§1 GRAPHS

- $G = (V, E)$ is a (simple) graph if $V$ is a finite set (of vertices) and $E$ is a set of edges of the form $\{u, v\}$, with $u, v \in V$ and $u \neq v$.

- If $\{u, v\} \in E$, say $u, v$ are joined by $\{u, v\}$ — and often write $uv$ for $\{u, v\}$.

**Write:**

- $n$ or $\nu$ for $\#(V)$.

- $e$ or $\varepsilon$ for $\#(E)$.

- $\rho(v)$ or $d(v)$ for the degree of $v$,
  $\text{defn}$ the no. of edges of $G$ with endpoint $v$.

**Theorem 1.1 — The Handshaking Lemma:**

\[
\sum_{v \in V} d(v) = 2\varepsilon
\]
• $G_1, G_2$ are isomorphic iff there is a bijective $\vartheta : V_1 \to V_2$ such that

$$uv \in E_1 \iff \vartheta(u)\vartheta(v) \in E_2, \quad \text{for all } u, v \in V_1.$$ 

• In which case — say $\vartheta$ is an isomorphism, and write $G_1 \cong G_2$.

• The adjacency matrix of a simple graph $G$ with vertices $V = \{v_1, \ldots, v_\nu\}$ is $A_G = (a_{i,j})$ where

$$a_{i,j} = \begin{cases} 1 & \text{if } v_i v_j \in E, \\ 0 & \text{otherwise}. \end{cases}$$
§4 CONNECTED GRAPHS

• Let $v_0, \ldots, v_k \in V$ and $v_i v_{i+1} \in E$, each $i = 0, \ldots, k - 1$. Then say $\{v_0 v_1, v_1 v_2, \ldots, v_{k-1} v_k\}$ is an edge sequence from $v_0$ to $v_k$.

• And — if all the $v$’s are distinct (except for possibly $v_0 = v_k$) — say the edge sequence is a path.

• While if $v_0 = v_k$, say the edge sequence is closed — and, if also a path, is a circuit or cycle.

• Say $u$ is connected to $v$ iff there is an edge sequence from $u$ to $v$ —

• Which gives an equivalence relation on $V$, dividing $G$ into components.

• Say $G$ is connected iff it has just one component — i.e., iff each pair $u, v \in V$ is connected.
**Lemma 4.1**: Let $v_1$ be a vertex of $G$. Then $G$ is connected iff for every other $v \in V$ there is an edge sequence from $v_1$ to $v$.

- This gives the *connectedness algorithm*:

  **Step 1.** Choose $X[1] = \{v_1\}$, $V_1 = V - X[1]$.

  **Step I+1.** Assume $V_I$ already defined. Let
  
  $X[I] = \{u | u \in V_I \& uv \in E, some u \in V - V_I\}$.

  **Is $V_I = \emptyset$?** — Yes: Stop — $G$ connected.

  **No:** **Is $X[I] = \emptyset$?** — Yes: $G$ not connected.

  **No:** Define $V_{I+1} = V_I - X[I]$. Go to step $I+2$.

**Lemma 4.2**: Let $G$ be a graph with $\nu$ vertices and $\varepsilon$ edges, and with each vertex of degree $\geq 2$ — Then $\nu \leq \varepsilon$.

**Lemma 4.3**: Let $G$ be a connected graph with $\nu$ vertices and $\varepsilon$ edges — Then $\nu - 1 \leq \varepsilon$. 
• A tree is a connected graph with no circuits.

**Theorem 5.1:** Let $G$ be a graph with $\nu$ vertices. The following are equivalent:

(a) $G$ is a tree.

(b) $G$ has no circuits and has $\nu - 1$ edges.

(c) $G$ is connected and has $\nu - 1$ edges.

(d) $G$ is connected, but the removal of any one edge disconnects the graph.

(e) Each pair of vertices is connected by just one path.

(f) $G$ has no circuits, but the addition of any one edge creates a circuit.

• A subgraph of $G = (V, E)$ is a graph $G' = (V', E')$ for which $V' \subseteq V$ and $E' \subseteq E$.

• A spanning subgraph $G'$ of $G$ is one for which $V' = V$ — and $G'$ is a spanning tree for $G$ if it is also a tree.
• If a weight function $\mu : E \to \mathbb{R}^{\geq 0}$ associates a non-negative real number with each edge of $G$ — say $G$ is a weighted graph.

• If $G' = (V', E')$ is a subgraph of $G$, the weight (or measure) of $G'$ is
  $$M(G') = \sum_{\varepsilon \in E'} \mu(\varepsilon).$$

• A minimal connector for $G$ is a connected spanning subgraph of $G$ least possible weight.

**Kruskal’s Algorithm:** Given weighted graph $G$ with weight function $\mu : E \to \mathbb{R}^{\geq 0}$:

For $k = 1, \ldots, \nu - 1$ choose $e_k$ to be an edge s.t.

• $e_k$ is different from $e_1, \ldots, e_{k-1}$,

• There is no circuit made up of edges from $\{e_1, \ldots, e_{k-1}, e_k\}$, and

• $\mu(e_k)$ is as small as possible.

Then — the edges $E' = \{e_1, \ldots, e_{\nu-1}\}$ give a minimal connector for $G$. 
• The complete graph on \( n \) vertices — written \( K_n \) — is the graph on \( n \) vertices with all possible edges.

• A graph \( G \) is planar iff it can be embedded in the plane so that edges meet only at vertices.

• In which case the regions into which the plane is divided by this embedding are called the faces.

**NOTATION:** Write \( \varphi \) or \( f \) for the number of faces in the embedding of \( G \).

• A cut edge — or isthmus — is an edge whose removal disconnects the graph.

**Theorem 7.1 Euler’s Formula for Planar Graphs:** Let \( G \) be a connected planar graph with \( \nu \) vertices and \( \varepsilon \) edges. Then if \( G \) is embedded in \( \mathbb{R}^2 \) with \( \varphi \) faces have

\[
\nu + \varphi = \varepsilon + 2.
\]
**Theorem 7.2**: Let $G$ be a connected planar graph with $\nu$ vertices and $\varepsilon$ edges. If $\nu \geq 3$, then

$$\varepsilon \leq 3\nu - 6.$$ 

**Theorem 7.3**: Let $G$ be a connected planar graph with $\nu$ vertices and $\varepsilon$ edges, in which each circuit has at least $k$ edges. If $\nu \geq \frac{k+2}{2}$, then

$$\varepsilon \leq \frac{k}{k-2}(\nu - 2).$$