Hardy spaces, inner and outer functions, shift-invariant subspaces, Toeplitz and Hankel operators

Jonathan R. Partington*

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1 Background on Hardy spaces

For $1 \le p < \infty$ the Hardy space H^p is defined as the space of all analytic functions f in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for which the norm

$$\|f\|_p = \sup_{r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{1/p}$$

is finite. The space H^∞ consists of all bounded analytic functions f in the disc, and the norm is now

$$\|f\|_\infty = \sup_{|z|<1} |f(z)|.$$

For functions in H^p for $1 \le p \le \infty$ the radial limit

$$\tilde{f}(e^{it}) = \lim_{r \to 1} f(re^{it})$$

exists almost everywhere in t (Fatou's theorem), and indeed $\tilde{f} \in L^p(\mathbb{T})$, where $\mathbb{T}$ denotes the unit circle which we equip with normalized Lebesgue measure: moreover $\|f\|_{H^p} = \|\tilde{f}\|_{L^p}$. We normally identify f with $\tilde{f}$, and can thus regard H^p as a closed subspace of $L^p(\mathbb{T})$.

It is also possible to start by defining H^p directly as the subspace of those $L^p(\mathbb{T})$ functions for which the negative Fourier coefficients vanish, that is

$$\frac{1}{2\pi} \int_0^{2\pi} \tilde{f}(e^{it}) e^{-int} \, dt = 0$$

for all $n < 0$. Then a function $\tilde{f}$ with $\tilde{f}(e^{it}) \sim \sum_{n=0}^\infty a_n e^{int}$ can be naturally identified with the power series $f(z) = \sum_{n=0}^\infty a_n z^n$, defining an analytic function f in $\mathbb{D}$. One can also obtain the extension from $\tilde{f}$ to f by integrating with the Poisson kernel K_r, namely

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} K_r(\theta - t) \tilde{f}(e^{it}) \, dt,$$

*School of Mathematics, University of Leeds, Leeds LS2 9JT, U.K. J.R.Partington@leeds.ac.uk.

where

$$K_r(\theta - t) = \frac{1-r^2}{1+r^2-2r\cos(\theta-t)} = \operatorname{Re}\left(\frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}\right).$$

The case $p = 2$ is simpler, since for a function $f : z \mapsto \sum_{n=0}^{\infty} a_n z^n$ we have $\|f\|_2 = \left(\sum_{n=0}^{\infty} |a_n|^2\right)^{1/2}$. We use P_{H^2} to denote the orthogonal projection from $L^2(\mathbb{T})$ onto H^2, so that

$$P_{H^2} : \sum_{n=-\infty}^{\infty} a_n e^{int} \mapsto \sum_{n=0}^{\infty} a_n e^{int}.$$

2 Inner and outer functions

2.1 The canonical factorization

In this section we are concerned with the multiplicative structure of the Hardy spaces, in that we want to factorize a general Hardy class function as the product of two somewhat simpler functions, an inner factor and an outer factor. Here are their definitions (a simpler characterization of outer functions appears later, in Corollary 3.1).

Definition 2.1 *An* inner function *is an H^∞ function that has unit modulus almost everywhere on $\mathbb{T}$. An* outer function *is a function $f \in H^1$ which can be written in the form*

$$f(re^{i\theta}) = \alpha \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}} k(e^{it})\, dt\right), \tag{1}$$

for $re^{i\theta} \in \mathbb{D}$, where k is a real-valued integrable function and $|\alpha| = 1$.

Proposition 2.1 *Let f be an outer function, satisfying (1). Then $\log|f(e^{i\theta})| = k(e^{i\theta})$ almost everywhere.*

Proof: By taking logarithms we can obtain an expression using the Poisson kernel, namely,

$$\log|f(re^{i\theta})| = \operatorname{Re}\log f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} K_r(\theta-t)k(e^{it})\, dt,$$

and now the result follows since $\lim_{r\to 1}\log|f(re^{i\theta})|$ equals (a.e.) both $k(e^{i\theta})$, by properties of the Poisson kernel, and $\log|f(e^{i\theta})|$, by Fatou's theorem. □

Clearly, an outer function can have no zeroes in the disc, since it the exponential of something. Any function that is invertible in H^∞ is outer (e.g. $z - a$ where $|a| > 1$); in fact $z - a$ is also outer when $|a| = 1$.

Examples of inner functions include Blaschke products (Example 2.1 below) which have zeroes, but also some functions without zeroes, such as $\exp((z-1)/(z+1))$. This last function is just e^{-s} where $s = (1-z)/(1+z)$; the mapping from z to s takes $\mathbb{D}$ to the right half plane $\mathbb{C}_+$, and $\mathbb{T} \setminus \{-1\}$ to the imaginary axis $i\mathbb{R}$.

Example 2.1 *A* finite Blaschke product *is a function of the form*

$$B(z) = e^{i\varphi}\prod_{j=1}^{n} \frac{z-z_j}{1-\overline{z}_j z},$$

where $\varphi \in \mathbb{R}$ *and* $|z_j| < 1$ *for* $j = 1, \ldots, n$. *It is easy to verify that* B *has the following properties.*

(i) B *is analytic in* $\mathbb{D}$ *and continuous in* $\overline{\mathbb{D}}$.

(ii) B *is inner.*

(iii) B *has zeroes at* $z_1, \ldots, z_n$ *only, and poles at* $1/\overline{z}_1, \ldots, 1/\overline{z}_n$ *only.*

Theorem 2.1 (Inner–outer factorization). *Let* f *be a nonzero function in* H^1. *Then* f *has a factorization* $f = \theta \cdot u$, *where* θ *is inner and* u *is outer. This factorization is unique up to a constant of modulus one.*

Proof: We define u by

$$u(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\log|f(e^{it})|\,dt\right),$$

which is an outer function as in (1). Now $\theta := f/u$ is analytic in the disc and $|\theta(z)| = 1$ a.e. for $|z| = 1$, and thus θ is inner. The factorization is unique, as if we have two outer functions u, u_1 with $|u| = |u_1| = |f|$ a.e. on $\mathbb{T}$, then u/u_1 and u_1/u are both inner. By the maximum modulus principle, $|u/u_1| \le 1$ and $|u_1/u| \le 1$ everywhere in the disc, which implies that $u = \alpha u_1$ for some constant α of modulus 1. □

The next thing we want to do is to break the inner part into two factors, an inner function with zeroes (which will be an infinite Blaschke product) and an inner function without zeroes (a so-called *singular inner function*). To do this we need to understand the properties of the zero set of a function in H^p.

Theorem 2.2 (G. Szegő) *Let* $f \in H^1$ *be such that* f *is not identically zero. Then the zeroes* (z_n) *of* f *are countable in number and satisfy the Blaschke condition*

$$\sum_1^\infty (1 - |z_n|) < \infty.$$

Proof: Without loss of generality, we may suppose that $f(0) \neq 0$, since otherwise we can consider $f(z)/z^k$ for a suitable $k \ge 1$. Now take $r < 1$, and consider the zeroes $z_1, \ldots, z_m$ in $\{|z| < r\}$, repeated according to multiplicity, supposing that none satisfy $|z_k| = r$; there can only be finitely many, since they are isolated.

The function $f_r(z) = f(rz)$ is analytic in $\{|z| \le 1\}$, and has zeroes at the points $z_1/r, \ldots, z_m/r$. Thus we can write

$$f(rz) = g(z)\prod_1^m \frac{z - z_k/r}{1 - \overline{z}_k z/r},$$

where g is analytic and non-zero in an open set containing $\overline{\mathbb{D}}$. Thus

$$\log g(0) = \frac{1}{2\pi}\int_0^{2\pi} \log g(e^{i\theta})\,d\theta.$$

Taking real parts, we obtain

$$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi}\log|g(e^{i\theta})|\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta.$$

That is,

$$\log|f(0)| + \sum_{|z_k|<r} \log\left(\frac{r}{|z_k|}\right) \quad = \quad \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta. \tag{2}$$

Now Jensen's inequality (see, for example, [11], Chapter 1) asserts that

$$\varphi\left(\int_E h(x)\,d\mu(x)\right) \le \int_E \varphi(h(x))\,d\mu(x)$$

whenever $\varphi : [a,b] \to \mathbb{R}$ is convex, $h : E \to [a,b]$ is measurable, μ is a probability measure on E, and both integrals exist. Hence, in our case,

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta \le \log\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\,d\theta$$

since $(-\log x)$ is a convex function. Thus

$$\log|f(0)| + \sum_{|z_k|<r} \log\left(\frac{r}{|z_k|}\right) \le \log\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\,d\theta \le \log\|f\|_{H^1}\,. \tag{3}$$

Finally, letting $r \to 1$, we see that $\sum_1^\infty \log 1/|z_k| < \infty$. This is equivalent to the assertion that $\sum_1^\infty (1-|z_k|) < \infty$, since

$$1-|z| \le \log 1/|z| \le 2(1-|z|)$$

for $1/2 \le |z| \le 1$. □

Theorem 2.3 *Let $f \in H^1$. Then the infinite Blaschke product*

$$B(z) = z^m \prod_{|z_n|\neq 0} \frac{\overline{z}_n}{|z_n|}\frac{z_n - z}{1-\overline{z}_n z},$$

where (z_n) are the zeroes of f, m of them being at 0, converges uniformly on compact sets to an H^∞ function the only zeroes of which are the (z_n), with the correct multiplicities. Moreover, $|B(z)| \le 1$ and $|B(e^{i\theta})| = 1$ almost everywhere.

Proof: It will be sufficient to prove the result for $f(z)/z^m$. Write

$$b_n(z) = \frac{\overline{z}_n}{|z_n|}\frac{z_n - z}{1-\overline{z}_n z},$$

where the first term is a factor chosen to make $b_n(0) > 0$.

Then $\prod b_n$ converges to an analytic function with the correct zeroes if and only if $\sum \log|b_n|$ converges locally uniformly; this happens if and only if

$$\sum |1-b_n|$$

converges locally uniformly.

However,

$$\begin{aligned} |1 - b_n(z)| &= \left| 1 + \frac{\overline{z}_n}{|z_n|} \frac{z - z_n}{1 - \overline{z}_n z} \right| \\ &= \frac{(1 - |z_n|)(\overline{z}_n z + |z_n|)}{|z_n|.|1 - \overline{z}_n z|} \\ &\leq \frac{(1 - |z_n|)(1 + |z|)}{1 - |z|}, \end{aligned}$$

which gives convergence, by Szegő's theorem.

Thus $B(z) \in H^\infty$, and $\|B\|_{H^\infty} \leq 1$, so that the boundary function satisfies $|B(e^{i\theta})| \leq 1$ almost everywhere. But, writing $B_n = \prod_1^n b_k$, we see that B/B_n is another Blaschke product, and so

$$\left| \frac{B(0)}{B_n(0)} \right| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{B(e^{i\theta})}{B_n(e^{i\theta})} \right| d\theta = \frac{1}{2\pi} \int_0^{2\pi} \left| B(e^{i\theta}) \right| d\theta.$$

Letting $n \to \infty$, we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} |B(e^{i\theta})| \, d\theta = 1,$$

and so $|B(e^{i\theta})| = 1$ almost everywhere. □

For any H^p function we can remove a Blaschke factor which accounts for the zeroes, as follows.

Lemma 2.1 (F. Riesz) *Let $f \in H^p$, $f \not\equiv 0$, and let $B(z)$ be the (possibly infinite) Blaschke product formed using the zeroes (z_n) of f. Then $f(z) = g(z)B(z)$ for some $g \in H^p$ with $\|f\|_p = \|g\|_p$.*

Proof: Let $g(z) = f(z)/B(z)$ and $g_n(z) = f(z)/B_n(z)$, where B_n is the Blaschke product corresponding to the first n zeroes of f. If $r < 1$ and $1 \leq p < \infty$, then

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |g_n(re^{i\theta})|^p \, d\theta &\leq \lim_{R \to 1} \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(Re^{i\theta})|^p}{|B_n(Re^{i\theta})|^p} \, d\theta \\ &= \lim_{R \to 1} \frac{1}{2\pi} \int_0^{2\pi} |f(Re^{i\theta})|^p \, d\theta \end{aligned}$$

since $|B_n(Re^{i\theta})| \to 1$ uniformly as $R \to 1$. Hence

$$\frac{1}{2\pi} \int_0^{2\pi} |g_n(re^{i\theta})|^p \, d\theta \leq \|f\|_{H^p}^p.$$

But $|g_n|$ increases to $|g|$ as $n \to \infty$, and so, by the monotone convergence theorem,

$$\frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^p \, d\theta \leq \|f\|_{H^p}^p.$$

However, $|g(z)| \geq |f(z)|$ for all $z \in \mathbb{D}$, so we have equality.

A similar (easier) argument holds for the case $p = \infty$. □

Corollary 2.1 *Any nonzero function $f \in H^1$ can be written as $f = B \cdot S \cdot u$, where B is a Blaschke product, S a singular inner function, and u an outer function. This factorization is unique up to constants of modulus 1.*

Proof: This follows from Theorem 2.1 and Lemma 2.1. □

In order to study singular inner functions, we recall the following result. It has two equivalent formulations, since any harmonic function in the disc is the real part of an analytic function.

Theorem 2.4 (G. Herglotz). *A complex-valued harmonic function u in the disc is the Poisson integral of a finite positive measure μ on the circle, that is,*

$$u(re^{i\theta}) = \int_{\mathbb{T}} K_r(t-\theta)\, d\mu(t),$$

if and only if it is non-negative. If h is an analytic function in the unit disc with values in the right-hand half-plane, such that $h(0) > 0$, then

$$h(re^{i\theta}) = \int_{\mathbb{T}} \frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}}\, d\mu(t)$$

for some positive measure μ defined on $\mathbb{T}$.

The next result explains why an inner function without zeroes is called a singular inner function.

Corollary 2.2 *Let g be an inner function without zeroes. Then there is a unique positive measure μ, singular with respect to Lebesgue measure, and a constant α of modulus 1, such that*

$$g(re^{i\theta}) = \alpha \exp\left(-\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}}\, d\mu(t) \right). \tag{4}$$

Formula (4) is very similar to (1), except that the integral is now taken with respect to a singular measure, rather than $k(t)\, dt$.

Proof: Since g has no zeroes, and is inner, we can write it as $g = \alpha \exp(-h)$, where h is analytic, takes values in $\mathbb{C}_+$, and satisfies $h(0) > 0$ (thus α is chosen to make $g(0)/\alpha$ real and positive). By Herglotz's Theorem 2.4 we have expression (4), except that we need to show that μ is singular with respect to Lebesgue measure. This follows since the nontangential limits of $h(z)$ are a.e. purely imaginary as $|z| \to 1$. But

$$\operatorname{Re} h(re^{i\theta}) = \int_{\mathbb{T}} K_r(t-\theta)\, d\mu(t),$$

and its nontangential limit is $\frac{1}{2\pi}\frac{d\mu}{dt}$, which must therefore vanish a.e. Hence μ is a singular measure. □

2.2 Consequences

It is easy to check (using the Cauchy–Schwarz inequality) that the product of two H^2 functions is always in H^1. The converse, which is harder, is also true; namely, that any H^1 function can be written as the product of two H^2 functions.

Theorem 2.5 (The Riesz factorization theorem) *A function f is in H^1 if and only if there exist g, $h \in H^2$ with $f = g \cdot h$ and $\|f\|_1 = \|g\|_2\|h\|_2$.*

Proof: Note that if g and h are in H^2 then $g \cdot h \in H^1$ and $\|g \cdot h\|_1 \le \|g\|_2\|h\|_2$, by the Cauchy–Schwarz inequality.

Conversely, given $f \in H^1$, write $f(z) = f_1(z)B(z)$, where B is as in Theorem 2.1, $\|f_1\|_{H^1} = \|f\|_{H^1}$, and f_1 has no zeroes in $\mathbb{D}$.

Since f_1 is never zero it has an analytic square root g (see, for example, [10]); that is, we can write

$$f_1(z) = g(z)^2.$$

Now $f(z) = g(z)g(z)B(z)$ and $\|f\|_{H^1} = \|g\|^2_{H^2}$, so $\|f\|_1 = \|g\|_2\|gB\|_2$ since $\|gB\|_2 = \|g\|_2$. □

We are now ready to look at the boundary behaviour of H^p functions.

Theorem 2.6 *Suppose that $f \in H^1$ and that f is not identically zero. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|\,d\theta > -\infty,$$

and hence $f(e^{i\theta}) \neq 0$ almost everywhere.

Proof: It is sufficient to prove the result for $f \in H^2$, and then invoke Theorem 2.5. Without loss of generality, we may suppose that $f(0) \neq 0$, as otherwise we may consider $f(z)/z^n$ for some suitable n. Writing $f_r(z) = f(rz)$ for $r < 1$, we know from Fatou's theorem that $f_r(e^{i\theta}) \to f(e^{i\theta})$ almost everywhere as $r \to 1$. We recall from the proof of Theorem 2.2 that, if (z_k) are the zeroes of f, then we have

$$\log|f(0)| + \sum_{|z_k|<r} \log(r/|z_k|) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta,$$

and so

$$\log|f(0)| \le \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta.$$

Let us write $\log(x) = \log^+(x) - \log^-(x)$ for $x \ge 0$, where

$$\log^+(x) = \max(0, \log x) \qquad \text{and} \qquad \log^-(x) = \max(0, -\log x).$$

Then, since $\log^+(x) \le x^2$, it follows that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|\,d\theta &\le \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2\,d\theta \\ &= \|f_r\|_2^2 \le \|f\|_2^2.\end{aligned}$$

Hence

$$\frac{1}{2\pi}\int_0^{2\pi} \log^-|f(re^{i\theta})|\,d\theta \le \|f\|_2^2 - \log|f(0)|$$

for each r. Thus, by Fatou's Lemma, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \log^-|f(e^{i\theta})|\,d\theta \le \|f\|_2^2 - \log|f(0)|,$$

and hence

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|\,d\theta > -\infty,$$

as required. □

3 Invariant subspaces of the shift operator

3.1 Introduction

The functions $e_n(z) = z^n$ for $n \in \mathbb{Z}$ form an orthonormal basis in $L^2(\mathbb{T})$. The orthonormal expansions

$$f = \sum_{n\in\mathbb{Z}} \hat{f}(n)e_n, \qquad \hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})e^{-int}\,dt, \quad n \in \mathbb{Z}$$

are just the classical Fourier series. Since $(zf)\hat{\ }(n) = \hat{f}(n-1)$, for $n \in \mathbb{Z}$, the action of the operator $f \mapsto zf$ can be considered as a *right translation*, or *shift*.

Let us consider those subspaces of the Hilbert space $L^2(\mathbb{T})$ that are invariant under this shift operator. (Subspaces will always be assumed to be closed, unless otherwise stated; we use the notation 'clos' for the closure, keeping the bar notation for complex conjugation.)

So let $E \subseteq L^2(\mathbb{T})$, with $E = \operatorname{clos} E$ be a linear subspace such that $zE \subseteq E$, that is, $f \in E \Longrightarrow zf \in E$. What does such an E look like? We can distinguish two separate cases:

$$zE = E \qquad \text{or} \qquad zE \neq E.$$

1- and 2-invariant subspaces. We note that $zE = E$ if and only if $\overline{z}E = E$, since $z\overline{z} = |z|^2 = 1$. In the case when $zE \subseteq E$ and $\overline{z}E \subseteq E$ a subspace E is called *doubly invariant*, or (*2-invariant* or *reducing*), and in the other case, when $zE \subset E$, $zE \neq E$, we say that E is *simply invariant* or *1-invariant*.

3.2 Doubly invariant subspaces

We begin with the simplest case of 2-invariant subspaces. From now on it will be convenient to write $d\mu$ for normalized Lebesgue measure on the circle, i.e., $d\mu = \frac{1}{2\pi}dt$.

Theorem 3.1 (N. Wiener). *Let $E \subseteq L^2(\mathbb{T})$ satisfy $zE = E$. Then there is a unique measurable set $\sigma \subseteq \mathbb{T}$ such that $E = \chi_\sigma L^2(\mathbb{T}) = \{f \in L^2(\mathbb{T}) : f = 0$ a.e. outside $\sigma\}$, where χ_σ is the characteristic function (indicator function) of σ.*

Proof: Let $\chi = P_E 1$, $\chi \in E$, where P_E is the orthogonal projection from $L^2(\mathbb{T})$ onto E. We have the following:

$$1 - \chi = (I - P_E)1 \in E^{\perp},$$

and so

$$z^n \chi \perp 1 - \chi, \qquad \forall n \in \mathbb{Z},$$

that is,

$$\int_{\mathbb{T}} z^n \chi (1 - \overline{\chi})\, d\mu = 0 \qquad \forall n \in \mathbb{Z}.$$

Since $\chi(1-\overline{\chi}) \in L^1(\mathbb{T})$, the product $\chi(1-\overline{\chi})d\mu$ is a finite complex Borel measure on $\mathbb{T}$ which annihilates the set $\mathcal{T}$ of trigonometric polynomials, the set of finite linear combinations of powers z^n with $n \in \mathbb{Z}$. But $\mathcal{T}$ is dense in $C(\mathbb{T})$, so $\chi(1-\overline{\chi}) = 0$ a.e.

Hence $\chi = |\chi|^2$, a.e., and this implies that χ takes only the values 0 and 1. Let $\sigma = \{t : \chi(t) = 1\}$; this set is well-defined up to a null set (i.e., a set of measure zero).

Since $\chi \in E$, we have $z^n \chi \in E$ for all $n \in \mathbb{Z}$, and then $\mathcal{T}\chi \subseteq E$ and $\operatorname{clos}(\chi\mathcal{T}) \subseteq E$. On the other hand, $\operatorname{clos}(\chi\mathcal{T}) = \chi L^2(\mathbb{T})$, since $\operatorname{clos}\mathcal{T} = L^2(\mathbb{T})$. Thus $\chi L^2(\mathbb{T}) \subseteq E$, and it only remains to show that these two spaces are equal.

Let $f \in E$ with $f \perp \chi z^n$ $\forall n \in \mathbb{Z}$. Then $z^n f \in E$ for all n, and $1 - \chi \perp z^n f$ $\forall n \in \mathbb{Z}$, and these imply that

$$\int_{\mathbb{T}} f \chi z^{-n}\, d\mu = 0 \qquad \text{and} \qquad \int_{\mathbb{T}} z^n f(1-\chi)\, d\mu = 0, \qquad (n \in \mathbb{Z}).$$

Hence $f\chi = f(1-\chi) = 0$ a.e., and so finally $f = 0$ a.e., i.e., $\chi L^2(\mathbb{T}) = E$. □

3.3 Simply invariant subspaces

The principal example of a simply invariant subspace is H^2, the closed linear span of $\{z^n : n \geq 0\}$. The following theorem shows that all the simply invariant subspaces have a somewhat similar structure.

Theorem 3.2 (A. Beurling, H. Helson). *Let $E \subseteq L^2(\mathbb{T})$, with $zE \subset E$, $zE \neq E$. Then there exists a measurable function θ (unique up to a constant) such that $|\theta| = 1$ a.e. on $\mathbb{T}$ and $E = \theta H^2$.*

Note that θH^2 is a closed subspace, since $f \mapsto \theta f$ is an *isometry* and even a *unitary operator* on $L^2(\mathbb{T})$.

Proof: We use the same method of orthogonal projection as in Theorem 3.1. zE is a proper closed subspace of E, i.e., $zE \neq E$. We consider the orthogonal complement of zE in E: $E \ominus zE$ is a nontrivial subspace of E, so take $\theta \in E \ominus zE$ with $\|\theta\| = 1$.

Then $\theta \in E$ and $\theta \perp zE$, and so $z^n\theta \in zE$, for $n \geq 1$, implying that $\theta \perp z^n\theta$, i.e.,

$$\begin{aligned} \int_{\mathbb{T}} \overline{\theta}\theta z^n\, d\mu &= 0, \qquad n \geq 1, \qquad \text{i.e.,} \\ \int_{\mathbb{T}} |\theta|^2 z^n\, d\mu &= 0, \qquad n \geq 1. \end{aligned}$$

Taking complex conjugates:

$$\int_{\mathbb{T}} |\theta|^2 \overline{z}^n \, d\mu = 0, \qquad n \geq 1,$$

i.e., $(|\theta|^2)\hat{}(n) = 0$ for $n \in \mathbb{Z} \setminus \{0\}$. Thus $|\theta|^2 = \text{const} = c$ a.e.

Since

$$1 = \|\theta\|_2^2 = \int_{\mathbb{T}} |\theta|^2 \, d\mu = c\mu(\mathbb{T}) = c,$$

we have $|\theta| = 1$ a.e.

Thus $f \mapsto \theta f$ is an isometry in $L^2(\mathbb{T})$. Now we have $z^n\theta \in E$, for $n \geq 0$. The linear span has the same property: write $\mathcal{P}$ for the set of polynomials in z, so

$$\mathcal{P}\theta \subseteq E, \qquad \text{and} \qquad \operatorname{clos}(\theta\mathcal{P}) = \theta \operatorname{clos}\mathcal{P} = \theta H^2 \subseteq E.$$

Thus we have a closed subspace of E,

$$\theta H^2 \subseteq E,$$

and we want it to coincide with E. To show this, consider $f \in E$, $f \perp \theta H^2$. We need to show that $f = 0$. We have:

$$f \perp \theta H^2 \qquad \Longrightarrow \qquad f \perp \theta z^n, \quad n \geq 0,$$

and

$$f \in E \qquad \Longrightarrow \qquad z^n f \in zE, \quad n \geq 1 \qquad \Longrightarrow \qquad z^n f \perp \theta, \quad n \geq 1.$$

It follows that

$$\begin{aligned} \int_{\mathbb{T}} f\overline{\theta}\overline{z}^n \, d\mu &= 0, \qquad n \geq 0, \qquad \text{and} \\ \int_{\mathbb{T}} f\overline{\theta} z^n \, d\mu &= 0, \qquad n \geq 1. \end{aligned}$$

Thus $(f\overline{\theta})\hat{}(n) = 0, \forall n \in \mathbb{Z}$ and $f\overline{\theta} \equiv 0$. But $|\theta| = 1$ a.e., and so $f = 0$ a.e. and $E = \theta H^2$.

To show uniqueness, let $\theta_1 H^2 = \theta_2 H^2$, where $|\theta_1| = |\theta_2| = 1$ a.e. on $\mathbb{T}$. Then $(\theta_1\overline{\theta}_2)H^2 = H^2$, so $\theta_1\overline{\theta}_2 \in H^2$, and, by symmetry, $\theta_2\overline{\theta}_1 \in H^2$, or $\theta_1\overline{\theta}_2 \in \overline{H^2}$.

But $H^2 \cap \overline{H^2} = \{\text{ const }\}$, as is clear for several reasons: for example, $f \in H^2 \Longrightarrow \hat{f}(n) = 0$, $n < 0$; and $\overline{f} \in H^2 \Longrightarrow \overline{f}\hat{}(n) = \overline{\hat{f}(-n)} = 0$, $n < 0 \Longrightarrow f = \text{const}$. □

Note that Theorem 3.2 implies a particular result about closed shift-invariant subspaces of H^2, generally referred to as Beurling's theorem: *any closed shift-invariant subspace E of H^2 has the form $E = \theta H^2$, where θ is inner.* For clearly, if $\theta H^2 \subseteq H^2$ and $|\theta| = 1$ a.e. on $\mathbb{T}$, then θ is in H^2, and hence inner. We may use Beurling's theorem to deduce a fairly user-friendly characterization of outer functions.

Corollary 3.1 *A function $u \in H^\infty$ is outer if and only if the subspace uH^2 is dense in H^2.*

Proof: By Beurling's theorem, $\operatorname{clos} uH^2$ is either H^2 or θH^2 for some nontrivial inner function θ, which must clearly divide u. Now, if u is outer, then it has no nontrivial inner factor θ and so $\operatorname{clos} uH^2 = H^2$. Conversely, if u is not outer, then it has an inner factor θ and so $uH^2 \subseteq \theta H^2$, which is a proper closed subspace since $1 \notin \theta H^2$, and thus $\operatorname{clos} uH^2 \subseteq \theta H^2$. □

4 Toeplitz and Hankel operators

In this section we shall review briefly certain operators defined on the Hardy space H^2 of the disc. Most of the results that we prove have their natural counterparts on the half-plane—in most cases they can be obtained directly from the results about the disc, or by an easy modification of the corresponding proofs: see [8], for example.

4.1 Laurent and Toeplitz operators

Let $P_{H^2} : L^2(\mathbb{T}) \to H^2$ be the orthogonal projection defined by

$$P_{H^2}\left(\sum_{n=-\infty}^{\infty} a_n e^{int}\right) = \sum_{n=0}^{\infty} a_n e^{int}.$$

Clearly, $\|P_{H^2} f\|_2 \le \|f\|_2$ for all $f \in L^2(\mathbb{T})$.

Definition 4.1 *Let* $\varphi \in L^\infty(\mathbb{T})$. *Then the* Laurent *(or* multiplication operator*)* $M_\varphi : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ *is given by*

$$(M_\varphi f)(e^{it}) = \varphi(e^{it}) f(e^{it}). \tag{5}$$

Theorem 4.1 *Let* $\varphi \in L^\infty(\mathbb{T})$. *Then* M_φ *is a bounded operator and its norm is given by* $\|M_\varphi\| = \|\varphi\|_\infty$. *Moreover,*

$$\sup\{\|M_\varphi f\|_2 : f \in H^2, \|f\|_2 = 1\} = \|\varphi\|_\infty .$$

If φ *is a measurable function on* $\mathbb{T}$ *which is not in* $L^\infty(\mathbb{T})$, *then* M_φ *is not a bounded operator on* L^2.

Proof: Clearly, $\|M_\varphi\| \le \|\varphi\|_\infty$, since

$$\begin{aligned} \|M_\varphi f\|_2^2 &= \frac{1}{2\pi}\int_0^{2\pi} |\varphi(e^{it}) f(e^{it})|^2 \, dt \\ &\le \|\varphi\|_\infty^2 \frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^2 \, dt = \|\varphi\|_\infty^2 \|f\|_2^2. \end{aligned}$$

The converse is more complicated. Given $\varepsilon > 0$ we can find a set $A_\varepsilon \subset \mathbb{T}$ of positive measure such that $|\varphi(e^{it})| > \|\varphi\|_\infty - \varepsilon$ on A_ε.

Writing $\chi = \chi_{A_\varepsilon}$ for the function that is equal to 1 on A_ε and 0 on its complement, we have that $\chi \in L^2(\mathbb{T})$ and $\|\chi\|_2^2 = \mu(A_\varepsilon)$, where μ is again normalized Lebesgue measure on the circle.

Also

$$\begin{aligned} \|M_\varphi \chi\|_2^2 &= \frac{1}{2\pi}\int_0^{2\pi} |\varphi(e^{it})|^2 \chi(e^{it})^2 \, dt \\ &> (\|\varphi\|_\infty - \varepsilon)^2 \mu(A_\varepsilon). \end{aligned}$$

Thus $\|M_\varphi \chi\|_2 / \|\chi\|_2 > \|\varphi\|_\infty - \varepsilon$ and $\|M_\varphi\| > \|\varphi\|_\infty - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this shows that $\|M_\varphi\| \ge \|\varphi\|_\infty$, and we have equality.

Now $\chi = \chi_{A_\varepsilon}$ need not be in H^2, but, if we write $\chi(e^{it}) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$, then the sequence of functions (f_m) given by

$$f_m(e^{it}) = e^{imt}\chi(e^{it}) = \sum_{n=-\infty}^{\infty} c_k e^{i(k+m)t}$$

satisfies $\|f_m\|_2 = \|\chi\|_2$ and

$$\|M_\varphi f_m\|_2 = \|M_{e^{imt}} M_\varphi \chi\|_2 > (\|\varphi\|_\infty - \varepsilon)\|\chi\|_2.$$

Now

$$\begin{aligned} \|P_{H^2} f_m - f_m\|_2 &= \left\| \sum_{k=-m}^{\infty} c_k e^{i(k+m)t} - \sum_{k=-\infty}^{\infty} c_k e^{i(k+m)t} \right\|_2 \\ &= \left(\sum_{k=-\infty}^{-m-1} |c_k|^2 \right)^{1/2} \to 0 \qquad \text{as } m \to \infty. \end{aligned}$$

Hence $\|P_{H^2} f_m\|_2 \to \|\chi\|_2$ and $\|M_\varphi P_{H^2} f_m - M_\varphi f_m\|_2 \to 0$ so that

$$\|M_\varphi P_{H^2} f_m\|_2 \to \|M_\varphi \chi\|_2 \qquad \text{as } m \to \infty.$$

Note that $P_{H^2} f_m \in H^2$ and that $\|M_\varphi P_{H^2} f_m\|_2 / \|P_{H^2} f_m\|_2 > (\|\varphi\|_\infty - \varepsilon)$ for m sufficiently large, which gives the converse inequality.

Finally, if φ is essentially unbounded, then we can take functions $\varphi_n \in L^\infty(\mathbb{T})$ such that $|\varphi_n(e^{it})|$ increases monotonically to $|\varphi(e^{it})|$ almost everywhere and $\|\varphi_n\|_\infty \to \infty$ (just replace φ by zero at those points at which its absolute value is greater than n). Then $\|M_\varphi f\| \geq \|M_{\varphi_n} f\|$ for any $f \in H^2$ and

$$\|M_{\varphi_n}\| = \|\varphi_n\|_\infty \to \infty,$$

so that M_φ is unbounded on H^2. □

Corollary 4.1 *Suppose that $\varphi \in H^\infty$. Then $M_\varphi : H^2 \to H^2$ defined by (5) satisfies $\|M_\varphi\| = \|\varphi\|_\infty$.*

Proof: This follows from Theorem 4.1 once we verify that $\varphi \cdot f \in H^2$ (and not just $L^2(\mathbb{T})$ for $\varphi \in H^\infty$ and $f \in H^2$. One can do this directly by multiplying power series or, alternatively, computing the inner product:

$$(\varphi \cdot f, e^{ikt}) = (\varphi, \overline{f} e^{ikt}) = 0 \qquad \text{for } k < 0,$$

since $\varphi \in H^2$ and $\overline{f}(e^{it})e^{ikt}$ has only negative Fourier coefficients. □

Matrix notation. Write $(e_n)_{n=-\infty}^{\infty}$ for the orthonormal basis of $L^2(\mathbb{T})$, so that $e_n(e^{it}) = e^{int}$ or $e_n(z) = z^n$. Recall that $(e_n)_{n=0}^{\infty}$ is an orthonormal basis for H^2.

Now $M_\varphi e_n = \sum_{k=0}^{\infty} d_k e^{ikt} e^{int}$, where $\varphi(z) = \sum_{k=0}^{\infty} d_k z^k$ (convergence being understood to be in the L^2 sense at least) and $\varphi \in H^\infty$. Thus we obtain the infinite matrix

$$\begin{pmatrix} d_0 & 0 & 0 & 0 & \dots \\ d_1 & d_0 & 0 & 0 & \dots \\ d_2 & d_1 & d_0 & 0 & \dots \\ d_3 & d_2 & d_1 & d_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

A matrix that is constant on northwest to southeast diagonals, such as the one above, is called a *Toeplitz matrix*.

There is a way of obtaining an operator with a general Toeplitz matrix, not necessarily lower triangular, and we now examine this.

Definition 4.2 *For $\varphi \in L^\infty(\mathbb{T})$, the* Toeplitz operator *with* symbol φ *is the operator $T_\varphi : H^2 \to H^2$ defined by $T_\varphi f = P_{H^2}(M_\varphi f)$.*

Clearly, since $\|P_{H^2}\| = 1$ and $\|M_\varphi\| = \|\varphi\|_\infty$ we have that T_φ is a bounded operator on H^2 and satisfies $\|T_\varphi\| \le \|\varphi\|_\infty$. In the case in which $\varphi \in H^\infty$, we note also that T_φ is the same as M_φ. We shall see soon that $\|T_\varphi\| = \|\varphi\|_\infty$ for every φ.

Let $\varphi(z) = \sum_{k=-\infty}^{\infty} d_k z^k$. Then

$$\begin{aligned} T_\varphi e_n &= P_{H^2}\Big(\sum_{k=-\infty}^{\infty} d_k e^{ikt} e^{int}\Big) \\ &= \sum_{p=0}^{\infty} d_{p-n} e^{ipt}, \end{aligned}$$

where $p = n + k$. This gives the Toeplitz matrix of T_φ; namely, the matrix

$$\begin{pmatrix} d_0 & d_{-1} & d_{-2} & d_{-3} & \dots \\ d_1 & d_0 & d_{-1} & d_{-2} & \dots \\ d_2 & d_1 & d_0 & d_{-1} & \dots \\ d_3 & d_2 & d_1 & d_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Some special cases of importance are $\varphi = 1$, where T_φ is the identity matrix; $\varphi(z) = z$, when T_φ can be regarded as the right shift; and $\varphi(z) = 1/z$, when T_φ can be regarded as the left shift. Clearly, $T_\varphi = 0$ if and only if $\varphi = 0$.

Theorem 4.2 *For $\varphi \in L^\infty(\mathbb{T})$ one has $\|T_\varphi\| = \|\varphi\|_\infty$; that is,*

$$\sup\big\{\|T_\varphi f\|_2 : f \in H^2,\ \|f\|_2 = 1\big\} = \|\varphi\|_\infty.$$

Proof: Clearly, $\|T_\varphi\| = \|P_{H^2} M_\varphi\| \le \|M_\varphi\| = \|\varphi\|_\infty$, so we need only prove that '$\ge$' holds.

Given any $\varepsilon > 0$, there is a function $f \in H^2$ with $\|f\|_2 = 1$ and

$$\|M_\varphi f\|_2 > \|\varphi\|_\infty - \varepsilon,$$

by Theorem 4.1. Indeed, we may suppose without loss of generality that f is a polynomial

$$p(e^{it}) = \sum_{k=0}^{N} c_k e^{ikt}$$

since we can always find a sequence of polynomials p_n such that

$$\|p_n - f\|_2 \to 0 \qquad \text{and} \qquad \|M_\varphi p_n - M_\varphi f\|_2 \to 0.$$

We show first that, for any $k \geq 0$, one has that $\|(T_\varphi - M_\varphi)(e^{imt}e^{ikt})\|_2 \to 0$ as $m \to \infty$. For, with φ having Fourier coefficients (d_n) as above, this quantity is simply

$$\begin{aligned} \left\| (I - P_{H^2}) \sum_{r=-\infty}^{\infty} d_r e^{irt} e^{imt} e^{ikt} \right\|_2 &= \left\| \sum_{r=-\infty}^{-1-m-k} d_r e^{i(r+m+k)t} \right\|_2 \\ &= \left(\sum_{r=-\infty}^{-1-m-k} |d_r|^2 \right)^{1/2} \to 0 \end{aligned}$$

as $m \to \infty$. It follows that

$$\|(T_\varphi - M_\varphi)(e^{imt}p(e^{it}))\|_2 \leq \sum_{k=0}^{N} |c_k| \|(T_\varphi - M_\varphi)e^{imt}e^{ikt}\|_2 \to 0$$

as $m \to \infty$.

Now $\|e^{imt}p\|_2 = \|p\|_2$ and

$$\|T_\varphi(e^{imt}p)\|_2 \to \|M_\varphi(e^{imt}p)\|_2 = \|e^{imt}\varphi p\|_2 = \|M_\varphi p\|_2.$$

But $\|M_\varphi p\|_2 > \|\varphi\|_\infty - \varepsilon$, which gives the result, since $\varepsilon > 0$ was arbitrary. □

The above result is of great interest and importance. In general, given an infinite matrix, there is no formula that tells us its norm as an operator on ℓ^2.

Although Toeplitz operators may be bounded, they are not, in general compact.

Proposition 4.1 *The only compact Toeplitz operator is $T_0 = 0$.*

Proof: Let (e_n) be any orthonormal basis of H^2. It is easy to see that if S is any finite-rank operator, then $\|Se_n\| \to 0$ as $n \to \infty$, since we can write

$$Se_n = \sum_{k=1}^{r} (e_n, x_k) y_k,$$

where $(x_k)_{k=1}^r$ and $(y_k)_{k=1}^r$ are finite sequences in H^2. Then

$$\|Se_n\| \leq \sum_{k=1}^{r} |(e_n, x_k)| \, \|y_k\| \to 0$$

as $n \to \infty$, since $(e_n, x_k) \to 0$ for each k.

Now, for any compact operator S on H^2, the same is true, as we can write S as the norm limit of a sequence (S_k) of finite-rank operators and observe that $\|Se_n\| \leq \|S - S_k\| + \|S_k e_n\|$. Given $\varepsilon > 0$ one can make the first term at most $\varepsilon/2$ by choosing k large and then make the second term at most $\varepsilon/2$ by choosing n large.

However, it is easy to see that, for a Toeplitz operator T_φ, one cannot have $\|T_\varphi e_n\| \to 0$ as $n \to \infty$ unless T_φ is the zero operator. For $\|T_\varphi e_n\| \geq |d_m|$ as soon as d_m appears in the $(n+1)$st column, which is when $n \geq -m$. So T_φ is not compact unless every entry is zero. □

4.2 Hankel operators

We now consider a closely related class of operators, also introduced originally by Toeplitz. Let us write $(H^2)^\perp$ for the orthogonal complement of H^2 in $L^2(\mathbb{T})$; that is, the closed subspace spanned by the basis vectors of negative index,

$$e_n(e^{it}) = e^{int} \qquad (n < 0).$$

Definition 4.3 *For $\varphi \in L^\infty(\mathbb{T})$ the* Hankel operator Γ_φ *with* symbol φ *is defined to be the operator from H^2 to $(H^2)^\perp$ given by*

$$\Gamma_\varphi f = (I - P_{H^2})M_\varphi f;$$

that is, $M_\varphi = T_\varphi + \Gamma_\varphi$ in the obvious sense.

There are several possible alternative definitions which, under some circumstances, can be regarded as more convenient, but we shall work with the one just given.

Proposition 4.2 *The inequality $\|\Gamma_\varphi\| \le \|\varphi\|_\infty$ holds. Moreover, if $\varphi(e^{i\theta})$ has the Fourier series $\sum_{k=-\infty}^{\infty} d_k e^{ik\theta}$, then the matrix of Γ_φ with respect to the orthonormal bases $(e_n)_{n=0,1,2,\ldots}$ of H^2 and $(e_n)_{n=-1,-2,\ldots}$ of $(H^2)^\perp$ is*

$$\begin{pmatrix} d_{-1} & d_{-2} & d_{-3} & d_{-4} & \cdots \\ d_{-2} & d_{-3} & d_{-4} & d_{-5} & \cdots \\ d_{-3} & d_{-4} & d_{-5} & d_{-6} & \cdots \\ d_{-4} & d_{-5} & d_{-6} & d_{-7} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{6}$$

which is constant on diagonals sloping southwest to northeast (a Hankel matrix). Furthermore, $\Gamma_\varphi = 0$ if and only if $\varphi \in H^\infty$.

Proof: First, $\|\Gamma_\varphi f\|_2 = \|(I - P_{H^2})M_\varphi f\|_2 \le \|M_\varphi f\|_2 \le \|\varphi\|_\infty \|f\|_2$. Also

$$\begin{aligned} \Gamma_\varphi e_k &= (I - P_{H^2}) \sum_{n=-\infty}^{\infty} d_n e^{int} e^{ikt} \\ &= \sum_{n=-\infty}^{-1} d_n e^{i(n+k)t} = \sum_{r=1}^{\infty} d_{-r-k} e^{-irt}, \end{aligned}$$

obtained by setting $n + k = -r$. This gives the matrix as required, and clearly $\Gamma_\varphi = 0$ if and only if $d_n = 0$ for all $n < 0$, which occurs precisely when φ lies in H^∞. □

Theorem 4.3 (Z. Nehari) *Suppose that $\Gamma_\varphi : H^2 \to (H^2)^\perp$ is a Hankel operator. Then there is a function $\psi \in L^\infty(\mathbb{T})$ with $\Gamma_\varphi = \Gamma_\psi$ and $\|\Gamma_\psi\| = \|\psi\|_\infty$. Hence*

$$\|\Gamma_\varphi\| = \inf\{\|\varphi + h\|_\infty : h \in H^\infty\} = \operatorname{dist}(\varphi, H^\infty)$$

and the infimum is achieved at $h = \psi - \varphi$.

Proof: Observe first that, if $\varphi(e^{it}) = \sum_{k=-\infty}^{\infty} d_k e^{ikt}$ and n, $m > 0$, then

$$(\Gamma e^{int}, e^{-imt}e^{-it}) = d_{-n-m-1} = (\Gamma e^{int}e^{imt}, e^{-it}).$$

Hence

$$\left(\Gamma \sum_{n=0}^{N} b_n e^{int}, \sum_{m=0}^{M} \overline{c}_m e^{-imt}e^{-it}\right) = \left(\Gamma\left(\sum_{n=0}^{N} b_n e^{int} \sum_{m=0}^{M} c_m e^{imt}\right), e^{-it}\right).$$

Thus we can define a linear functional on polynomials by

$$\alpha(f) = (\Gamma f, e^{-it})$$

and if f factorizes as a product of polynomials by $f = f_1 f_2$, then

$$\alpha(f) = (\Gamma(f_1 f_2), e^{-it}) = (\Gamma f_1, \overline{f}_2 e^{-it}),$$

and so

$$|\alpha(f_1 f_2)| \le \|\Gamma\| \, \|f_1\|_2 \|f_2\|_2.$$

However, it follows from Theorem 2.5 that products of polynomials are dense in H^1 and so α has a unique extension to H^1 by

$$\alpha(f_1 f_2) = (\Gamma f_1, \overline{f}_2 e^{-it}) \qquad \text{for } f_1, f_2 \in H^2,$$

with $|\alpha(f_1 f_2)| \le \|\Gamma\| \, \|f_1\|_2 \|f_2\|_2$, or (using Theorem 2.5)

$$|\alpha(g)| \le \|\Gamma\| \, \|g\|_1 \qquad \text{for } g \in H^1.$$

We can now use the Hahn–Banach theorem to extend the domain of definition of the linear functional α to the whole of $L^1(\mathbb{T})$, while keeping the condition $|\alpha(g)| \le \|\Gamma\| \, \|g\|_1$ for $g \in L^1(\mathbb{T})$. This implies that there is a representation

$$\alpha(g) = \frac{1}{2\pi} \int_0^{2\pi} g(e^{i\theta})\beta(e^{i\theta}) \, d\theta,$$

for some $\beta \in L^\infty(\mathbb{T})$ with

$$\|\beta\|_\infty = \|\alpha\| \le \|\Gamma\|.$$

Moreover, for $k \ge 0$,

$$(\beta, e^{-ikt}) = \frac{1}{2\pi} \int_0^{2\pi} e^{ikt}\beta(e^{it}) \, dt = \alpha(e^{ikt}) = (\Gamma e^{ikt}, e^{-it}) = d_{-k-1}.$$

Hence the Fourier coefficients (β_n) of β satisfy $\beta_{-k} = d_{-k-1}$ for $k \ge 0$. We now take $\psi(e^{it}) = e^{-it}\beta(e^{it})$, so that the Fourier coefficients (ψ_n) of ψ satisfy $\psi_{-k-1} = d_{-k-1}$ for $k \ge 0$. Moreover,

$$\|\psi\|_\infty = \|\beta\|_\infty \le \|\Gamma\|,$$

as required.

Now since $\Gamma_\varphi = \Gamma_\psi$ if and only if the function h, defined by $h = \psi - \varphi$, is in H^∞, we have that

$$\|\Gamma_\varphi\| \le \inf\{\|\varphi + h\|_\infty : h \in H^\infty\}$$

and there is a function ψ such that $\psi = \varphi + h$ and

$$\|\psi\|_\infty \le \|\Gamma_\psi\| \le \|\psi\|_\infty.$$

Hence $\|\psi\|_\infty = \|\Gamma_\psi\|$ as required. □

Remark 4.1 The above proof actually shows rather more; namely, that any bounded operator T from H^2 to $(H^2)^\perp$ given by a Hankel matrix is actually a Hankel operator—that is, $T = \Gamma_\psi$ for some $\psi \in L^\infty$ with $\|\psi\|_\infty = \|T\|$.

We saw earlier (Theorem 2.6) that, for every function $f \in H^2$ except the identically zero function, one has $f(e^{i\theta}) \neq 0$ almost everywhere. This enables us to give an explicit solution, in a wide range of cases, to the Nehari problem of finding a minimal-norm symbol for a Hankel operator; or, equivalently, of approximating a bounded function by an analytic one.

Theorem 4.4 (D. Sarason) *Suppose that $\Gamma : H^2 \to (H^2)^\perp$ is a Hankel operator for which there exists $f \in H^2$, $f \neq 0$ with $\|\Gamma f\|_2 = \|\Gamma\|\|f\|_2$. Then there is a unique symbol $\psi \in L^\infty$ for Γ such that $\|\psi\|_\infty = \|\Gamma\|$. Moreover, ψ is given by $\psi = (\Gamma f)/f$; that is, $\psi(e^{i\theta}) = (\Gamma f)(e^{i\theta})/f(e^{i\theta})$ almost everywhere. Also, the identity $|\psi(e^{i\theta})| = \|\Gamma\|$ holds almost everywhere.*

Proof: Let ψ be a symbol of minimal norm, which exists by Nehari's theorem. Since $\Gamma_\psi f = (I - P_{H^2})M_\psi f$ we have that

$$\|\Gamma_\psi\|\|f\|_2 = \|\Gamma f\|_2 \leq \|\psi \cdot f\|_2 \leq \|\psi\|_\infty \|f\|_2.$$

Since $\|\Gamma_\psi\| = \|\psi\|_\infty$, there is equality throughout and so we necessarily have $\Gamma f = \psi \cdot f$. Thus $\psi = (\Gamma f)/f$ almost everywhere, since $f(e^{i\theta}) \neq 0$ almost everywhere. Moreover, since $\|\psi \cdot f\|_2 = \|\psi\|_\infty \|f\|_2$, the identity $|\psi(e^{i\theta})| = \|\psi\|_\infty$ must hold almost everywhere. □

The problem of finding a minimal norm symbol for a Hankel operator can be recast usefully as one of approximation of L^∞ functions by H^∞ functions, as we now see.

Corollary 4.2 *Suppose that $\varphi \in L^\infty(\mathbb{T})$ is such that Γ_φ attains its norm (that is,*

$$\|\Gamma_\varphi f\|_2 = \|\Gamma_\varphi\|\|f\|_2$$

for some $f \in H^2$, $f \neq 0$). Then φ has a unique best approximant $h \in H^\infty$; that is, a function satisfying $\|\varphi - h\|_\infty = \operatorname{dist}(\varphi, H^\infty)$. Moreover, $h = \varphi - (\Gamma f)/f$ and $|(\varphi - h)(e^{i\theta})| = \|\varphi - h\|_\infty = \|\Gamma_\varphi\|$ almost everywhere.

Proof: If $g \in H^\infty$ then $\Gamma_{\varphi - g} = \Gamma_\varphi$, and so

$$\|\varphi - g\|_\infty \geq \|\Gamma_{\varphi - g}\| = \|\Gamma_\varphi\|.$$

But there is a function $\psi \in L^\infty(\mathbb{T})$, given by $\psi = (\Gamma f)/f$, such that $\Gamma_\psi = \Gamma_\varphi$, and

$$\|\psi\|_\infty = \|\Gamma_\psi\| = \|\Gamma_\varphi\|.$$

Thus $h = \varphi - \psi$ lies in H^∞ (since Γ_h is the zero operator) and

$$\|\varphi - h\|_\infty = \|\psi\|_\infty = \|\Gamma_\varphi\|,$$

so that h is a best approximant to φ. However, ψ is unique and hence so is h. Also, the function $\varphi - h = \psi$ has constant modulus almost everywhere. □

Since functions corresponding to Hankel operators that attain their norm have unique best approximants in H^∞, we know that any function corresponding to a Hankel operator that is finite-rank, or even just compact, has this desirable property. Thus we would like to be able to say when this happens.

Theorem 4.5 (L. Kronecker) *The Hankel operator Γ with matrix given by Equation (6) has finite rank if and only if $f(z) = \sum_{k=-\infty}^{-1} d_k z^k$ is a rational function of z. Its rank is the number of poles of f (which will necessarily lie in $\mathbb{D}$).*

Proof: If the rank of Γ is r, then the first $(r+1)$ columns of Γ are linearly dependent; that is, there exist scalars $\lambda_1, \ldots, \lambda_{r+1}$, not all zero, such that

$$\sum_{k=1}^{r+1} \lambda_k d_{-k-m} = 0 \qquad \text{for all } m \geq 0.$$

This implies that

$$(\lambda_1 + \lambda_2 z + \ldots + \lambda_{r+1} z^r) \left(\frac{d_{-1}}{z} + \frac{d_{-2}}{z^2} + \ldots \right)$$

is a polynomial of degree at most $(r-1)$ in z, since the coefficient of a negative power z^{-m-1} is equal to

$$\lambda_1 d_{-m-1} + \ldots + \lambda_{r+1} d_{-m-r-1},$$

which is zero. Hence f is rational of degree at most r.

Conversely, if $P(z) \sum_{-\infty}^{-1} d_k z^k = Q(z)$ for some polynomials P and Q with the degree of P less than or equal to r, then working backwards we see that the rank of Γ is at most r.

Thus the rank is actually equal to the number of poles. Note that f is the projection of an L^2 function (namely any bounded symbol for Γ) onto $(H^2)^\perp$. Since $g(z) = (1/z)f(1/z)$ lies in H^2 and thus has poles outside $\overline{\mathbb{D}}$, it follows that the poles of f lie within $\mathbb{D}$. □

We write R_k for the set of rational functions $f(z)$ which have at most k poles, all lying in the open disc $\mathbb{D}$.

Corollary 4.3 *The Hankel operator Γ_φ has rank at most k if and only if $\varphi \in H^\infty + R_k$.*

Proof: This follows since $\Gamma_\varphi = \Gamma_\psi$ if and only if $\varphi - \psi \in H^\infty$. □

We finish with a characterization of the symbols of compact Hankel operators, omitting the proof, which can be found, for example, in [4] or [9].

Theorem 4.6 (P. Hartman) *The Hankel operator Γ_φ is compact if and only if $\varphi \in H^\infty + C(\mathbb{T})$.*

The following result on the mysterious space $H^\infty + C(\mathbb{T})$ is quite difficult but has great theoretical importance.

Theorem 4.7 (D. E. Sarason) *The space $H^\infty + C(\mathbb{T})$ is a closed subalgebra of the Banach algebra $L^\infty(\mathbb{T})$.*

5 Background reading

The ideas of Section 1 are standard and can be found in [3, 5, 10], for example.

Section 2 is based on a number of sources, including [1, 5, 9].

The material of Section 3 is adapted from some lecture notes of Nikolski [6]. A somewhat more advanced treatment can be found in [7].

Most of Section 4 is taken from [9]. Much of it may also be found in [2, 4, 7].

References

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