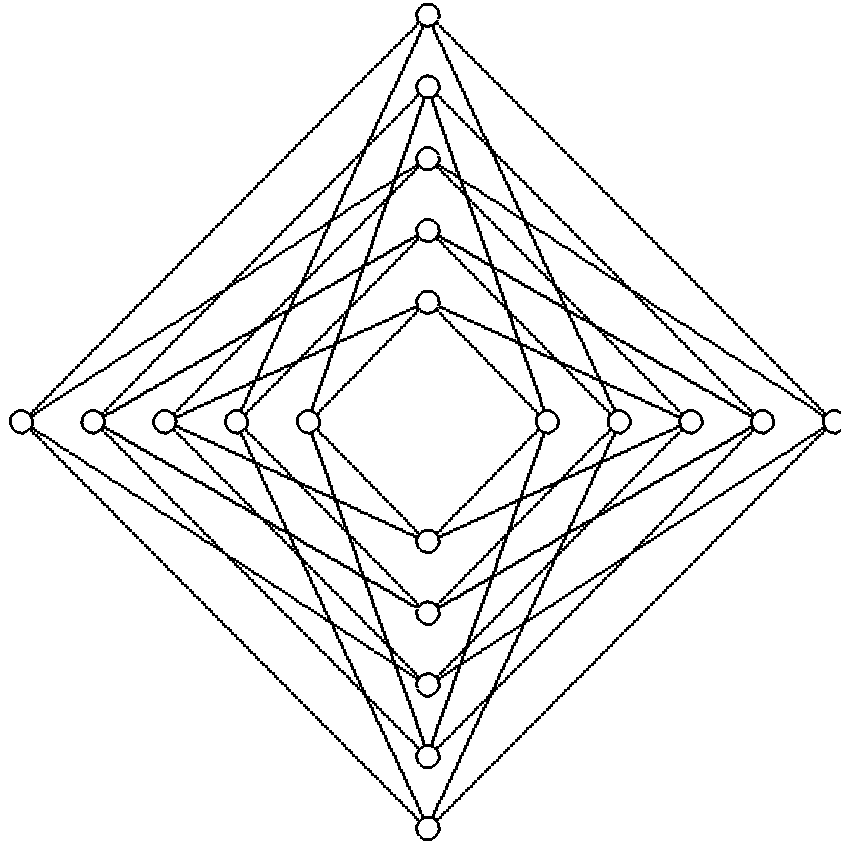
Assuming Euler's formula:

$$\nu - \varepsilon + \varphi = 2,$$

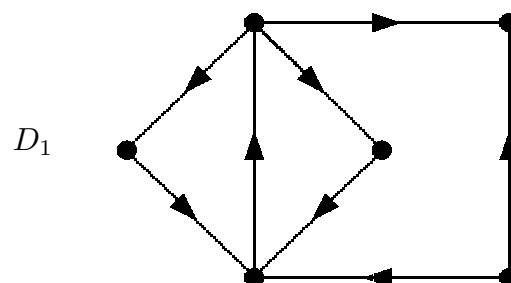
or otherwise, deduce that

$$\varepsilon \leq \frac{k(\nu - 2)}{(k - 2)}.$$

Deduce that the Folkman graph (below) is non-planar:



4. (a) Define the terms *strongly connected* and *dicomponent* for a digraph D . Find the dicomponents for the digraph:



- (b) Define: D is an *Eulerian* digraph.

State (but do not prove) a necessary and sufficient condition on the in-degrees and out-degrees of the vertices of a connected digraph D for D to be Eulerian.

Let D be a connected digraph with $x, y \in V(D)$ such that

(i) $d^{\rightarrow}(x) - d^{\leftarrow}(x) = k = d^{\leftarrow}(y) - d^{\rightarrow}(y)$,

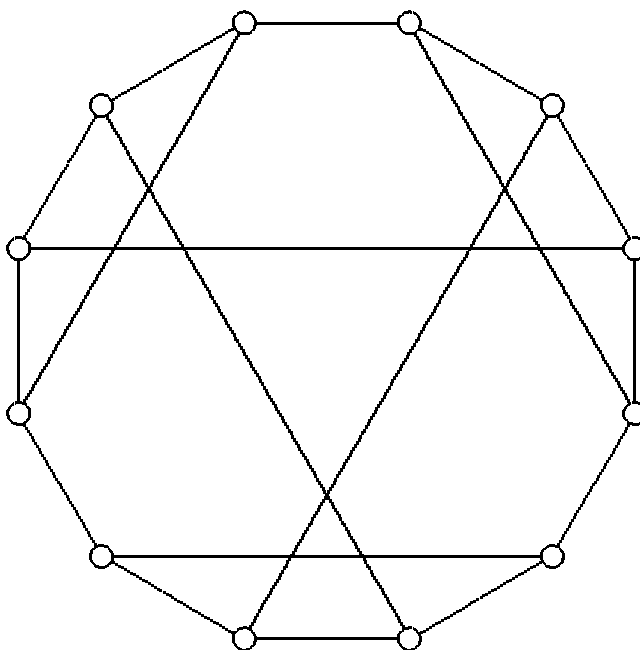
and (ii) $d^{\rightarrow}(v) = d^{\leftarrow}(v)$ for every $v \in V(D) - \{x, y\}$.

Show that there exist k dichains from x to y in D , with no arc in common to any pair of these k dichains.

(c) Show by induction on the number of vertices of D , or otherwise, that a loopless digraph D has an independent set S such that each vertex of D not in S is reachable from a vertex in S by a dipath of length ≤ 2 .

Deduce that a tournament T contains a vertex v from which every other vertex is reachable by a dipath of length ≤ 2 .

- 5. (a) Show that any Eulerian map G is 2-face-colourable.
- (b) Let G be a plane graph in which all the faces are bounded by triangles. Show that if G is Eulerian, then G is 3-colourable.
- (c) Find an embedding of the Franklin graph (below) on the Klein bottle.



Deduce that the chromatic number $\chi(\text{Klein bottle})$ of a Klein bottle is at least 6. Use Heawood's inequality

$$\chi(S) \leq \lceil \frac{1}{2}(7 + \sqrt{49 - 24n}) \rceil$$

for a surface S of Euler characteristic $n < 2$ to show that $6 \leq \chi(\text{Klein bottle}) \leq 7$.

END