This question paper consists of 5 printed pages, each of which is identified by the reference MATH–011101

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Examination for the Module MATH–0111
(January 2011)
Elementary Differential Calculus (Version 1)

Time allowed: 2 hours

Attempt all questions in Section A and any three questions from Section B.

Each question in Section A carries 2 marks, each question in section B carries 20 marks.
You must show your working in answer to all questions.
A formula sheet is supplied with this paper.

SECTION A

Attempt all the questions in Section A

A1. Expand $(2x - 1)(2 + 7x)$.

A2. Factorise $x^2 - 9x + 20$.

A3. Solve the equation $x^3 - 2x^2 + x = 0$.

A4. Simplify $\frac{(ab)^4 \times a^3 \times b^6}{b^4 \times a^3}$, expressing your answer in index form.

A5. Evaluate $(\sqrt[4]{16})^{-5}$, leaving your answer as a whole number or fraction.

A6. Find the equation of the straight line joining the points $(-3, -1)$ and $(2, 4)$.

A7. Find the equation of the line which passes through $(2, 5)$ and is parallel to the line through $(2, 3)$ and $(1, 5)$.

A8. Find $\log_{16} 2$.

A9. Convert $225^\circ$ to radians.

A10. The angle $\theta$ lies between 0 and $\pi/2$, and $\cos \theta = \frac{1}{5}$. Find $\sin \theta$ and $\tan \theta$ leaving your answers as exact expressions involving square roots.

A11. Differentiate $y = 5x^{2/3}$. 
A12. Find \( \frac{dy}{dx} \) when \( y = x^3 + 3x + 1 \).

A13. Differentiate \( y = \sqrt[3]{3x^3 + 1} \).

A14. Find \( \frac{dy}{dx} \) when \( y = \frac{x^3 + 7}{7x^2 - 1} \).

A15. Find \( \frac{dy}{dx} \) when \( y = \cos x^5 \).

A16. Find \( \frac{dy}{dx} \) when \( y = e^{7x} \sin x^3 \).

A17. Find \( \frac{dy}{dx} \) when \( y = \tan(\ln x + 8) \).

A18. Find \( \frac{d^2y}{dx^2} \) when \( y = 3x^3 + 12x^2 \).

A19. Find the tangent to the curve \( y = x^2 + 4x - 5 \) at the point \((-2, -9)\).

A20. Without using a calculator, find an exact expression for \( \cos(5\pi/6) \) involving square roots.

SECTION B

Attempt three questions in Section B

B1. (a) Sketch (on the same diagram) the graphs of \( y = \cos \theta \) and \( y = \sec \theta \), for \( \theta \) in the range \( 0 \leq \theta \leq 2\pi \), labelling the values of \( \theta \) where \( y = \cos \theta \) crosses the horizontal axis, and where \( y = \sec \theta \) has minimum or maximum values.

(b) Find all values of \( \theta \) (in radians) between 0 and \( 2\pi \), such that \( \cos \theta = \frac{1}{\sqrt{2}} \).

(c) Find all values of \( \theta \) (in radians) in the range \( 0 \leq \theta \leq \pi \) such that \( 2\sin^2 \theta = 1 - \cos \theta \).

B2. (a) Let \( f(x) = \frac{1 + 3x}{1 - 3x} \). Find:

(i) the gradient of the graph of \( y = f(x) \) at the point \((1, -2)\);
(ii) the equation of the normal to the curve at this point.

(b) Let \( g(x) = x(x^2 - 12) \). Find the stationary points of \( y = g(x) \) and, for each one, say whether it is a maximum, minimum or point of inflexion.

Sketch the graph of \( y = g(x) \), showing the values of \( y \) at any stationary points, and any points where the curve intersects the \( x \)-axis.
B3. Differentiate each of the following functions with respect to $x$:

(i) $y = \frac{(x^3 - 1)^3}{\sqrt{x - 3}}$;

(ii) $y = (x^{100} + \tan x) \cos(3x - 1)$;

(iii) $y = \frac{x^2 + x + 1}{e^{x^2} + 5}$;

(iv) $y = \ln(\arcsin x)$;

(v) $y = \log_{10} x$.

B4. (a) Draw the graph of $y = \arcsin x$, where $-\pi/2 \leq x \leq \pi/2$.

Find the value of $\arcsin \frac{1}{\sqrt{2}}$.

Find $\cos(\arcsin x)$ in terms of $x$ (giving a formula without trigonometric functions).

Sketch the graph of $y = \arctan x$, and show that $\frac{d}{dx}(\arctan x) = \frac{1}{1 + x^2}$.

(b) If $x^3 + 2xy - 3y^3 = 0$, find the derivative of $y$ in terms of $x$ and $y$.

Hence find the tangent to the curve $x^3 + 2xy - 3y^3 = 0$ at the point $(1, 1)$.

END
Elementary Differential Calculus

FORMULA SHEET

Exponents
\[ x^a \cdot x^b = x^{a+b}, \quad a^x \cdot b^x = (ab)^x, \quad (x^a)^b = x^{ab}, \quad x^0 = 1. \]

Logarithms
\[ \log_c xy = \log_c x + \log_c y, \quad \log_c x^a = a \log_c x, \quad \log_c 1 = 0, \quad c^{\log_c x} = x, \quad \log_c c^y = y, \]
\[ \ln xy = \ln x + \ln y, \quad \ln x^a = a \ln x, \quad \ln 1 = 0, \quad e^{\ln x} = x, \quad \ln e^y = y, \]
\[ a^x = e^{x \ln a}. \]

Trigonometry
\[ \cos 0 = \sin \frac{\pi}{2} = 1, \quad \sin 0 = \cos \frac{\pi}{2} = 0, \]
\[ \cos^2 \theta + \sin^2 \theta = 1, \quad \cos(-\theta) = \cos \theta, \quad \sin(-\theta) = -\sin \theta, \]
\[ \cos(A + B) = \cos A \cos B - \sin A \sin B, \quad \cos 2\theta = \cos^2 \theta - \sin^2 \theta, \]
\[ \sin(A + B) = \sin A \cos B + \cos A \sin B, \quad \sin 2\theta = 2 \sin \theta \cos \theta, \]
\[ \tan \theta = \frac{\sin \theta}{\cos \theta}, \quad \sec \theta = \frac{1}{\cos \theta}, \quad 1 + \tan^2 \theta = \sec^2 \theta. \]

Inverse Functions
\[ y = \sin^{-1} x \text{ means } x = \sin y \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, \]
\[ y = \cos^{-1} x \text{ means } x = \cos y \text{ and } 0 \leq y \leq \pi. \]
\[ y = \tan^{-1} x \text{ means } x = \tan y \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}. \]
\[ y = x^{1/n} \text{ means } x = y^n. \quad y = \ln x \text{ means } x = e^y. \]

Alternative Notation
\[ \arcsin x = \sin^{-1} x, \quad \arccos x = \cos^{-1} x, \quad \arctan x = \tan^{-1} x, \quad \log_or x = \ln x. \]
Note: \[ \sin^{-1} x \neq (\sin x)^{-1}, \quad \cos^{-1} x \neq (\cos x)^{-1}, \quad \tan^{-1} x \neq (\tan x)^{-1}. \]
However: \[ \sin^2 x = (\sin x)^2, \quad \cos^2 x = (\cos x)^2, \quad \tan^2 x = (\tan x)^2. \]

Lines
The line \( y = mx + c \) has gradient \( m \).
The line through \((x_1, y_1)\) with gradient \( m \) has equation \( y - y_1 = m(x - x_1) \).
The line through \((x_1, y_1)\) and \((x_2, y_2)\) has gradient \( m = \frac{y_2 - y_1}{x_2 - x_1} \) and equation \( \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \).
The line \( y = mx + c \) is perpendicular to the line \( y = m'x + c' \) if \( mm' = -1 \).

Circles
The distance between \((x_1, y_1)\) and \((x_2, y_2)\) is \( \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \).
The circle with centre \((a, b)\) and radius \( r \) is given by \((x - a)^2 + (y - b)^2 = r^2 \).

Quadratics
If \( ax^2 + bx + c = 0 \), with \( a \neq 0 \), then \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \).
Differential Calculus
If \( y = u + v \) then \( \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \). If \( y = uv \) then \( \frac{dy}{dx} = \frac{du}{dx}v + u \frac{dv}{dx} \).

If \( y = \frac{u}{v} \) then \( \frac{dy}{dx} = \left( \frac{\frac{du}{dx}v - u \frac{dv}{dx}}{v^2} \right) \).

If \( y \) is a function of \( u \) where \( u \) is a function of \( x \), then \( \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \).

Standard Derivatives
If \( y = x^a \) then \( \frac{dy}{dx} = ax^{a-1} \).

If \( y = \sin x \) then \( \frac{dy}{dx} = \cos x \); and if \( y = \cos x \) then \( \frac{dy}{dx} = -\sin x \).

If \( y = \tan x \) then \( \frac{dy}{dx} = \sec^2 x \).

If \( y = e^x \) then \( \frac{dy}{dx} = e^x \).

If \( y = \ln x \) then \( \frac{dy}{dx} = \frac{1}{x} \).

If \( y = \sin^{-1} x \) then \( \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \); and if \( y = \cos^{-1} x \) then \( \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} \).

If \( y = \tan^{-1} x \) then \( \frac{dy}{dx} = \frac{1}{1+x^2} \).