MATH221001

This question paper consists of 4 printed pages, each of which is identified by the reference MATH2210

Only approved basic scientific calculators may be used.

Examination for the Module MATH2210
(May/June 2000)

INTRODUCTION TO DISCRETE MATHEMATICS

Time allowed : 2 hours

Do not answer more than FOUR questions
All questions carry equal marks.

1. (**a**) A bridge hand consists of 13 cards drawn from a pack of 52 cards.

i) How many different bridge hands are there?

ii) How many different bridge hands are there which do not contain any aces?

iii) How many different bridge hands are there which contain no card higher than a nine (with aces counting high)? Such a hand is called a *Yarborough*.

Assuming that a bridge hand is dealt at random, what is the probability of getting a Yarborough?

(**b**) Let $A_1, \ldots, A_n$ be finite sets.

State (without proof) the Inclusion-Exclusion Principle giving $\#(A_1 \cup \cdots \cup A_n)$ in terms of numbers of the form $\#(A_{i_1}, \cap \cdots \cap A_{i_k})$, with $1 \leq i_1, < \cdots < i_k \leq n$, $1 \leq k \leq n$.

How many integers are there in the range from 1 to 10^{12}, which are either perfect squares, or perfect cubes, or both?

(**c**) Let k be a number between l and n, and let D_k = the set of permutations of $\{1, 2, \ldots, n\}$ taking k to its original position.

Show that

$$\#\Big(\bigcup_{k=1}^{n} D_k\Big) = n! \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k!} \,.$$

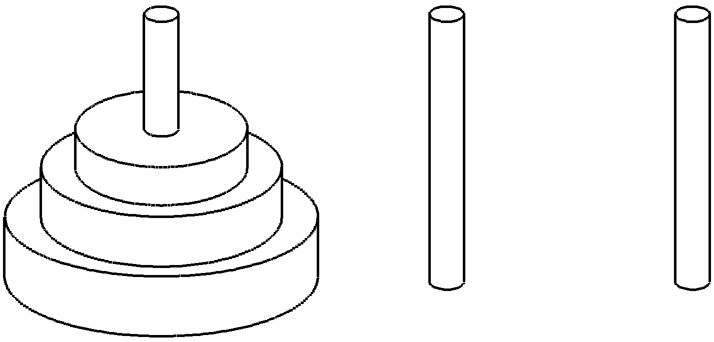
If two packs of cards are dealt one by one simultaneously, what is the probability that there is at least one coincidence of cards?

2. (**a**) Solve the recurrence relation

$$a_{n+1} = 3a_n + 8n, \;\; n \geq 1$$

subject to the initial condition $x_1 = 6$.

(**b**) There are three pegs on a stand. n disks, all of different sizes, are placed on one peg in order of increasing size downwards.



Let a_k = the number of moves of a disk needed to transfer k disks from one peg to another, only ever putting one disk on top of a larger disk.

Show that $a_{k+1} = 2a_k + 1$, with $a_1 = 1$, and hence find the total number of moves needed to transfer 64 disks to a different peg.

3. (**a**) Determine whether the graph given by the following adjacency matrix is connected.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
v_1	0	1	0	1	0	0	0	0	0	0	0	0
v_2	1	0	0	0	1	0	0	0	0	0	0	0
v_3	0	0	0	1	0	0	1	0	0	0	0	0
v_4	1	0	1	0	0	0	1	0	0	0	0	0
v_5	0	1	0	0	0	1	0	0	1	0	0	0
v_6	0	0	0	0	1	0	0	0	1	1	0	0
v_7	0	0	1	1	0	0	0	0	0	0	0	0
v_8	0	0	0	0	0	0	0	0	1	0	1	1
v_9	0	0	0	0	1	1	0	1	0	1	0	1
v_{10}	0	0	0	0	0	1	0	0	1	0	0	0
v_{11}	0	0	0	0	0	0	0	1	0	0	0	1
v_{12}	0	0	0	0	0	0	0	1	1	0	1	0

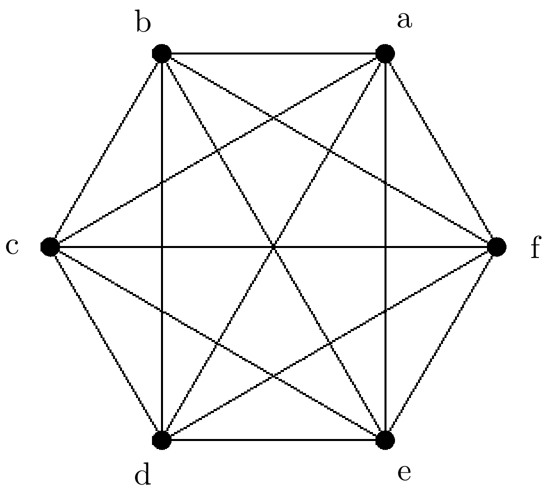
(**b**) **i**) Define what is meant by saying that a graph is a *tree*.

ii) List all the non-isomorphic trees with at most 4 vertices.

iii) Prove that for all $\nu \geq 1$, a tree with ν vertices has $\nu - 1$ edges.

(**c**) Find a minimal connector G' for the following graph and specify the value of $M(G')$.

Question 3 continues . . .



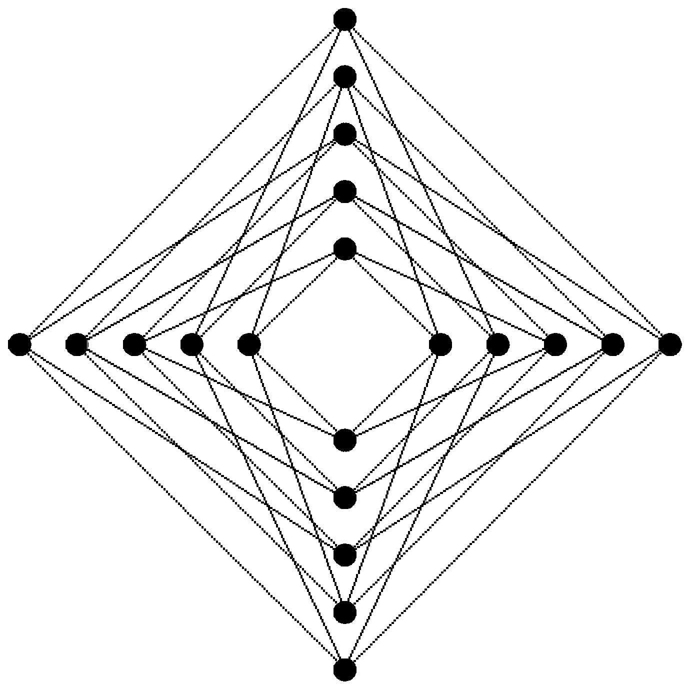
edge	μ
ab	8
ac	9
ad	16
ae	17
af	19
bc	10
bd	14
be	15
bf	7
cd	11
ce	13
cf	4
de	5
df	12
ef	20

4. (**a**) State (without proof) *Euler's Formula* for a connected planar graph with ν vertices and ε edges which is drawn in the plane with φ faces.

(**b**) Prove that if G is a connected planar graph with ν vertices with $\nu \geq 3$, and ε edges, in which all closed paths contain at least 4 edges, then

$$\varepsilon \leq 2\nu - 4\,.$$

(**c**) Hence, or otherwise, deduce that the Folkman graph (below) is not planar.



Question 4 continues ...

(**d**) Prove that a connected planar graph in which all closed paths contain at least 4 edges must have a vertex of degree 3 or lower.

(**e**) Give an example of a connected graph *with at most 8 vertices* in which all closed paths contain at least 4 edges and in which all vertices have degree at least 4.

5. (**a**) Consider the URM program:

1. $J(1,4,9)$
2. $S(3)$
3. $J(1,3,7)$
4. $S(2)$
5. $S(3)$
6. $J(1,1,3)$
7. $T(2,1)$

i) Draw the flow chart corresponding to this program.

ii) Give the full trace table of the URM computation using this program for the single number inputs: $0, 1$.

iii) Find the output of the computation using this program for input 4.

iv) Describe the function $f : \mathbb{N} \to \mathbb{N}$ computed by this program.

(**b**) Devise a URM program to compute the function $f : \mathbb{N}^2 \to \mathbb{N}$ given by

$$f(m,n) = \begin{cases} 0 & \text{if } m = n \\ 5 & \text{if } m \neq n\,. \end{cases}$$

6. (**a**) Prove that if $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ are both URM-computable functions, then so also is the function $h : \mathbb{N} \to \mathbb{N}$ given by

$$h(n) = g(f(n))\,.$$

(**b**) Show that the function $f(m,n) = m + n$ is primitive recursive.

Assuming that proper subtraction

$$m \dot{-} n = \begin{cases} m - n & \text{if } m > n \\ 0 & \text{otherwise} \end{cases}$$

is primitive recursive, show that so is

$$\max(m,n) = \begin{cases} m & \text{if } m \geq n \\ n & \text{otherwise.} \end{cases}$$

(**c**) Prove that for every URM-computable function $f : \mathbb{N} \to \mathbb{N}$ there is a strictly increasing URM-computable function $g : \mathbb{N} \to \mathbb{N}$ which dominates f.

END