MATH221001

This question paper consists of 4 printed pages, each of which is identified by the reference MATH221001

Only approved basic scientific calculators may be used.

Examination for the Module MATH2210
(May/June 2003)

INTRODUCTION TO DISCRETE MATHEMATICS

Time allowed : 2 hours

Do not answer more than FOUR questions

All questions carry equal marks.

1. (**a**) To enter the National Lottery of Ruritania, you must choose 10 different integers in the range from 1 to 100. After the lottery has been closed, the Lottery draw chooses 12 numbers selected at random. You win prizes as follows:

Jackpot: All 10 of your numbers occur among the 12 that are drawn by the Lottery.

Consolation prize: Exactly 8 or 9 of your numbers occur among the 12 numbers that are drawn.

i) Calculate the probability that in one play of the Lottery, involving just one choice of 10 numbers, you win the jackpot.

ii) Also calculate your chances of winning a consolation prize.

iii) If entrants to the Ruritanian Lottery choose their numbers at random and 10 million plays are made, what is the probability that no-one will win the jackpot?

(**b**) A dice is thrown 12 times. What is the probability that each of the numbers 1, 2, 3, 4, 5, 6 comes up exactly twice?

(**c**) Let $A_1, \ldots, A_n$ be finite sets.

State (without proof) the Inclusion-Exclusion Principle giving $\#(A_1 \cup \cdots \cup A_n)$ in terms of numbers of the form $\#(A_{i_1}, \cap \cdots \cap A_{i_k})$, with $1 \leq i_1, < \cdots < i_k \leq n$, $1 \leq k \leq n$.

How many integers are there in the range from 1 to 10^{12}, which are either perfect squares, or perfect cubes, or both?

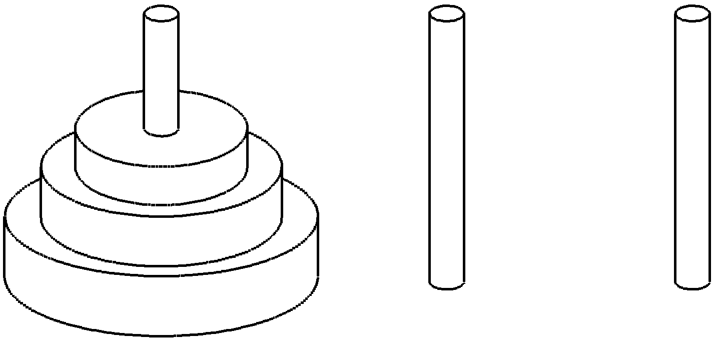
2. (**a**) Let e_n be the number of sequences of length n made up of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, in which the digit 0 occurs an even number of times (noting that 0 counts as an even number).

Find a recurrence relation satisfied by e_n.

Solve this recurrence relation, so finding a general formula for e_n in terms of n.

Question 2 continues ...

(**b**) There are three pegs on a stand. n disks, all of different sizes, are placed on one peg in order of increasing size downwards.



Let a_k = the number of moves of a disk needed to transfer k disks from one peg to another, only ever putting one disk on top of a larger disk.

Show that $a_{k+1} = 2a_k + 1$, with $a_1 = 1$, and hence find the total number of moves needed to transfer 64 disks to a different peg.

3. (**a**) Use the connectedness algorithm to determine whether the graph given by the following adjacency matrix is connected.

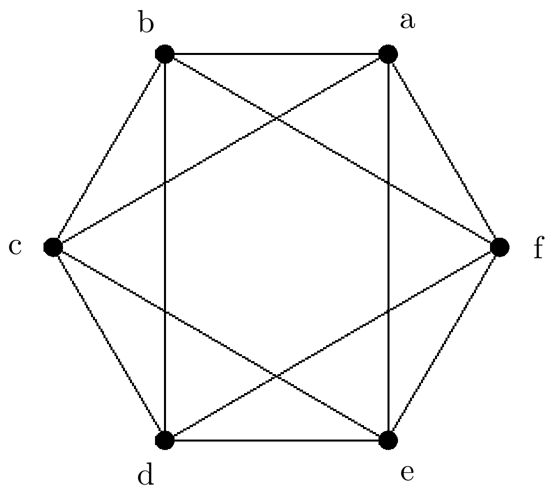
	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
v_1	0	1	0	0	0	1	1	0	1	0
v_2	1	0	0	0	1	1	1	0	1	0
v_3	0	0	0	1	1	0	0	1	0	1
v_4	0	0	1	0	0	0	0	1	0	1
v_5	0	1	1	0	0	1	0	0	0	0
v_6	1	1	0	0	1	0	1	0	0	0
v_7	1	1	0	0	0	1	0	0	1	0
v_8	0	0	1	1	0	0	0	0	0	1
v_9	1	1	0	0	0	0	1	0	0	0
v_{10}	0	0	1	1	0	0	0	1	0	0

(**b**) **i)** Define what is meant by saying that a graph is a *tree*.

ii) Let G be a graph with ν vertices. Show that if G is a tree then G has exactly $\nu - 1$ edges.

(**c**) Show that a minimal connector for a weighted graph G must be a spanning tree for G.

Find a minimal connector G' for the following graph and specify the value of $M(G')$.



edge	μ
ab	3
ac	8
ae	11
af	5
bc	7
bd	7
bf	6
cd	5
ce	12
de	10
df	14
ef	15

4. (**a**) Let G be a connected planar graph with ν vertices and ε edges which is drawn in the plane with φ faces. Show that

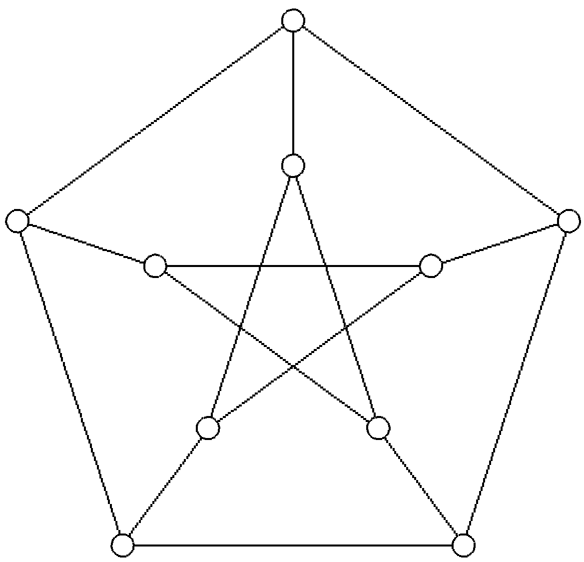
$$\nu + \varphi = \epsilon + 2.$$

(**b**) Deduce that if $\nu \geq 4$ and each closed path in this connected planar graph G has at least 5 edges, then

$$3\varepsilon \leq 5\nu - 10\,.$$

(**c**) Show further that if every vertex of G has degree 3, then G must have at least 20 vertices.

(**d**) Determine whether or not the **Petersen Graph** (shown below) is planar or not.



continued . . .

5. (**a**) Consider the URM program:

1. $J(1,3,7)$
2. $S(2)$
3. $S(2)$
4. $S(2)$
5. $S(3)$
6. $J(1,1,1)$
7. $T(2,1)$

i) Draw the flow chart corresponding to this program.

ii) Give the full trace table of the URM computation using this program for the single number inputs 0 and 1.

iii) Find the output of the computation using this program for input 3.

iv) Describe the function $f : \mathbb{N} \to \mathbb{N}$ computed by this program.

(**b**) Devise a URM program to compute the function $f : \mathbb{N}^2 \to \mathbb{N}$ given by

$$f(m,n) = 3m + 4n$$

6. (**a**) Prove that if $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ are both URM-computable functions, then so also is the function $h : \mathbb{N} \to \mathbb{N}$ given by

$$h(n) = g(f(n)) \,.$$

(**b**) Show that the functions $f(m,n) = m \times n$ and $g(m) = m!$ are primitive recursive.

Assuming that proper subtraction

$$m \dot{-} n = \begin{cases} m - n & \text{if } m > n \\ 0 & \text{otherwise} \end{cases}$$

is primitive recursive, show that so are

$$\min(m,n) = \begin{cases} n & \text{if } m \geq n \\ m & \text{otherwise.} \end{cases}$$

and

$$\max(m,n) = \begin{cases} m & \text{if } m \geq n \\ n & \text{otherwise.} \end{cases}$$

(**c**) Prove that for every URM-computable function $f : \mathbb{N} \to \mathbb{N}$ there is a strictly increasing URM-computable function $g : \mathbb{N} \to \mathbb{N}$ which dominates f.

END