

MATH 2040 MATHEMATICAL LOGIC I

Problems 5

Models, prenex form, and formal proofs

1. Let φ_{PO} be the sentence characterizing partial orders:

$$\forall x \underline{P}(x, x) \wedge \forall x \forall y (\underline{P}(x, y) \wedge \underline{P}(y, x) \rightarrow x = y) \wedge \\ \forall x \forall y \forall z (\underline{P}(x, y) \wedge \underline{P}(y, z) \rightarrow \underline{P}(x, z)).$$

Describe what all the models are of the following sentences, and give an example for each:

- (a) $\varphi_{PO} \wedge \forall x \forall y (\underline{P}(x, y) \vee \underline{P}(y, x))$,
- (b) $\varphi_{PO} \wedge \forall x \forall y (\underline{P}(x, y) \vee \underline{P}(y, x)) \\ \wedge \forall x \forall y (\underline{P}(x, y) \wedge x \neq y \rightarrow \exists z (z \neq x \wedge z \neq y \wedge \underline{P}(x, z) \wedge \underline{P}(z, y)))$,
- (c) $\varphi_{PO} \wedge \exists x \forall y \underline{P}(x, y)$,
- (d) $\varphi_{PO} \wedge \exists x \forall y \underline{P}(y, x)$,
- (e) $\varphi_{PO} \wedge \exists x \forall y (y \neq x \rightarrow \neg \underline{P}(y, x))$.

2. Find prenex formulae logically equivalent to the following:

- (a) $(\forall x \underline{P}(x, y) \rightarrow \underline{Q}(z)) \vee \exists x (\underline{R}(z) \rightarrow \forall y \underline{S}(x, y, z))$
- (b) $\neg(\exists x \underline{P}(x) \wedge \forall y (\underline{P}(y) \vee \underline{Q}(y, z)) \rightarrow \exists w \underline{R}(y, z, w))$
- (c) $\underline{P}(0) \wedge (\forall n) (\underline{P}(n) \rightarrow \underline{P}(\underline{S}n)) \rightarrow (\forall n) \underline{P}(n)$.

3. What is wrong with the ‘proof’ shown? Show that the formula which has been ‘proved’ is not logically valid.

$$\frac{\frac{[\exists y \underline{A}(x, y)]^1 \vee I}{\forall x \exists y \underline{A}(x, y)} \vee E}{\exists y \underline{A}(y, y)} \rightarrow I^1$$

4. Prove the soundness of $\exists I$ and $\exists E$.

5. Find natural deduction proofs of the following:

- (a) $\vdash (\exists x)(\forall y) \underline{A}(x, y) \rightarrow (\forall y)(\exists x) \underline{A}(x, y)$.
- (b) $\vdash \neg(\forall x)(\exists y)(\neg \underline{A}(x) \wedge \underline{A}(y))$,
- (c) $\vdash (\exists x)(\underline{A}(x) \vee \underline{B}(x)) \leftrightarrow (\exists x) \underline{A}(x) \vee (\exists x) \underline{B}(x)$.

HAND IN SOLUTIONS TO TWO QUESTIONS.
--