Analysis (wavelets): Some useful results and formulae

J.R. Partington, School of Mathematics, University of Leeds

December 1996, revised June 1997

Please let me know of any errors, omissions or obscurities.

L_p spaces: functions in $L_p(\mathbb{R})$ are measurable and satisfy $\|f\|_p = \left(\int_{-\infty}^{\infty} |f(t)|^p \, dt \right)^{1/p} < \infty$, regarding two functions as the same if they are equal almost everywhere. These are Banach spaces, and $L_2(\mathbb{R})$ is a Hilbert space with inner-product $\langle f, g \rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\, dt$. For $L_\infty(\mathbb{R})$, take the norm to be the (essential) supremum of $|f(t)|$.

Fourier series: $L_2(0,T)$ has an orthonormal basis, $e_n(t) = T^{-1/2} \exp(2\pi i n t/T)$, $n \in \mathbb{Z}$, so $f = \sum_{n=-\infty}^{\infty} \langle f, e_n \rangle e_n$, converging in L_2 norm.

Fourier transforms: For $f \in L_1(\mathbb{R})$ define $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-2\pi i \omega t)\, dt$, and write $\mathcal{F} f = \hat{f}$. This extends to a bounded linear operator $\mathcal{F}$ from $L_2(\mathbb{R})$ onto $L_2(\mathbb{R})$, and $\|\hat{f}\|_2 = \|f\|_2$ (Plancherel). Moreover $(\hat{f})\hat{}\,(t) = f(-t)$, or $f(t) = \int_{-\infty}^{\infty} \hat{f}(\omega) \exp(2\pi i \omega t)\, d\omega$, a.e., at least if $\hat{f} \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$. We get as a corollary that $\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$, for f, $g \in L_2(\mathbb{R})$. Write $\check{F}$ for the inverse F.T. of F.

Haar wavelets: Let $V_0 \subset L_2(\mathbb{R})$ be the closed subspace of all functions f which are constant on all intervals $(k, k+1)$, $k \in \mathbb{Z}$. Let $\phi(t) = \chi_{(0,1)}(t)$ and $\phi_k(t) = \phi(t-k)$. Then $(\phi_k)_{k \in \mathbb{Z}}$ is an orthonormal basis of V_0. Now for $j \in \mathbb{Z}$ let V_j be the space of functions constant on all intervals $(k/2^j, (k+1)/2^j)$, $k \in \mathbb{Z}$.

Then $\ldots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots$; $\overline{\bigcup V_j} = L_2(\mathbb{R})$, and $\bigcap V_j = \{0\}$. Also $f(t) \in V_j \iff f(2^{-j} t) \in V_0$, and V_j has o.n.b. consisting of $\phi_{j,k}(t) = 2^{j/2} \phi(2^j t - k)$, $k \in \mathbb{Z}$. Any chain of subspaces with these properties is called a *multiresolution approximation* or *multiresolution analysis (MRA)* of $L_2(\mathbb{R})$.

Define the *Haar wavelet* $\psi(t) = \phi(2t) - \phi(2t-1)$. The functions $\psi_k(t) = \psi(t-k)$ form an o.n.b. for a space W_0 such that $V_0 \oplus W_0 = V_1$ (direct sum of orthogonal spaces). Then $V_j \oplus W_j = V_{j+1}$ and W_j has o.n.b. $\psi_{j,k}(t) = 2^{j/2} \psi(2^j t - k)$, $k \in \mathbb{Z}$. Finally $L_2(\mathbb{R}) = \ldots \oplus W_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus \ldots$ and has orthonormal basis $(\psi_{j,k})_{j,k \in \mathbb{Z}}$. Hence $f = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \langle f, \psi_{j,k} \rangle \psi_{j,k}$, converging in L_2 norm.

Band limited functions: $f \in L_2(\mathbb{R})$ is *time-limited*, if $\exists T > 0$ s.t. $f(t) = 0$ a.e. for $|t| > T$; f is *band-limited* if $\hat{f}$ is time-limited.

Paley-Wiener space: $PW(b) = \{ f \in L_2(\mathbb{R}) : \hat{f}(\omega) = 0 \text{ a.e. for } |\omega| > b \}$. Such functions are the restrictions to $\mathbb{R}$ of entire functions satisfying $|f(z)| \le \|\hat{f}\|_1 e^{2\pi b y}$, $z = x + iy \in \mathbb{C}$. Hence the only function both time- and band-limited is 0 a.e.

Whittaker-Kotel'nikov-Shannon sampling theorem: for $f \in PW(b)$,
$f(t) = (1/2b) \sum_{n=-\infty}^{\infty} f(n/2b) k_{n/2b}(t)$, where $k_s(t) = \sin 2\pi b(t-s)/(\pi(t-s))$, the *reproducing kernel function.* Moreover the functions $(1/\sqrt{2b}) k_{n/2b}(t)$ form an o.n.b. of $PW(b)$ and $\langle f, k_s \rangle = f(s)$ for all $s \in \mathbb{R}$ – this is the r.k.H.s. property.

Other r.k.H.s. include the Hardy space H_2 on $\mathbb{D}$, where $k_w(z) = 1/(1 - \overline{w} z)$; also the space $\mathbb{C}^n$ on the set $\{1, \ldots, n\}$. In general $k_s(t) = \overline{k_t(s)}$ and $k_s(s) > 0$ unless all functions vanish at s.

Define $V_0 = PW(1/2)$ and $V_j = PW(2^{j-1})$ in general. These give a multiresolution approximation, as defined above, and an orthonormal basis of V_j is given by $2^{j/2}\phi(2^j t - k)$, $k \in \mathbb{Z}$, where $\phi(t) = \sin \pi t/(\pi t) = k_0(t)$.

Define the *Littlewood-Paley wavelet* $\psi(t)$ so that $\hat{\psi}(\omega) = \chi_{(1/2,1)}(\omega)$, i.e. $\psi(t) = (\sin 2\pi t - \sin \pi t)/(\pi t)$. Then as before we obtain an o.n.b. of $L_2(\mathbb{R})$ consisting of wavelets $2^{j/2}\psi(2^j t - k)$, j, $k \in \mathbb{Z}$, where for fixed j they form an o.n.b. of a space W_j s.t. $V_j \oplus W_j = V_{j+1}$.

A *Riesz basis* (ϕ_k) in a Hilbert space H satisfies (i) H is the closed linear span of the (ϕ_k) and (ii) $\exists A$, $B > 0$ s.t. $A\left(\sum_{k=1}^{\infty} |a_k|^2\right)^{1/2} \leq \|\sum_{k=1}^{\infty} a_k \phi_k\| \leq B\left(\sum_{k=1}^{\infty} |a_k|^2\right)^{1/2}$ for all $(a_k) \in \ell_2$. (ϕ_k) is a Riesz basis $\iff \exists U : H \to H$ bounded, invertible, such that $(U\phi_k)$ is an orthonormal basis. A (finite) R.b. in $\mathbb{C}^n$ is any basis in the usual sense.

A *frame* (ϕ_k) in H satisfies $\exists A$, $B > 0$ s.t. $A\|\phi\|^2 \leq \sum_{k=1}^{\infty} |\langle \phi, \phi_k \rangle|^2 \leq B\|\phi\|^2$ for all $\phi \in H$. Then o.n.b. $\implies$ R.b. $\implies$ frame. The numbers $\langle \phi, \phi_k \rangle$ are the *frame coefficients* of ϕ. A (finite) frame in $\mathbb{C}^n$ is any finite spanning set.

We have a frame (ϕ_k) iff there is a bounded linear map $T : \ell_2 \to H$ (ONTO) such that $T((a_k)) = \sum a_k \phi_k$ for all $(a_k) \in \ell_2$. Define the mapping $S = TT^* : H \to H$ by $S\psi = \sum \langle \psi, \phi_k \rangle \phi_k$. Now S is positive, bounded, invertible and $(\tilde{\phi}_k) = (S^{-1}\phi_k)$ is a frame, the *dual frame* to (ϕ_k). The dual frame to $(\tilde{\phi}_k)$ is (ϕ_k) again. Then $\psi = \sum \langle \psi, \phi_k \rangle \tilde{\phi}_k$. We can invert S using $S^{-1} = (2/(B+A)) \sum_{n=0}^{\infty} (I - 2S/(B+A))^n$, which converges rapidly.

Windowed Fourier transforms: Take $g \in L_2(\mathbb{R})$ of norm 1, often real and positive; write $f_t(u) = f(u)\overline{g(u-t)}$, for $f \in L_2(\mathbb{R})$ and $t \in \mathbb{R}$. Define $\tilde{f}(\omega, t) = \hat{f}_t(\omega)$. This is the same as $e^{-2\pi i \omega t}\tilde{\hat{f}}(-t, \omega)$, where the RHS is formed using the window $\hat{g}$ (time-frequency duality). *Inversion formula:* $f(u) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(\omega, t) g(u-t) e^{2\pi i \omega u}\, d\omega\, dt$ when all integrals converge absolutely. Also $\|\tilde{f}\|_2 = \|f\|_2$, so we get a norm-preserving linear operator $W : L_2(\mathbb{R}) \to L_2(\mathbb{R}^2)$, not onto. Can be discretised: $\tilde{f}(m\omega_0, nt_0) = \langle f, g_{mn} \rangle$, where $g_{mn}(u) = e^{2\pi i m \omega_0 u} g(u - nt_0)$. For suitable ω_0, $t_0 > 0$, these may form a frame and we can invert it.

Heisenberg's inequality: For $f \in L_2(\mathbb{R})$, $\|tf(t)\|_2 \|\omega \hat{f}(\omega)\|_2 \geq \|f\|_2^2/(4\pi)$ – equality for the Gaussian functions.

Wavelet transform: Let $\psi \in L_2(\mathbb{R})$ satisfy $C_\psi = \int_{-\infty}^{\infty} (|\hat{\psi}(\omega)|^2/|\omega|)\, d\omega < \infty$ and write $\psi^{x,y}(t) = |x|^{-1/2}\psi((t-y)/x)$, x, $y \in \mathbb{R}$, $x \neq 0$. Define $f^\circ(x,y) = \langle f, \psi^{x,y} \rangle$. Better 'zooming' properties than the W.F.T. Then $\int\int f^\circ(x,y)\overline{g^\circ(x,y)}\, dx\, dy/x^2 = C_\psi \langle f, g \rangle$, giving a weak inversion formula. Also the transform maps $L_2(\mathbb{R})$ isometrically into a subspace H of $L_2(\mu)$ where $d\mu = dx\, dy/(C_\psi x^2)$, which is a r.k.H.s. on $\mathbb{R}^2$ with kernel $k_{(s,t)}(x,y) = \langle \psi^{x,y}, \psi^{s,t} \rangle$.

Discretising: write $\psi_{j,k}(t) = a^{j/2}\psi(a^j t - kb)$ for some fixed $a > 1$, $b > 0$ and consider inner-products $\langle f, \psi_{j,k} \rangle$. Again we would like a frame (e.g. for the Mexican hat function $C(1 - x^2)e^{-x^2/2}$, $a = 2$, $b = 1$ will do).

Meyer wavelets: Let $\theta : [0,1] \to \mathbb{R}$ be C^d and satisfy $\theta = 0$ on $[0, 1/3]$; $\pi/2$ on $[2/3, 1]$; and $\theta(u) + \theta(1-u) = \pi/2$. Define $\hat{\phi}(\omega) = \cos(\theta(|\omega|))$, C^d and supported on $[-2/3, 2/3]$. The condition $\langle \phi_k, \phi_\ell \rangle = \delta_{k,\ell}$ is equivalent to $K(\omega) = \sum_{k=-\infty}^{\infty} |\hat{\phi}(\omega + k)|^2 = 1$ a.e. (for any $\phi \in L_2(\mathbb{R})$) and this is used to show that the Meyer function ϕ gives a multiresolution analysis.

General construction of wavelets: Given a MRA let ϕ be the father wavelet, so that $\phi_{0,k}$ is an o.n.b. of V_0. We seek $\psi \in V_1$ such that (ψ_k) form an o.n.b. of W_0 and $V_0 \oplus W_0 = V_1$, orthogonal sum. Write $\phi = \sum_{k=-\infty}^{\infty} h_k \phi_{1,k}$ and $h(t) = \sum h_k \exp(2\pi i k t) \in L_2(0,1)$. Note that

$\sum_{k=-\infty}^{\infty} \overline{h}_k h_{k+2\ell} = \delta_{\ell,0}$ and that $\chi = \sum_{k=-\infty}^{\infty} g_k \phi_{1,k} \in W_0 \iff \sum_{k=-\infty}^{\infty} \overline{g}_k h_{k+2\ell} = 0\ \forall \ell \in \mathbb{Z}$. Now $|h(t)|^2 + |h(t+1/2)|^2 = 2$ a.e. and $g(t) = \sum_{k=-\infty}^{\infty} g_k \exp(2\pi i k t)$ corresponds to $\chi \in W_0$ iff $g(t) = \exp(2\pi i t)\mu(2t)\overline{h(t+1/2)}$, where μ has period 1. One such choice takes $g_k = (-1)^k \overline{h}_{1-k}$ and now $\psi = \sum_{k=-\infty}^{\infty} g_k \phi_{1,k}$ (**) is the mother wavelet. Also $\hat{\phi}(\omega) = (1/\sqrt{2})h(\omega/2)\hat{\phi}(\omega/2)$.

Main Theorem: Let ν be any function of period 1, satisfying $|\nu| = 1$ a.e. Then the mother wavelet ψ defined by $\hat{\psi}(\omega) = (1/\sqrt{2})e^{i\pi\omega}\nu(\omega)\overline{h(\omega/2 + 1/2)}\hat{\phi}(\omega/2)$ generates an orthonormal basis (ψ_k) for W_0 and hence an o.n.b. (ψ_{jk}) of $L_2(\mathbb{R})$.

Spline wavelets: A *spline* function of degree n with knots at $a_0, \ldots, a_N$ is C^{n-1} on $[a_0, a_N]$ and given by a (probably different) polynomial of degree n on each $[a_j, a_{j+1}]$. The *B-splines* are particular (basis) splines with knots at the integers: for $n = 0$ we get Haar wavelets again; for $n = 1$ take $\phi(x) = \max(0, 1 - |x|)$ and $\phi(x) = (1/2)\phi(2x+1) + \phi(2x) + (1/2)\phi(2x-1)$. Now the $K(\omega)$ function above isn't constantly 1 but is bounded above and away from zero. Hence (ϕ_k) give a Riesz basis for their closed linear span, which we take as V_0. To get an orthonormal basis define $\phi^\sharp$ by $\hat{\phi}^\sharp(\omega) = \hat{\phi}(\omega)/\sqrt{K(\omega)}$. For $n = 1$, $\hat{\phi}(\omega) = (\sin(\pi\omega)/(\pi\omega))^2$ and $K(\omega) = 2/3 + 1/3\cos(2\pi\omega)$. Now $(\phi_k^\sharp)$ gives an o.n.b. for V_0 and we can construct the *Battle-Lemarié wavelets.* Note that $\phi^\sharp$ is not compactly supported, but has exponential decay and is still a spline.

Change and decay: If $f, \ldots, f^{(n)}$ are in $L_1(\mathbb{R})$ then $\hat{f}(\omega) = O(|\omega|^{-n})$. If $\hat{f}(\omega) = O(|\omega|^{-n})$ then f is C^{n-2}. Similarly, swapping f and $\hat{f}$. So one being rapidly decreasing corresponds to the other being C^∞.

Daubechies wavelets: Let ϕ, h be related as before, and h a trigonometric polynomial; then $h(0) = \sqrt{2}$ and $\hat{\phi}(\omega) = \hat{\phi}(0)\prod_{m=1}^{\infty}\left(2^{-1/2}h(\omega/2^m)\right)$. If ϕ is a compactly supported function generating a MRA then ψ will be a compactly supported wavelet if we take ψ as in (**) above. Fix $N \geq 1$. Write $z = e^{2\pi i\omega}$. We take $h(\omega) = \sqrt{2}P(z)$, where P is a polynomial of degree $2N-1$, $P(z) = \left(\frac{1+z}{2}\right)^{N-1} W(z)$, $\deg W = N$, $W(1) = 1$. Then $\hat{\phi}(k) = \delta_{0,k}$. We need $|P(z)|^2 + |P(-z)|^2 = 1$ on $|z| = 1$. We can do this by taking $|P(z)|^2 = \sum_{k=0}^{N-1}\binom{2N-1}{k} C^{4N-2-2k}S^{2k}$, where $C(z) = \cos\pi\omega$, $S(z) = \sin\pi\omega$ (Strichartz, 1993). Requires a *spectral factorization* to determine P.

Finally ϕ and hence ψ will be supported on $[0, 2N-1]$, and they become smoother as N increases. For $N = 1$ we get Haar wavelets; for $N = 2$, $P(z) = (1+z)^2[(1 \pm \sqrt{3}) + (1 \mp \sqrt{3})z]/8$ and ϕ, ψ are continuous; after that a computer is needed.

Wavelet	Properties of $\psi(t)$	Properties of $\hat{\psi}(\omega)$
Haar	Compact support, discontinuous,	$O(1/\omega)$, C^∞
Littlewood-Paley	$O(1/t)$, C^∞	Compact support, discontinuous
Meyer	Rapidly-decreasing, C^∞	Compact support, can be C^∞
Battle-Lemarié	Rapidly-decreasing, C^k	$O(1/\omega^k)$, C^∞
Daubechies	Compact support, C^k	$O(1/\omega^k)$, C^∞

Compact support of both ψ and $\hat{\psi}$ is always impossible; exponential decay and smoothness for both isn't possible for o.n.b. of wavelets, but is for frames (e.g. Mexican hat).