MATH–204001

This question paper consists of 2 printed pages, each of which is identified by the reference MATH–2040

Only approved basic scientific calculators may be used.

© UNIVERSITY OF LEEDS

Examination for the Module MATH–2040
(January 2005)

Mathematical Logic 1
Time allowed : 2 hours

Answer not more than **four** questions. All questions carry equal marks.

1. (i) The following formula is written in Polish notation:

$$\wedge \wedge \neg \rightarrow pq\neg \rightarrow r\neg p\neg \rightarrow q\neg r.$$

Draw up its parse tree, and rewrite in the usual notation. State with reasons whether it is a tautology, satisfiable, or contradictory.

(ii) What is meant by saying that a formula φ is a *logical consequence* of a set of formulae Γ (written $\Gamma \models \varphi$)? Let Γ be the set of formulae $\{p \rightarrow (q \wedge r), q \rightarrow (r \wedge p), r \rightarrow (p \wedge q)\}$. Determine with reasons whether (a) $\neg p \vee q$, (b) $\neg q \wedge r$, is a logical consequence of Γ.

(iii) Prove from the definition of 'logical consequence' that for any Γ, φ, ψ, we have $\Gamma \cup \{\varphi\} \models \psi$ if and only if $\Gamma \models \varphi \rightarrow \psi$.

2. (i) Write the following argument in propositional logic, and determine whether or not it is valid:

If interest rates are not increased, more goods will be imported. If interest rates are increased, the rate of inflation will rise or there will be more unemployment. If more goods are imported, the balance of trade will get worse. If the balance of trade gets worse or the rate of inflation rises, the value of the pound will fall. Hence, if the value of the pound does not fall, there will be more unemployment.

(ii) Define what it means to say that a set of connectives is *adequate*. Prove that $\{\neg, \rightarrow, \wedge\}$ is adequate, but that $\{\rightarrow, \vee\}$ is not. (You may assume without proof that every truth function is represented by some formula in disjunctive normal form.)

(iii) Find a formula which only uses the connective *nor* which is logically equivalent to

$$(p \rightarrow q) \wedge (\neg p \rightarrow r).$$

3. (i) Show that in any boolean algebra $\mathbb{B}$, the relation $\leq$ defined by $a \leq b$ if $a \wedge b = a$ is a partial ordering of $\mathbb{B}$. State what this partial ordering is in the following cases:

(a) $\mathbb{B}$ is the set of all subsets of a non-empty set X, under the usual operations,

(b) $\mathbb{B}$ is the family of truth functions on a given set of propositional variables, with operations corresponding to the logical connectives, $\wedge$, $\vee$, $\neg$.

(ii) Find natural deduction proofs as follows:

(a) $p \to q, p \to r \vdash p \to (q \wedge r)$.

(b) $p \to r, q \to r \vdash (p \vee q) \to r$.

(c) $\vdash (p \wedge \neg q) \to \neg(p \wedge q)$.

(iii) Define *maximal consistent* set of formulae of propositional logic. Show that if Γ is a maximal consistent set, then for each formula φ, exactly one of φ, $\neg\varphi$ lies in Γ.

4. (i) (a) Find the parse tree for the term $\underline{f}(\underline{g}(\underline{S}\,\underline{0}, v_0), \underline{g}(\underline{0}, \underline{S}v_0), \underline{S}\,\underline{S}v_1)$.

(b) List free and bound occurrences of variables in the formula

$$\forall x \exists y(\underline{P}(x,z) \wedge \underline{Q}(y,u)) \to \forall z(\underline{P}(x,z) \to \exists u \underline{Q}(y,u)).$$

(ii) Express each of the following sentences in predicate logic, stating clearly what any relation symbols stand for:

(a) Every number is either prime or composite,

(b) Every positive integer can be expressed as the sum of four squares,

(c) Some numbers are more interesting than others.

(iii) Find assignments of values to the variables of the formula $\forall v_0 \exists v_2(v_0^2 + v_1^2 = v_2^2)$ which make it (a) true, (b) false, when interpreted in the natural numbers.

5. (i) Define *logical equivalence* of formulae φ, ψ in predicate logic, written $\varphi \equiv \psi$. Let $\varphi(x)$ and ψ be formulae such that x does not occur in ψ. Show that $\exists x\varphi(x) \to \psi \equiv \forall x(\varphi(x) \to \psi)$ holds, but (by means of a suitable counter-example) that $\forall x\varphi(x) \to \psi \equiv \forall x(\varphi(x) \to \psi)$ does not.

(ii) Find formulae in prenex normal form logically equivalent to

$$(\exists x \underline{A}(x) \wedge \underline{B}(y)) \vee (\forall y \underline{A}(y) \wedge \underline{C}(z)) \vee (\exists z \underline{B}(z) \wedge \underline{C}(x)),$$

and $$(\exists x \underline{P}(x,y) \to \forall z \underline{Q}(z)) \to \forall u \underline{R}(u).$$

(iii) The following sentences express '$\underline{R}$ is a partial ordering':

$$\varphi_1 : \forall x \underline{R}(x,x), \qquad \varphi_2 : \forall x \forall y(\underline{R}(x,y) \wedge \underline{R}(y,x) \to x = y),$$

$$\varphi_3 : \forall x \forall y \forall z(\underline{R}(x,y) \wedge \underline{R}(y,z) \to \underline{R}(x,z)).$$

Describe all models of the following sets of sentences:

(a) $\{\varphi_1, \varphi_2, \varphi_3, \exists x \forall y \underline{R}(y,x)\}$

(b) $\{\varphi_1, \varphi_2, \varphi_3, \exists x \forall y(x \neq y \to \neg\underline{R}(x,y))\}$

and give an example of a model of the second which is not a model of the first.

END