MATH-011101

This question paper consists of 3 printed pages, each of which is identified by the reference MATH-011101

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Examination for the Module MATH-0111

(October 2010)

Elementary Differential Calculus (Version 1)

Time allowed: 2 hours

Attempt all questions in Section A and any three questions from Section B.

Each question in Section A carries 2 marks; each question in section B carries 20 marks.

You must show your working in your answers to all questions.

A formula sheet is supplied with this paper.

SECTION A

Attempt all the questions in Section A

A1. Expand \((\frac{5}{2} + 4x)(2x - 4)\).

A2. Evaluate \(8^{-2/3}\).

A3. Write \(x^{2/3}y^{-2}x^{-2}y^4\) in the form \(ax^by^c\).

A4. Find \(\log_4 \frac{1}{8}\).

A5. Factorise \(x^2 + x - 6\).

A6. Solve the equation \(x^2 + x - 1 = 0\).

A7. Find the equation of the straight line through the point \((2, 1)\) which is perpendicular to the line \(x - 3y + 4 = 0\).

A8. What is the distance between the points \((2, -3)\) and \((3, 2)\)?

A9. The angle \(\theta\) lies between \(-\pi/2\) and \(\pi/2\) and \(\sin\theta = -1/3\). Find \(\cos\theta\) and \(\tan\theta\) leaving your answers as exact values involving square roots.

A10. Find the equation of the circle with centre \((-3, 1)\) and radius 2. (You need not simplify your answer.)
A11. Find \( \frac{dy}{dx} \) when \( y = x^{-3/4} \).

A12. Find \( \frac{dy}{dx} \) when \( y = \frac{3}{2}x^2 - 2x^3 - 9 \).

A13. Find \( \frac{dy}{dx} \) when \( y = \sqrt{x^6 + 1} \).

A14. Find \( \frac{dy}{dx} \) when \( y = \tan(x^3) \).

A15. Find \( \frac{dy}{dx} \) when \( y = \frac{x^3 - 1}{2x^2 - 3x} \).

A16. Find \( \frac{dy}{dx} \) when \( y = e^{2x} \cos(x^2) \).

A17. Find \( \frac{dy}{dx} \) when \( y = \ln(\sin x + 6) \).

A18. Find \( \frac{d^2y}{dx^2} \) when \( y = 4x^2 - 5x^4 \).

A19. Find the equation of the tangent to the curve \( y = x^4 - 3x^2 + 5 \) at the point \((-1,3)\).

A20. Without using a calculator, find the value of \( \cos(9\pi/4) \) giving an exact value involving a square root.

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SECTION B

Attempt three questions from Section B

B1. (a) Sketch the graph of \( y = \cos \theta \) for \( \theta \) in the range \(-2\pi \leq \theta \leq 2\pi\), labelling the values of \( \theta \) where the graph crosses the horizontal axis and where \( y \) has minimum and maximum values.

(b) Find all values of \( \theta \) (in radians) between \(-2\pi\) and \(2\pi\), such that \( 2 \cos^2 \theta - 5 \cos \theta = 3 \).

(c) Use the result from the previous part of the question and the Pythagorean identity to find all values of \( \theta \) between 0 and \(5\pi\) such that \( 2 \sin^2 \theta + 5 \cos \theta + 1 = 0 \) (note the range of \( \theta \)).

B2. (a) The points \( A, B \) and \( C \) have coordinates \((1, -2), (2, 2)\) and \((1, 0)\), respectively. Find

(i) the equation of the line \( AB \);
(ii) the equation of the line through \( C \) perpendicular to \( AB \);
(iii) the point where the above two lines meet;
(iv) the distance from \( C \) to the line \( AB \).

continued ...
(b) A circle has centre at the point $O = (1, -1)$ and passes through the point $P(-2, -3)$. Find

(i) the radius of the circle;
(ii) the equation of the circle;
(iii) the gradient of the line $OP$;
(iv) the equation of the tangent to the circle at $P$.

B3. Differentiate each of the following functions with respect to $x$.

(i) $y = (x^4 + 3)^{5/2} + (x^3 + 2)^{-3/2}$;
(ii) $y = (x^2 - 3) \sin(3x^4)$;
(iii) $y = \ln \cos(3x^2 + 2)$;
(iv) $y = e^{x+1/x}$;
(v) $y = \arctan \left( \frac{x^2 - 2}{x} \right)$.

B4. (a) Find the stationary points of the function given by $y = x^3 + x^2 - x + 3$ and determine whether they are local maxima or minima. Find the values of the function at these points.

(b) Find the global maximum and minimum values of the function $f(x) = 2x^2 - 4x + 5$ over the interval $0 \leq x \leq 3$.

(c) If $y$ is given as a function of $x$ by $xy^3 + x^2y = (x + 1)$, find $\frac{dy}{dx}$ in terms of $x$ and $y$.
Elementary Differential and Integral Calculus

FORMULA SHEET

Exponents
$$x^a \cdot x^b = x^{a+b}, \quad a^x \cdot b^x = (ab)^x, \quad (a^x)^b = x^{ab}, \quad x^0 = 1.$$

Logarithms
$$\ln xy = \ln x + \ln y, \quad \ln x^n = n \ln x, \quad \ln 1 = 0, \quad e^{\ln x} = x, \quad \ln e^y = y,$$
$$a^x = e^{x \ln a}.$$

Trigonometry
$$\cos 0 = \sin \frac{\pi}{2} = 1, \quad \sin 0 = \cos \frac{\pi}{2} = 0,$$
$$\cos^2 \theta + \sin^2 \theta = 1, \quad \cos(-\theta) = \cos \theta, \quad \sin(-\theta) = -\sin \theta,$$
$$\cos(A + B) = \cos A \cos B - \sin A \sin B, \quad \cos 2\theta = \cos^2 \theta - \sin^2 \theta,$$
$$\sin(A + B) = \sin A \cos B + \cos A \sin B, \quad \sin 2\theta = 2 \sin \theta \cos \theta,$$
$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \quad \sec \theta = \frac{1}{\cos \theta}, \quad 1 + \tan^2 \theta = \sec^2 \theta.$$

Inverse Functions
$$y = \sin^{-1} x$$ means $$x = \sin y$$ and $$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2},$$
$$y = \cos^{-1} x$$ means $$x = \cos y$$ and $$0 \leq y \leq \pi,$$
$$y = \tan^{-1} x$$ means $$x = \tan y$$ and $$-\frac{\pi}{2} < y < \frac{\pi}{2},$$
$$y = x^{1/n}$$ means $$x = y^n, \quad y = \ln x$$ means $$x = e^y.$$

Alternative Notation
$$\arcsin x = \sin^{-1} x, \quad \arccos x = \cos^{-1} x, \quad \arctan x = \tan^{-1} x, \quad \log x = \log_e x = \ln x.$$
Note: $$\sin^{-1} x \neq (\sin x)^{-1}, \quad \cos^{-1} x \neq (\cos x)^{-1}, \quad \tan^{-1} x \neq (\tan x)^{-1},$$
However: $$\sin^2 x = (\sin x)^2, \quad \cos^2 x = (\cos x)^2, \quad \tan^2 x = (\tan x)^2.$$

Lines
The line $$y = mx + c$$ has slope $$m.$$ The line through $$(x_1, y_1)$$ with slope $$m$$ has equation $$y - y_1 = m(x - x_1).$$ The line through $$(x_1, y_1)$$ and $$(x_2, y_2)$$ has slope $$m = \frac{y_2 - y_1}{x_2 - x_1}$$ and equation $$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$ The line $$y = mx + c$$ is perpendicular to the line $$y = m'x + c'$$ if $$mm' = -1.$$

Circles
The distance between $$(x_1, y_1)$$ and $$(x_2, y_2)$$ is $$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$ The circle with centre $$(a, b)$$ and radius $$r$$ is given by $$(x - a)^2 + (y - b)^2 = r^2.$$ 

Triangles
In a triangle $$ABC:$$
$$(\text{Sine Rule}) \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}; \quad (\text{Cosine Rule}) \quad a^2 = b^2 + c^2 - 2bc \cos A.$$
Pascal’s Triangle

\[(x + y)^2 = x^2 + 2xy + y^2, \quad (x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3\] and so on.

The coefficients in \((x + y)^n\) form the \(n\)th row of Pascal’s triangle:

\[
\begin{array}{ccccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
\end{array}
\]

and so on.

Quadratics

If \(ax^2 + bx + c = 0\), with \(a \neq 0\), then \(x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\).

Calculus

If \(y = u + v\) then \(\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}\). If \(y = uv\) then \(\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}\).

If \(y = \frac{u}{v}\) then \(\frac{dy}{dx} = \left(\frac{du}{dx}v - u\frac{dv}{dx}\right)/v^2\).

\[
\int (u + v)\,dx = \int u\,dx + \int v\,dx. \quad \int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx.
\]

If \(y\) is a function of \(u\) where \(u\) is a function of \(x\), then

\[
\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \text{and} \quad \int y\frac{du}{dx}\,dx = \int y\,du.
\]

Standard Derivatives and Integrals

If \(y = x^a\) then \(\frac{dy}{dx} = ax^{a-1}\), and \(\int x^a\,dx = \frac{x^{a+1}}{a+1} + \text{constant} \quad (a \neq -1)\).

If \(y = \sin x\) then \(\frac{dy}{dx} = \cos x\), and \(\int \sin x\,dx = -\cos x + \text{constant}\).

If \(y = \cos x\) then \(\frac{dy}{dx} = -\sin x\), and \(\int \cos x\,dx = \sin x + \text{constant}\).

If \(y = \tan x\) then \(\frac{dy}{dx} = \sec^2 x\), and \(\int \tan x\,dx = \ln|\sec x| + \text{constant}\).

If \(y = e^x\) then \(\frac{dy}{dx} = e^x\), and \(\int e^x\,dx = e^x + \text{constant}\).

If \(y = \ln x\) then \(\frac{dy}{dx} = \frac{1}{x}\), and \(\int \frac{1}{x}\,dx = \ln |x| + \text{constant}\).

If \(y = \sin^{-1} x\) then \(\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}\), and \(\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + \text{constant}\).

If \(y = \cos^{-1} x\) then \(\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}\).

If \(y = \tan^{-1} x\) then \(\frac{dy}{dx} = \frac{1}{1+x^2}\), and \(\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + \text{constant}\).
MATH0111/0131 JANUARY 2010, SOLUTIONS

SECTION A [2 MARKS EACH]

1. \((\frac{3}{2} + 4x)(2x - 4) = 8x^2 + 5x - 16x - 10 = 8x^2 - 11x - 10.\)
2. \(8^{-2/3} = 2^{-2} = 1/4.\)
3. \(x^{2/3}y^{-2}x^{-2}y^4 = x^{2/3-2}y^{-2+4} = x^{-4/3}y^2.\)
4. \(\log_4 \frac{1}{2} = -3/2 \text{ since } 4^{-3/2} = \frac{1}{2}.\)
5. \(x^2 + x - 6 = (x + 3)(x - 2).\)
6. \(x^2 + x - 1 = 0 \text{ if and only if } x = \frac{-1 \pm \sqrt{1^2 + 4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.\)
7. The line \(x - 3y + 4 = 0\) has gradient 1/3 so the required line has gradient -3 and so is \(y - 1 = -3(x - 2), \text{ i.e., } y + 3x - 7 = 0.\)
8. The distance between the points \((2, -3)\) and \((3, 2)\) is \(\sqrt{(3 - 2)^2 + (2 + 3)^2} = \sqrt{26}.\)
9. With \(\sin \theta = -1/3,\) we have \(\cos \theta = \pm\sqrt{1 - 1/9} = \pm\sqrt{8}/3.\) But since \(-\pi/2 \leq \theta \leq \pi/2,\) \(\cos \theta\) is positive and equals \(\sqrt{8}/3.\) Hence \(\tan \theta = \frac{-1/3}{\sqrt{8}/3} = -\frac{1}{\sqrt{8}}.\)
10. The circle with centre \((-3, 1)\) and radius 2 has equation \((x + 3)^2 + (y - 1)^2 = 4.\)
11. When \(y = x^{-3/4}\) we have \(\frac{dy}{dx} = -\frac{3}{4}x^{-7/4}.\)
12. When \(y = \frac{3}{2}x^2 - 2x^3 - 9\) we have \(\frac{dy}{dx} = 3x - 6x^2.\)
13. When \(y = \sqrt{x^6 + 1} = (x^6 + 1)^{1/2}\) we have \(\frac{dy}{dx} = \frac{1}{2}(x^6 + 1)^{-1/2}(6x^5) = \frac{3x^5}{\sqrt{x^6 + 1}}.\)
14. When \(y = \tan(x^3)\) we have \(\frac{dy}{dx} = 3x^2 \sec^2(x^3).\)
15. When \(y = \frac{x^3 - 1}{2x^2 - 3x}\) we have \(\frac{dy}{dx} = \frac{3x^2(2x^2 - 3x) - (x^3 - 1)(4x - 3)}{(2x^2 - 3x)^2} = \frac{2x^4 - 6x^3 + 4x - 3}{(2x^2 - 3x)^2}.\)
16. When \(y = e^{2x} \cos(x^2)\) we have \(\frac{dy}{dx} = 2e^{2x} \cos(x^2) + e^{2x}(-\sin(x^2))(2x) = e^{2x}(2 \cos(x^2) - 2x \sin(x^2)).\)
17. When \(y = \ln(x + 6)\) we have \(\frac{dy}{dx} = \frac{\cos x}{\sin x + 6}.\)
18. When \(y = 4x^2 - 5x^4\) we have \(\frac{dy}{dx} = 8x - 20x^3\) so that \(\frac{d^2y}{dx^2} = 8 - 60x^2.\)
19. When \( y = x^4 - 3x^2 + 5 \) we have \( \frac{dy}{dx} = 4x^3 - 6x \), when \( x = -1 \) this gives \( 4(-1)^3 - 6(-1) = 2 \).
Thus the tangent at \((-1, 3)\) has equation \( y - 3 = 2(x + 1) \), i.e., \( y = 2x + 5 \).
20. \( \cos(7\pi/4) = \cos(-\pi/4) = 1/\sqrt{2} \).

SECTION B [20 MARKS EACH]

1. (a) Graph should be clearly labelled with the values of \( \theta \) where the graph crosses the horizontal axis and where \( y \) has minimum and maximum values for full marks. [6]
(b) Factorizing gives \((2\cos\theta + 1)(\cos\theta - 3) = 0\) so that \( \cos\theta = -1/2 \) or \( \cos\theta = 3 \). The second of these has no solutions; the first has solutions \( \theta = \pm 2\pi/3 \) in the range \(-\pi \leq \theta \leq \pi \), giving solutions \(-2\pi/3, -2\pi/3 + 2\pi = 4\pi/3, 2\pi/3 \) and \(2\pi/3 - 2\pi = -4\pi/3\) in the range \(-2\pi \leq \theta \leq \mp2\pi \). [7]
(c) From the Pythagorean identity we have \( \sin^2\theta = 1 - \cos^2\theta \), so that the given equation becomes \( 2 - 2\cos^2\theta + 5\cos\theta + 1 = 0 \), i.e., \( 2\cos^2\theta - 5\cos\theta = 3 \) which we just solved, so that the solutions to the given equation in the range \( 0 \leq \theta \leq 5\pi \) are \( 2\pi/3, 4\pi/3, 2\pi/3 + 2\pi = 8\pi/3, 4\pi/3 + 2\pi = 10\pi/3, 2\pi/3 + 4\pi = 14\pi/3 \). [7]

2. (a) (i) The line \( AB \) has equation \( \frac{y - (-2)}{x - 1} = \frac{-2 - 2}{1 - 2} \), i.e., \( y = 4x - 6 \). [2]
(i) The gradient of \( AB \) is 4, so the gradient of the perpendicular line is \(-1/4\), hence it has equation
\[
y - 0 = -\frac{1}{4}(x - 1), \quad \text{i.e.,} \quad y = -\frac{1}{4}x + \frac{1}{4} \quad \text{i.e.} \quad x + 4y - 1 = 0. \] [2]
(iii) The lines meet when \( y = 4x - 6 \) and \( y = \frac{1}{4}x + \frac{1}{4} \), so \( 4x - 6 = \frac{1}{4}x + \frac{1}{4} \) giving \( x = \frac{25}{17} \)
so \( y = -\frac{2}{17} \), thus the lines meet at \( \left( \frac{25}{17}, -\frac{2}{17} \right) \). [3]
(iv) The distance from \( C \) to the line \( AB \) is
\[
\sqrt{\left(1 - \frac{25}{17}\right)^2 + \left(0 + \frac{2}{17}\right)^2} = \frac{\sqrt{8^2 + 2^2}}{17} = \frac{\sqrt{68}}{17}. \] [3]
(b) (i) The radius of the circle is \( \sqrt{(-2 - 1)^2 + (-3 + 1)^2} = \sqrt{13} \). [2]
(ii) The equation of the circle is \( (x - 1)^2 + (y + 1)^2 = 13 \). [2]
(iii) The gradient of the line \( OP \) is \( -\frac{3 + 1}{-2 - 1} = \frac{2}{3} \). [3]
(iv) The tangent to the circle has gradient $\frac{-3}{2}$ so its equation is $y + 3 = \frac{-3}{2}(x + 2)$, i.e., $y = \frac{-3}{2}x - 6$. [3]

3. (i) $y' = \frac{5}{2}(x^4 + 3)^{3/2}(4x^3) + \frac{3}{2}(x^3 + 3)^{-5/2}(3x^2) = 10x^3(x^4 + 3)^{3/2} - \frac{9}{2}x^2(x^3 + 2)^{-5/2}$. [4]
(ii) $y' = (2x)\sin(3x^4) + (x^2 - 3)\cos(3x^4)(12x^3) = 2x\sin(3x^4) + 12x^3(x^2 - 3)\cos(3x^4)$. [4]
(iii) $y' = \frac{1}{\cos(3x^2 + 2)}(-\sin(3x^2 + 2))(6x) = -6x\tan(3x^2 + 2)$. [4]
(iv) $y' = e^{x+1/x}(1 - \frac{1}{x^2})$. [4]
(v) $y' = \frac{1}{1 + [(x^2 - 2)/x]^2}\frac{2x - (x^2 - 2)1}{x^2} = \frac{2x^2 - x^2 + 2}{x^2 + (x^2 - 2)^2} = \frac{x^2 + 2}{x^4 - 3x^2 + 4}$. [4]

4. (a) With $y = x^3 + x^2 - x + 3$ we have $\frac{dy}{dx} = 3x^2 + 2x - 1 = (3x - 1)(x + 1)$.

Stationary points occur when $x = \frac{1}{3}$ or $-1$.

Now $\frac{d^2y}{dx^2} = 6x + 2$. When $x = \frac{1}{3}$, this is positive, so we have a minimum.

When $x = -1$, it is negative, so we have a maximum.

At $x = \frac{1}{3}$, $y = \frac{76}{27}$. At $x = -1$, $y = 4$. [5]

(b) $f'(x) = 4x - 4$, so stationary point when $x = 1$. This gives the value $f(1) = 2 - 4 + 5 = 3$.

At the end points, $f(0) = 5$ and $f(3) = 18 - 12 + 5 = 11$.

Hence the global maximum value of $f(x)$ on the interval $0 \leq x \leq 3$ is 11 and the global minimum value is 3. [7]

(c) Differentiating the given equation wrt $x$, we obtain

$$y^3 + x\frac{dy}{dx} + 2xy + x^2\frac{dy}{dx} = 1$$

so that

$$\frac{dy}{dx} = \frac{1 - y^3 - 2xy}{3xy^2 + x^2}.$$ [8]