MATH–011101

This question paper consists of 3 printed pages, each of which is identified by the reference MATH–011101

All calculators must carry an approval sticker issued by the School of Mathematics.

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Examination for the Module MATH–0111
(January 2013)

Elementary Differential Calculus

Time allowed: 2 hours

Attempt all questions in Section A and any three questions from Section B.

Each question in Section A carries 2 marks, each question in Section B carries 20 marks.

You must show your working in answer to all questions.
A formula sheet is supplied with this paper.

SECTION A

Attempt all the questions in Section A

A1. Express as a fraction with a single denominator:
\[ \frac{1}{x + 3} + \frac{1}{x - 1}. \]

A2. Evaluate \( \sqrt[3]{6}/(36)^{1/6} \).

A3. Evaluate \( 5x^{2/3}y^3x^{-2}y^{-4} \) when \( x = \frac{1}{8} \) and \( y = 3 \).

A4. Find \( \log_{125} 5 \).

A5. Factorise \( x^2 + 2x - 15 \).

A6. Solve the equation \( x^2 + 3x - 28 = 0 \).

A7. Find the equation of the line which passes through \( (2, 5) \) and is parallel to the line through \( (2, 3) \) and \( (1, 5) \).

A8. What is the distance between the points \( (2, -1) \) and \( (-3, 2) \)?

A9. The angle \( \theta \) lies between 0 and \( \pi/2 \) and \( \cos \theta = \frac{1}{5} \). Find \( \sin \theta \) and \( \tan \theta \) leaving your answers as exact expressions involving square roots.
A10. Find the equation of the circle with centre \((3, -1)\) and radius 5.

A11. Find \(\frac{dy}{dx}\) when \(y = x^{2/5}\).

A12. Find \(\frac{dy}{dx}\) when \(y = 8x^5 + x + \pi\).

A13. Find \(\frac{dy}{dx}\) when \(y = \sqrt[4]{x^7 + 7}\).

A14. Find \(\frac{dy}{dx}\) when \(y = \sin(x^{13})\).

A15. Find \(\frac{dy}{dx}\) when \(y = \frac{x^3 + 2x + 1}{8x - 1}\).

A16. Find \(\frac{dy}{dx}\) when \(y = e^{-x} \cos e^{2x}\).

A17. Find \(\frac{dy}{dx}\) when \(y = \ln(\arcsin x + 1)\).

A18. Find \(\frac{d^2y}{dx^2}\) when \(y = x^{10} + 7x\).

A19. Find the tangent to the curve \(y = 2x^2 + 5x - 1\) at the point \((1, 6)\).

A20. Without using a calculator, find an exact expression for \(\cos(5\pi/12)\).

SECTION B

Attempt three questions in Section B

B1. (a) Sketch, on the same diagram, the graphs of \(y = \cos 2\theta\) and \(y = \sec 2\theta\), for \(\theta\) in the range \(-\pi \leq \theta \leq \pi\) labelling the values of \(\theta\) where the graph of \(y = \cos 2\theta\) crosses the horizontal axis, and where the graphs have local minimum and maximum values.

(b) Define \(\tan \theta\) in terms of \(\sin \theta\) and \(\cos \theta\). Use your definition to show that

\[
\sec^2 \theta = \tan^2 \theta + 1.
\]

Suppose that \(0 < \theta < \pi/2\) and \(\tan \theta = 3\). Without using a calculator, find exact values of \(\sin \theta\), \(\cos \theta\) and \(\sin 2\theta\).
B2. (a) The points $A$ and $B$ have coordinates $(1, 4)$ and $(3, 3)$ correspondingly. Find:

(i) the equation of the line $AB$;
(ii) the equation of the line through the origin perpendicular to $AB$;
(iii) the point where the above two lines meet;
(iv) the distance from the origin to the line $AB$.

(b) A circle has centre at the point $C = (4, 2)$ and passes through the point $P(1, 6)$. Find:

(i) the radius of the circle;
(ii) the equation of the circle;
(iii) the gradient of the line $CP$;
(iv) the equation of the tangent to the circle at $P$.

B3. Differentiate each of the following functions with respect to $x$.

(i) $y = (3x^2 + 5)^{1/2}(x + 1)^{3/2}$;
(ii) $y = \frac{\tan x + 1}{x^3 + 1}$;
(iii) $y = \ln\left(\frac{e^x - 1}{e^x + 1}\right)$;
(iv) $y = \arctan(\sqrt{x^2 - 1})$;
(v) $y = 2\arcsin x$.

B4. (a) Find the stationary point of the function given by $y = x^3 - 3x^2 - 9x + 1$ and determine whether they are (local) maxima or minima.

Sketch the graph of the curve, showing any local maximum or minimum values, and any point where the curve cuts the $y$-axis.

(b) Find the normal to the curve $x^2 + xy^2 + y^3 = 7$ at the point $(2, 1)$.

END
Elementary Differential Calculus

FORMULA SHEET

Exponents
\[ x^a \cdot x^b = x^{a+b}, \quad a^x \cdot b^x = (ab)^x, \quad (x^a)^b = x^{ab}, \quad x^0 = 1. \]

Logarithms
\[ \log_c xy = \log_c x + \log_c y, \quad \log_c x^a = a \log_c x, \quad \log_c 1 = 0, \quad c^{\log_c x} = x, \quad \log_c e^y = y, \]
\[ \ln xy = \ln x + \ln y, \quad \ln x^a = a \ln x, \quad \ln 1 = 0, \quad e^{\ln x} = x, \quad \ln e^y = y, \]
\[ a^x = e^{x \ln a}. \]

Trigonometry
\[ \cos 0 = \sin \frac{\pi}{2} = 1, \quad \sin 0 = \cos \frac{\pi}{2} = 0, \]
\[ \cos^2 \theta + \sin^2 \theta = 1, \quad \cos(-\theta) = \cos \theta, \quad \sin(-\theta) = -\sin \theta, \]
\[ \cos(A + B) = \cos A \cos B - \sin A \sin B, \quad \cos 2\theta = \cos^2 \theta - \sin^2 \theta, \]
\[ \sin(A + B) = \sin A \cos B + \cos A \sin B, \quad \sin 2\theta = 2 \sin \theta \cos \theta, \]
\[ \tan \theta = \frac{\sin \theta}{\cos \theta}, \quad \sec \theta = \frac{1}{\cos \theta}, \quad 1 + \tan^2 \theta = \sec^2 \theta. \]

Inverse Functions
\[ y = \arcsin x \text{ means } x = \sin y \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}. \]
\[ y = \arccos x \text{ means } x = \cos y \text{ and } 0 \leq y \leq \pi. \]
\[ y = \arctan x \text{ means } x = \tan y \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}. \]
\[ y = x^{1/n} \text{ means } x = y^n. \quad y = \ln x \text{ means } x = e^y. \]

Alternative Notation
\[ \sin^{-1} x = \arcsin x, \quad \cos^{-1} x = \arccos x, \quad \tan^{-1} x = \arctan x, \quad \log_e x = \ln x. \]
Note: \[ \sin^{-1} x \neq (\sin x)^{-1}, \quad \cos^{-1} x \neq (\cos x)^{-1}, \quad \tan^{-1} x \neq (\tan x)^{-1}. \]
However: \[ \sin^2 x = (\sin x)^2, \quad \cos^2 x = (\cos x)^2, \quad \tan^2 x = (\tan x)^2. \]

Lines
The line \( y = mx + c \) has gradient \( m \).
The line through \((x_1, y_1)\) with gradient \( m \) has equation \( y - y_1 = m(x - x_1) \).
The line through \((x_1, y_1)\) and \((x_2, y_2)\) has gradient \( m = \frac{y_2 - y_1}{x_2 - x_1} \) and equation \[ \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}. \]
The line \( y = mx + c \) is perpendicular to the line \( y = m'x + c' \) if \( mm' = -1 \).

Circles
The distance between \((x_1, y_1)\) and \((x_2, y_2)\) is \[ \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \]
The circle with centre \((a, b)\) and radius \( r \) is given by \((x - a)^2 + (y - b)^2 = r^2\).

Quadratics
If \( ax^2 + bx + c = 0 \), with \( a \neq 0 \), then \[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \]
Differential Calculus

If \( y = u + v \) then \( \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \).

If \( y = uv \) then \( \frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx} \).

If \( y = \frac{u}{v} \) then \( \frac{dy}{dx} = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2} \).

If \( y \) is a function of \( u \) where \( u \) is a function of \( x \), then \( \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \).

Standard Derivatives

If \( y = x^a \) then \( \frac{dy}{dx} = ax^{a-1} \).

If \( y = \sin x \) then \( \frac{dy}{dx} = \cos x \); and if \( y = \cos x \) then \( \frac{dy}{dx} = -\sin x \).

If \( y = \tan x \) then \( \frac{dy}{dx} = \sec^2 x \).

If \( y = e^x \) then \( \frac{dy}{dx} = e^x \).

If \( y = \ln x \) then \( \frac{dy}{dx} = \frac{1}{x} \).

If \( y = \arcsin x \) then \( \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \); and if \( y = \arccos x \) then \( \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}} \).

If \( y = \arctan x \) then \( \frac{dy}{dx} = \frac{1}{1+x^2} \).