

## MATH221001

This question paper consists of 5 printed pages, each of which is identified by the reference MATH221001

Only approved basic scientific calculators may be used.

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Examination for the Module MATH2210

(May/June 2002)

**INTRODUCTION TO DISCRETE MATHEMATICS**

Time allowed : 2 hours

**Do not answer more than FOUR questions**

All questions carry equal marks.

1. (a) The National Lottery involves choosing 6 different integers in the range from 1 to 49. In each draw, 6 main numbers plus the bonus number are drawn at random. You win prizes as follows:

*Jackpot:* Choose all 6 of the main numbers.

*Second prize:* Choose 5 of the 6 main numbers and the bonus number.

*Third prize:* Choose 5 of the 6 main numbers.

*Fourth prize:* Choose 4 of the 6 main numbers.

*Fifth prize:* Choose 3 of the 6 main numbers.

- i) Calculate the probability that in one play of the lottery, involving just one choice of six numbers, you win the jackpot.
- ii) Also calculate your chances of winning each of the second and third prizes.

- (b) Let  $A_1, \dots, A_n$  be finite sets.

State (without proof) the Inclusion-Exclusion Principle giving  $\#(A_1 \cup \dots \cup A_n)$  in terms of numbers of the form  $\#(A_{i_1} \cap \dots \cap A_{i_k})$ , with  $1 \leq i_1 < \dots < i_k \leq n$ ,  $1 \leq k \leq n$ .

- i) Let  $\theta_n(s)$  denote the probability that if  $n$  objects are sampled with replacement, then each object has appeared after  $s$  samples.

Show that if  $s \geq n$ , then the value of  $\theta_n(s)$  is given by the formula:

$$\theta_n(s) = \frac{1}{n^s} \sum_{k=0}^n (-1)^k C(n, k) (n-k)^s,$$

where  $C(n, k)$  is the number of ways of choosing  $k$  objects from  $n$ .

- ii) Find the probability that in 8 throws of a dice each of the numbers 1,2,3,4,5,6 is thrown at least once.

2. (a) The *Fibonacci sequence* is defined as follows:

$$\begin{aligned} \text{FIB}(0) &= \text{FIB}(1) = 1, \\ \text{FIB}(n) &= \text{FIB}(n-1) + \text{FIB}(n-2), \quad \text{for } n \geq 2. \end{aligned}$$

Show that for all  $n \geq 0$

$$\text{FIB}(n) = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \right].$$

- (b) The plane is divided into a number of regions by the drawing of  $n$  circles. Let  $r_n$  denote the maximum possible number of such regions.

Find the values of  $r_1$ ,  $r_2$ ,  $r_3$  and  $r_4$ .

Find a recurrence relation of the form

$$r_{n+1} = r_n + g(n)$$

for  $r_n$ ,  $n \geq 1$ .

Solve this recurrence relation, giving an explicit value for  $r_n$  for all  $n \geq 1$ .

3. (a) i) Let  $G$  be a graph with  $\nu$  vertices and  $\varepsilon$  edges, and with each vertex being of degree  $\geq 2$ . Show that  $\nu \leq \varepsilon$ .

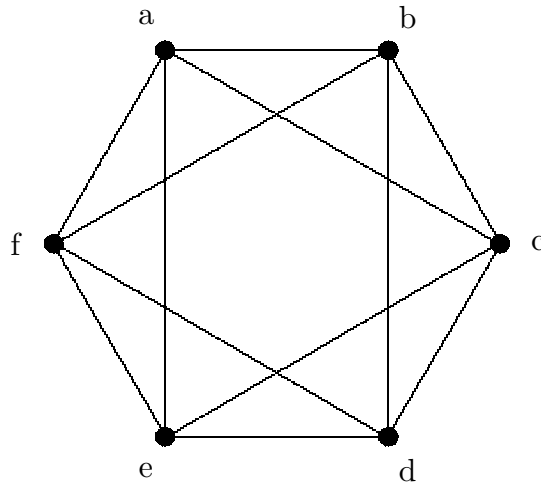
ii) Let  $G$  be a connected graph with  $\nu$  vertices and  $\varepsilon$  edges. Show that  $\nu - 1 \leq \varepsilon$ .

- (b) A graph  $G$  is given by the following adjacency matrix. Determine, *without drawing*  $G$ , whether  $G$  is connected or not.

$$\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \end{array} \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

- (c) Let  $G = (V, E)$  be a graph with a weight function  $\mu : E \rightarrow \mathbb{R}^{>0}$  which assigns a positive real number to each edge. Explain what is meant by a *minimal connector* for  $G$ .

Use Kruskal's algorithm to find a minimal connector  $G'$  for the following graph, and give the value of  $M(G')$ .



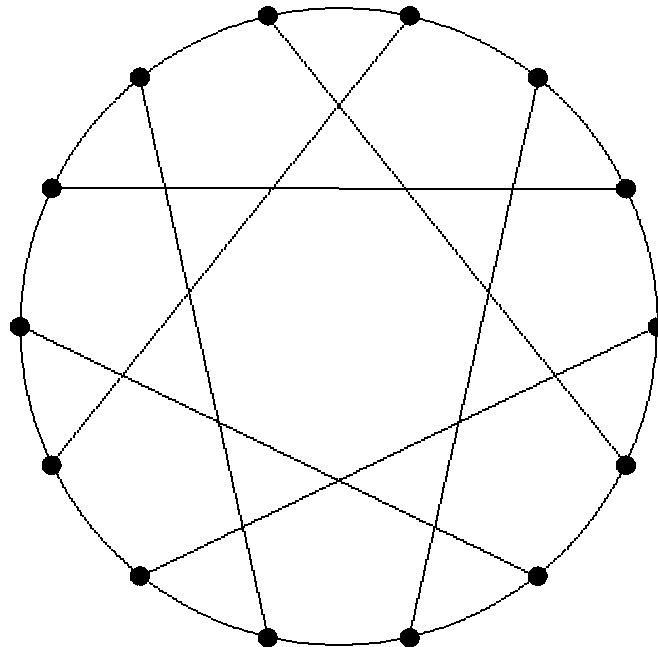
edge	$\mu$
$ab$	3
$ac$	8
$ae$	11
$af$	5
$bc$	7
$bd$	7
$bf$	6
$cd$	5
$ce$	12
$de$	10
$df$	14
$ef$	15

4. (a) State (without proof) *Euler's Formula* for a connected planar graph drawn in the plane with  $\nu$  vertices,  $\varepsilon$  edges and  $\varphi$  faces.

(b) Prove that if  $G$  is a connected planar graph with  $\nu$  vertices with  $\nu \geq 3$ , and  $\varepsilon$  edges, in which all closed paths contain at least 6 edges, then

$$2\varepsilon \leq 3\nu - 6.$$

(c) Hence, or otherwise, deduce that the Heawood graph (below) is not planar.



(d) i) Let  $G$  be a connected planar graph, with no vertex of degree  $< 2$ , in which all closed paths contain at least 6 edges. Show that  $G$  must have at least 6 vertices of degree *exactly* 2.

- ii) Let  $G$  be a connected planar graph, in which all closed paths contain at least 6 edges, and which has exactly  $\frac{3\nu - 6}{2}$  edges.

Show that  $G$  cannot have an odd number of vertices.

Show, however, that  $G$  can have any even number  $\nu = 2n + 2$  of vertices for  $n \geq 2$ .

5. (a) Consider the URM program  $P_f$ :

1.  $T(1, 3)$
2.  $T(2, 4)$
3.  $J(1, 4, 9)$
4.  $J(2, 3, 9)$
5.  $S(3)$
6.  $S(4)$
7.  $S(5)$
8.  $J(1, 1, 3)$
9.  $T(5, 1)$

i) Draw the flow chart corresponding to this program.

ii) Give the full trace table of the URM computation using  $P_f$  for the input  $(1, 1)$ .

iii) Find the output of the computation using  $P_f$  for input  $(1, 2)$ .

iv) Describe the function  $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  computed by  $P_f$ .

- (b) Devise a URM program  $P_g$  to compute the function  $g : \mathbb{N} \rightarrow \mathbb{N}$  given by

$$g(m) = 3m + 2.$$

By using the above programs  $P_f$  and  $P_g$ , or otherwise, write down a URM program which computes the function  $h : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  given by

$$h(m, n) = 3|m - n| + 2.$$

6. (a) Show that the predecessor function  $\text{pred}(x)$  given by

$$\text{pred}(m) = \begin{cases} 0 & \text{if } m = 0 \\ m - 1 & \text{if } m > 0 \end{cases}$$

is primitive recursive.

Deduce that proper subtraction  $\dot{-}$  is also primitive recursive, where

$$m \dot{-} n = \begin{cases} m - n & \text{if } m > n \\ 0 & \text{otherwise.} \end{cases}$$

Hence, or otherwise, show that the minimum  $\min(m, n)$  of  $m$  and  $n$ , defined by

$$\min(m, n) = \begin{cases} n & \text{if } m \geq n \\ m & \text{otherwise,} \end{cases}$$

is primitive recursive.

(b) The *Busy Beaver Function* is defined by

$B(n)$  = the maximum output, for input 0, of any URM program with at most  $n$  instructions.

- i) Show that  $B$  is strictly increasing.
- ii) Show that for all  $n \geq 1$ ,  $B(n + 5) \geq 2n$ .
- iii) Let  $g$  be a URM computable function, computed by a URM program  $P_g$  of length  $\ell(P_g) = k_0$ . Show that for each number  $n \geq 0$  there is a URM program with at most  $n + k_0$  instructions which gives output  $g(B(n))$  from input 0.

Deduce that  $B(n + k_0) \geq g(B(n))$ , each  $n \geq 0$ .

iv) Show that if  $g$  is known to be strictly increasing, then

$$B(n + 5 + k_0) \geq g(2n),$$

for each  $n \geq 1$ , and hence that  $B$  dominates  $g$ .

v) Deduce that  $B$  is not URM computable.

[You may assume that every URM computable function  $f$  is dominated by a strictly increasing URM computable function  $g$ .]

**END**