University of Leeds, School of Mathematics

Postgraduate course in Analysis 2009–2010

Problems 1

1. Find the Fourier series of the Haar wavelet ψ given by

$$\psi(t) = \chi_{(0,1/2)}(t) - \chi_{(1/2,1)}(t).$$

with respect to the orthonormal basis $(e_n)_{n\in\mathbb{Z}}$ of $L^2(0,1)$, where $e_n(t) = \exp(2\pi int)$.

2. Prove that if $(c_n)_{n\in\mathbb{Z}}$ is a sequence in $\ell^1(\mathbb{Z})$, then the function

$$f : t \mapsto \sum_{n=-\infty}^{\infty} c_n \exp(2\pi int)$$

is continuous on $\mathbb{R}$ and 2π-periodic. What can you say about f if (c_n) satisfies the stronger condition $\sum_{n=-\infty}^{\infty} |nc_n| < \infty$?

3. Let $(V_n)_{n\in\mathbb{Z}}$ be a chain of closed subspaces of a Hilbert space H, satisfying

$$\ldots \subseteq V_{-2} \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq V_2 \subseteq \ldots,$$

and also $\bigcap V_n = \{0\}$ and $\overline{\bigcup V_n} = H$. Write $U_n = V_n^\perp$. Show that

$$\ldots \supseteq U_{-2} \supseteq U_{-1} \supseteq U_0 \supseteq U_1 \supseteq U_2 \supseteq \ldots,$$

and also $\bigcap U_n = \{0\}$ and $\overline{\bigcup U_n} = H$.

4. Expand the function $\phi = \chi_{(0,1)}$ in terms of the orthonormal Haar basis $(\psi_{jk})_{j,k\in\mathbb{Z}}$, where

$$\psi_{jk}(t) = 2^{j/2}\psi(2^j t - k).$$

(Hint: many of the coefficients $\langle \phi, \psi_{jk} \rangle$ will be 0.)

5. Write down a necessary and sufficient condition, in terms of integrals of f over finite intervals, for $f \in L^2(\mathbb{R})$, to lie in $W_0 \oplus W_1 \oplus W_2 \oplus \ldots$, where $(W_n)_{n\in\mathbb{Z}}$ are the subspaces used in the construction of the Haar wavelets.

6. A function $f \in PW(\frac{1}{2})$ satisfies $f(0) = 1$, $f(1) = 2$ and $f(n) = 0$ for $n \in \mathbb{Z} \setminus \{0,1\}$. What is f? Find a different function $g \in PW(1)$ that satisfies the same interpolation conditions as f.

7. Given that for the Littlewood–Paley wavelet ψ we have

$$\hat{\psi}(w) = \begin{cases} 1 & \text{if } \frac{1}{2} \le |w| \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

calculate $\psi(t)$ by using the formula for the inverse Fourier transform.

8. Let H be a Hilbert space with orthonormal basis $(e_n)_{n=1}^{\infty}$. Show that the sequence

$$\{e_1, e_2, e_1 + e_2, \ldots, e_{2n+1}, e_{2n+2}, e_{2n+1} + e_{2n+2}, \ldots\}$$

forms a frame in H. Is it tight? Calculate the dual frame (hint: you only ever need to calculate in 2-dimensional spaces).

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Problems 2

1. Find the constant $C > 0$ such that the function $g(t) = C(1 - t^2)\chi_{[-1,1]}(t)$ satisfies $\|g\|_2 = 1$. Using this function as the window, calculate the windowed Fourier transform $\widetilde{f}(w, t)$ of the function $f(t) = 1/(1 + t^2)$.

2. Prove the shifted form of Heisenberg's inequality:

$$\|(t - t_0)g(t)\|_2 \|(w - w_0)\widehat{g}(w)\|_2 \geq \frac{1}{4\pi}\|g\|_2^2,$$

starting from the given version with $t_0 = w_0 = 0$ and using the substitution $f(t) = g(t + t_0)e^{-2\pi i w_0 t_0}$.

3. Let

$$\psi(t) = \chi_{[0,1/2]}(t) - \chi_{[1/2,1]}(t).$$

Use this wavelet to calculate the wavelet transform $\mathring{f}(x, y)$ of the function $f(t) = 1/(1 + t^2)$.

4. Show that $L^2(\mathbb{R}^2, dx\,dy/x^2)$ is not a reproducing kernel Hilbert space (i.e., point evaluations are discontinuous), even though the subspace H of wavelet transforms is known to be a r.k.H.s.

5. Find a C^1 spline function f with knots at $\{0, 1, 2, 3\}$, such that $f(k) = (-1)^k$ for $k = 0, 1, 2, 3$ and f is defined by polynomials of degree at most 2 on each interval $[0, 1]$, $[1, 2]$ and $[2, 3]$. Is the answer unique?

6. Let $\phi(x) = (1 - |x|)\chi_{[-1,1]}(x)$, as used in the construction of Battle–Lemarié wavelets. Prove the claims made in lectures that

$$\widehat{\phi}(w) = \left(\frac{\sin \pi w}{\pi w}\right)^2.$$

and $\phi(x) = \frac{1}{2}\phi(2x + 1) + \phi(2x) + \frac{1}{2}\phi(2x - 1)$.

7. Prove the Fejér–Riesz theorem on spectral factorization by completing the details of the following argument.

Suppose that $w(z) = \sum_{k=-n}^{n} c_k z^k$ is real and strictly positive whenever $|z| = 1$ and that $c_{-n} \neq 0$. Then the polynomial $z^n w(z)$ can be factorized as

$$c\prod_{j=1}^{n}(z - a_j)(z - 1/\overline{a_j}),$$

where $c \neq 0$ is a constant and $a_1, \ldots, a_n$ are its roots in the unit disc. So $w(z) = |p(z)|^2$ on the unit circle, where $p(z)$ is a suitable polynomial of degree n with roots $a_1, \ldots, a_n$.