Classical Wavelet Theory

Jonathan R. Partington, University of Leeds, School of Mathematics

April 29, 2010

`http://www.maths.leeds.ac.uk/~ pmt6jrp/gradcourse0910.html`

LECTURE 1

Books:
Ingrid Daubechies, Ten lectures on wavelets, SIAM.
C.K. Chui, An introduction to wavelets, Academic Press.
Gerald Kaiser, A friendly guide to wavelets, Birkhäuser.

1 Introduction

Fourier series and transforms

For suitable functions f defined on $[0,1]$ we can form a Fourier series

$$\begin{aligned} f(t) &\sim \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikt} \\ &= a_0 + \sum_{j=1}^{\infty} (a_j \cos 2\pi jt + b_j \sin 2\pi jt), \end{aligned}$$

and

$$c_k = \int_0^1 f(t) e^{-2\pi ikt}\, dt = \hat{f}(k), \qquad \text{say.}$$

Likewise for suitable functions f on the real line we have

$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty} \hat{f}(w) e^{2\pi iwt}\, dw, \qquad \text{where} \\ \hat{f}(w) &= \int_{-\infty}^{\infty} f(t) e^{-2\pi iwt}\, dt. \end{aligned}$$

These are useful formulae, but not *localised* – if you want to examine f near $t = t_0$, say, you still need all the information in $\hat{f}$.

Windowed Fourier transform (Gabor, 1946)

Let g be a function which is 'well-localised' – i.e., big near 0, say, and with $g(t)$ small for large t.
Examples:

- $g(t) = e^{-t^2/2}$;
- $g(t) = \chi_{[-1,1]}(t)$;
- $g(t)$ a *bump function*, $g \in C^\infty(\mathbb{R})$ and $g(t) = \begin{cases} 0 & \text{for} \quad |t| \geq 2, \\ 1 & \text{for} \quad |t| \leq 1. \end{cases}$

Let

$$T^{\text{win}} f(w, t') = \int_{-\infty}^{\infty} f(t) g(t - t') e^{-2\pi i w t} \, dt.$$

This contains information about f near t' and at frequency $2\pi\omega$.
Signal analysts can use a discrete form of this, say $t' = mt_0$ and $w = kw_0$ for $m, k \in \mathbb{Z}$, to get Fourier coefficients of a "slice" of f, i.e.,

$$T^{\text{win}}_{m,k}(f) = \int_{-\infty}^{\infty} f(t) g(t - mt_0) e^{-2\pi i k w_0 t} \, dt.$$

Wavelet transforms

Let ψ be a well-localised smooth function satisfying the technical condition

$$\int_{-\infty}^{\infty} \frac{|\hat{\psi}(w)|^2}{|w|} \, dw < \infty.$$

In particular, we want

$$\int_{-\infty}^{\infty} \psi(t) \, dt = \hat{\psi}(0) = 0.$$

For example, the *Mexican hat function*,

$$\psi(t) = (1 - t^2) \exp(-t^2/2),$$

which is basically the second derivative of the Gaussian function. Define

$$(T^{\text{wav}} f)(a, b) = |a|^{-1/2} \int_{-\infty}^{\infty} f(t) \psi\left(\frac{t - b}{a}\right) dt = \langle f, \psi^{a,b} \rangle,$$

where

$$\psi^{a,b}(t) = |a|^{-1/2} \psi\left(\frac{t - b}{a}\right).$$

The number $a \neq 0$ controls the resolution (scaling) and b the position of the function.
Then in fact there is an inversion formula

$$f = C_\psi^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^{a,b} \langle f, \psi^{a,b} \rangle \, \frac{da\,db}{a^2},$$

where

$$C_\psi = 2\pi \int_{-\infty}^{\infty} \frac{|\hat{\psi}(w)|^2}{|w|} \, dw.$$

Again, to reconstruct f we only need a discrete version. Let

$$\psi_{n,m}(t) = a_0^{-n/2} \psi(a_0^{-n} t - m b_0)$$

for suitable a_0, b_0, e.g., $a_0 = 2$ and $b_0 = 1$. Then let

$$T_{n,m}^{\text{wav}}(f) = \langle f, \psi_{n,m} \rangle.$$

Then for suitable a_0, b_0 (the ones above are OK for the Mexican hat), we can find 'dual' functions $\tilde{\psi}_{n,m}$ such that

$$f = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \langle f, \psi_{n,m} \rangle \tilde{\psi}_{n,m}.$$

Also, there exist $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} |\langle f, \psi_{n,m} \rangle|^2 \leq B\|f\|^2,$$

a condition which makes $(\psi_{n,m})$ a *frame*. Sometimes we can do better (e.g. Haar wavelets, Littlewood–Paley wavelets) and $(\psi_{n,m})$ form an orthonormal basis, i.e., $\tilde{\psi}_{n,m} = \psi_{n,m}$ and $A = B = 1$.

Multi-resolution approximations

Illustrated by a crude but simple example. We work in $L^2(\mathbb{R})$, square-integrable functions on $\mathbb{R}$.
Let $k \in \mathbb{Z}$ and let $V - k$ be the subspace of all functions that are constant on all intervals $\left(\frac{m}{2^k}, \frac{m+1}{2^k}\right)$ for $m \in \mathbb{Z}$. So

$$\ldots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots.$$

Also $\bigcap V_n = \{0\}$ and $\bigcup V_n$ is dense in $L^2(\mathbb{R})$.
We also note that $f(t) \in V_n \iff f(2t) \in V_{n+1}$ (for each n), and $f(t) \in V_0 \iff f(t-m) \in V_0$ (for each m).

In this case we also have that the function $g(t) = \chi_{(0,1)}(t)$ has the property that $(g(t-m))_{m\in\mathbb{Z}}$ is an orthonormal basis for V_0.
From this we will later construct "wavelets" and find a function ψ that gives an orthonormal basis of $L^2(\mathbb{R})$ of the form

$$\psi_{n,m}(t) = 2^{-n/2}\psi(2^n t - m)$$

such that $(\psi_{n,m})_{n\le N, m\in\mathbb{Z}}$ is an o.n.b. for V_N.

Applications

Quantum mechanics (Heisenberg group), Signal processing, Image processing, Numerical analysis.

LECTURE 2

2 L^p spaces and Fourier transforms

Wavelets give a technique for approximating and reconstructing functions, usually on $\mathbb{R}$.

2.1 *Definition*

Let $1 \le p < \infty$. We define the space $L^p(\mathbb{R})$ as the space of (measurable) functions $f : \mathbb{R} \to \mathbb{C}$ satisfying

$$\|f\|_p := \left(\int_{-\infty}^{\infty} |f(t)|^p \, dt\right)^{1/p} < \infty.$$

Similarly we define $L^p(a,b)$ for an interval (a,b) or $[a,b] \subset \mathbb{R}$. In the above we regard two functions f and g as equal if $f(t) = g(t)$ except on a set of measure 0 (i.e., *almost everywhere (a.e.)*). With this convention $L^p(\mathbb{R})$ is a normed vector space, and complete in the metric $d(f,g) = \|f-g\|_p$, hence a *Banach space.*
The space $L^2(\mathbb{R})$ is an inner-product space with the inner product being

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\, dt,$$

linear in the first argument, anti-linear in the second; indeed it is a complete inner-product space, or *Hilbert space.*
We write $L^\infty(\mathbb{R})$ for the space of essentially bounded measurable functions, with norm

$$\|f\|_\infty := \inf\{K \ge 0 : |f(t)| \le K \quad \text{a.e.}\}.$$

Again we regard two functions f, g as equal if $f(t) = g(t)$ a.e. Then $L^\infty(\mathbb{R})$, and more generally $L^\infty(a, b)$, is a Banach space.

Fourier series

We work on the interval $[0, 1]$; other intervals can be treated by a change of variables.
Let

$$e_n(t) = \exp(2\pi i n t),$$

for each $n \in \mathbb{Z}$.

2.2 *Theorem*

$(e_n)_{n\in\mathbb{Z}}$ is an *orthonormal basis* of $L^2(0, 1)$, that is, it satisfies:

1. $\langle e_n, e_m\rangle = \delta_{nm} = \begin{cases} 1 & \text{if} \quad n = m, \\ 0 & \text{for} \quad n \neq m. \end{cases}$
2. Every function $f \in L^2(0, 1)$ has an expansion $f = \sum_{n=-\infty}^{\infty} c_n e_n$, converging in L^2 norm, i.e.,

$$\|f - \sum_{n=-N}^{N} c_n e_n\|_2 \to 0 \qquad \text{as} \quad N \to \infty.$$

Moreover $\|f\|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2$ (*Parseval's formula*), and

$$c_n = \langle f, e_n\rangle = \int_0^1 f(t)e^{-2\pi i n t}\, dt$$

is the nth *Fourier coefficient* of f.

Proof: 1. Easy (exercise).
2. More tricky. Suppose that $c_n = \langle f, e_n\rangle$ and write $f_N = \sum_{n=-N}^{N} c_n e_n$.
We may verify that $\langle f - f_N, f_N\rangle = 0$, and so $\|f\|^2 = \|f - f_N\|^2 + \|f_N\|^2$, so that

$$\|f\|^2 \geq \|f_N\|^2 = \sum_{n=-N}^{N} |c_n|^2.$$

Hence $\sum_{N=-\infty}^{\infty} |c_N|^2 \leq \|f\|^2$, and

$$\|f_N - f_M\|^2 = \sum_{N<|n|\leq M} |c_n|^2, \qquad \text{if} \quad N < M,$$

from which we deduce that (f_N) is a Cauchy sequence, and so converges to g, say. The hard part is to show that $f = g$, i.e., (e_n) is a *complete orthonormal basis*, and then

$$\|f\|^2 = \lim \|f_N\|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

The proof is omitted (see, e.g., N. Young's *An introduction to Hilbert space*).

2.3 *Corollary (Riesz–Fischer theorem)*

A Fourier series $\sum_{n=-\infty}^{\infty} c_n e_n$ converges to a function in $L^2(0,1)$ if and only if $\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$.

2.4 *Remarks*

In fact $f_N \to f$ a.e. when $f \in L^2$, but this is much harder (Carleson, 1962). If f only lies in L^1, then it is possible for its Fourier series to diverge everywhere (Kolmogorov, 1926). Even for continuous periodic functions the Fourier series may diverge at some points, and certainly need not converge uniformly.

We have a norm preserving invertible operator F

$$\begin{array}{rcl} L^2(0,1) & \to & \ell^2(\mathbb{Z}) \\ f & \mapsto & (\hat{f}(n))_n := (\langle f, e_n \rangle)_n. \end{array}$$

Fourier transforms.

Fourier series represent functions on $[0,T]$ in terms of a discrete sequence of nodes $\exp 2\pi i n t/T$ – Fourier transforms represent functions on $\mathbb{R}$ in terms of nodes $\exp 2\pi i \omega t$, where $\omega \in \mathbb{R}$.

2.5 *Proposition*

Suppose that $f \in L^1(\mathbb{R})$. Then the integral

$$\int_{-\infty}^{\infty} f(t) \exp(-2\pi i \omega t)\, dt$$

converges absolutely for all $\omega \in \mathbb{R}$, and defines a function which we write as $\hat{f}(\omega)$, the Fourier transform of f. Moreover $\|\hat{f}\|_\infty \le \|f\|_1$.

LECTURE 3

2.6 *Theorem*

Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then the function $\hat{f}$ exists and satisfies $\|\hat{f}\|_2 = \|f\|_2$; thus the Fourier transform can be extended continuously to the whole of $L^2(\mathbb{R})$, and satisfies the following:

$$\begin{aligned} \|\hat{f}\|_2 &= \|f\|_2 \qquad \text{Plancherel identity;} \\ \hat{\hat{f}}(t) &= f(-t), \qquad \text{that is,} \\ f(t) &= \int_{-\infty}^{\infty} \hat{f}(w) \exp(2\pi i w t)\, dw \qquad \text{a.e.,} \end{aligned}$$

if f is sufficiently well-behaved that $\hat{f} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

(Proof sketched in lectures.)

2.7 *Corollary*

There is a linear operator $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ such that

$$(\mathcal{F}f)(w) = \int_{-\infty}^{\infty} f(t) \exp(-2\pi i w t)\, dt \qquad \text{a.e.}$$

for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Also $\mathcal{F}$ is norm-preserving and surjective (unitary), and

$$\langle f, g \rangle = \langle \mathcal{F}f, \mathcal{F}g \rangle$$

for $f, g \in L^2(\mathbb{R})$.

Proof: Use the *polarization identity*

$$\langle f, g \rangle = \frac{1}{4}(\|f+g\|^2 - \|f-g\|^2) + \frac{1}{4i}(\|f+ig\|^2 - \|f-ig\|^2).$$

If the operator $\mathcal{F}$ preserves norms, then it preserves inner products (i.e., replacing f and g by $\mathcal{F}f$ and $\mathcal{F}g$ doesn't change the RHS, and hence doesn't change the LHS).

$\square$

3 The Haar wavelet

This is the prototype for construction of wavelets, which here will mean orthonormal bases of functions related in a certain particular way. At present, we use discontinuous (step) functions; later we will construct smoother wavelets.

Let V_0 denote the subspace of $L^2(\mathbb{R})$ consisting of all functions f that are constant on every interval of the form $(n, n+1)$, where n is an integer.

3.1 *Lemma*

Every function $f \in V_0$ has the form

$$f(t) = \sum_{k=-\infty}^{\infty} a_k \phi(t-k), \qquad \text{with} \quad \|f\|_2^2 = \sum_{k=-\infty}^{\infty} |a_k|^2,$$

where

$$\phi(t) = \chi_{(0,1)}(t) = \begin{cases} 1 & \text{if } 0 < t < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, a_k is the value of f on $(k, k+1)$. Thus the functions $(\phi_k)_{k \in \mathbb{Z}}$, where $\phi_k(t) = \phi(t-k)$, form an orthonormal basis of V_0.

Proof: Simple verification.

□

3.2 *Theorem*

For $j \in \mathbb{Z}$ let V_j denote the space of functions f constant on all intervals of the form $\left(\frac{k}{2^j}, \frac{k+1}{2^j}\right)$, with $k \in \mathbb{Z}$. Then

$$\ldots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots.$$

Also $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$ and $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R})$.
A function f lies in V_j if and only if the function $t \mapsto f(2^{-j}t)$ lies in V_0, and V_j has an orthonormal basis consisting of the functions

$$\phi_{jk}(t) = 2^{j/2}\phi(2^j t - k), \qquad k \in \mathbb{Z}.$$

Proof: Since every interval $\left(\frac{m}{2^{j+1}}, \frac{m+1}{2^{j+1}}\right)$ is contained in an interval of the form $\left(\frac{k}{2^j}, \frac{k+1}{2^j}\right)$, a function in V_j is automatically in V_{j+1}.

If $f \in \bigcap_{j \in \mathbb{Z}} V_j$, then f is constant on $(0, 2^j)$ for each j. If this value is c then

$$\|f\|^2 \geq \int_0^{2^j} |c|^2 \, dt = 2^j |c|^2$$

for each j, and if $\|f\| < \infty$, this means that $c = 0$ and $f = 0$ a.e. on $(0, \infty)$. Similarly for $(-\infty, 0)$. So $f = 0$ a.e.

Now to show that $\bigcup_{j \in \mathbb{Z}} V_j$ is dense, take $f \in L^2(\mathbb{R})$ and $\epsilon > 0$. By standard results on L^2 functions we have that $\exists N \in \mathbb{N}$ and a continuous function g such that $g(t) = 0$ for $|t| \geq N$ and $\|f - g\|_2 < \epsilon/2$.

Now g is uniformly continuous on $[-N, N]$, so $\exists j > 0$ such that g doesn't vary by more than $\epsilon/\sqrt{8N}$ on any interval $(k/2^j, (k+1)/2^j)$. Hence there is a step function $h \in V_j$ such that

$$\|g - h\|_2 \leq \left(\int_{-N}^{N} \frac{\epsilon^2}{8N} \, dt \right)^{1/2} = \epsilon/2.$$

Thus $\|f - h\|_2 < \epsilon$ by the triangle inequality, and so $\bigcup_{j \in \mathbb{Z}} V_j$ is dense.

If $f \in V_j$ then $f(t)$ is constant for $t \in \left(\frac{k}{2^j}, \frac{k+1}{2^j}\right)$, for all $k \in \mathbb{Z}$.
This happens if and only if $f(2^{-j}t)$ is constant for $t \in (k, k+1)$, for all k, i.e., $f(2^{-j}t) \in V_0$.
Finally, any function in V_j has the form

$$f(2^{-j}t) = \sum_{k=-\infty}^{\infty} a_k \phi(t - k),$$

by Lemma 3.1, and so

$$f(t) = \sum_{k=-\infty}^{\infty} a_k \phi(2^j t - k).$$

Now

$$\langle \phi(2^j t - k), \phi(2^j t - \ell) \rangle = 2^{-j} \delta_{k\ell},$$

and so clearly $(2^{j/2} \phi(2^j t - k))_{k \in \mathbb{Z}}$ is an orthonormal basis of V_j.

□

LECTURE 4

Now (V_j) provides a *multi-resolution approximation* of $L^2(\mathbb{R})$, the 'resolution' being 2^{-j} at the jth level. The best approximation $f_j \in V_j$ to $f \in L^2(\mathbb{R})$ is given by

$$f_j = P_{V_j} f = \sum_{k=-\infty}^{\infty} \langle f, \phi_{jk} \rangle \phi_{jk},$$

where P_{V_j} is the orthogonal projection onto V_j. Also $\|P_{V_j} f - f\| \to 0$ as $j \to \infty$.

We now want to use the orthonormal bases for the V_j to get a grand orthonormal basis for $L^2(\mathbb{R})$.
To do this we introduce the *Haar wavelet* ψ given by

$$\psi(t) = \phi(2t) - \phi(2t - 1) = \chi_{(0,1/2)}(t) - \chi_{(1/2,1)}(t).$$

Clearly $\psi \in V_1 \setminus V_0$.

3.3 *Lemma*

The functions $t \mapsto \psi(t-k)$, $k \in \mathbb{Z}$, form an orthonormal basis of a space W_0 such that V_1 has the orthogonal decomposition $V_1 = V_0 \oplus W_0$.

Proof: Clearly $\langle \psi(t-k), \psi(t-\ell) \rangle = \delta_{kl}$ for all $k, \ell \in \mathbb{Z}$.
Also $\langle \psi(t-k), \phi(t-\ell) \rangle = 0$ for all $k, \ell \in \mathbb{Z}$, since

$$\begin{aligned} \psi(t-k)\phi(t-k) = & \quad \psi(t-k) \qquad \text{for each} \quad k, \quad \text{and} \\ \psi(t-k)\phi(t-\ell) = & \qquad 0 \qquad\quad \text{for} \quad k \neq \ell. \end{aligned}$$

Hence the collection of vectors $S = \{\phi(t-k), \psi(t-\ell) : k, \ell \in \mathbb{Z}\}$ is an orthonormal set. In fact it is an orthonormal basis of V_1: they all lie in V_1 and $\phi(t) + \psi(t) = 2\phi(2t)$, while $\phi(t) - \psi(t) = \phi(2t-1)$, and so

$$\begin{aligned} \phi(2t-2k) &= \frac{1}{2}(\phi(t-k) + \psi(t-k)), \qquad \text{and} \\ \phi(2t-2k-1) &= \frac{1}{2}(\phi(t-k) - \psi(t-k)), \qquad k \in \mathbb{Z}. \end{aligned}$$

Hence the linear span of S includes all the basis vectors of V_1, which is enough to show that it is an orthonormal basis of V_1.

3.4 *Lemma*

For each $j \in \mathbb{Z}$, $V_{j+1} = V_j \oplus W_j$, where W_j has the orthonormal basis $(2^{j/2}\psi(2^j t - k))_{k \in \mathbb{Z}}$.

Proof: A similar calculation using the fact that $f(t) \in V_j \iff f(2^{-j}t) \in V_0$. $\square$

3.5 *Theorem*

$L^2(\mathbb{R}) = \ldots \oplus W_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \ldots$ as an orthogonal direct sum; that is, the functions

$$\psi_{jk}(t) = 2^{j/2}\psi(2^j t - k), \qquad j, k \in \mathbb{Z}$$

form an orthonormal basis of $L^2(\mathbb{R})$.

Proof: Clearly, if $i < j$ and $f \in W_i$, $g \in W_j$, then $f \in V_{i+1} \subset V_j$ and so $\langle f, g \rangle = 0$, since $V_j \perp W_j$.
This implies that if we combine the orthonormal bases from all the W_j we get a sequence that is still orthonormal – but does it give a basis for $L^2(\mathbb{R})$?

LECTURE 5

Any closed subspace V of $L^2(\mathbb{R})$ has an *orthogonal complement* $V^\perp$, i.e., $L^2(\mathbb{R}) = V \oplus V^\perp$, where

$$V^\perp = \{f \in L^2(\mathbb{R}) : \langle f, g\rangle = 0 \quad \text{for all} \quad g \in V\}.$$

We define the orthogonal projection $P_V : L^2(\mathbb{R}) \to V$ by $P_V(v+w) = v$, where $v \in V$ and $w \in V^\perp$. It is the closest point in V to $v+w$.
Now,

$$\begin{aligned} L^2(\mathbb{R}) &= V_{n+1} \oplus V_{n+1}^\perp \\ &= V_n \oplus W_n \oplus V_{n+1}^\perp, \end{aligned}$$

so

$$\begin{aligned} \mathrm{I} &= P_{V_n} + P_{W_n} + P_{V_{n+1}^\perp} \\ &= P_{V_n} + P_{W_n} + (\mathrm{I} - P_{V_{n+1}}), \end{aligned}$$

where I is the identity mapping. Thus

$$P_{W_n} = P_{V_{n+1}} - P_{V_n}.$$

Now $P_{V_n} f \to f$ as $n \to \infty$, for all $f \in L^2(\mathbb{R})$, since $\overline{\bigcup_n V_n} = L^2(\mathbb{R})$. Also, $P_{V_n} f \to 0$ as $n \to -\infty$, since $\bigcap_n V_n = \{0\}$.
Hence

$$\begin{aligned} P_{W_{-N} \oplus \ldots \oplus W_0 \oplus \ldots W_N} f &= \sum_{n=-N}^{N} P_{W_n} f \\ &= \sum_{n=-N}^{N} (P_{V_{n+1}} - P_{V_n}) f \\ &= (P_{V_{N+1}} - P_{V_{-N}}) f \to f - 0 = f \end{aligned}$$

as $N \to \infty$. This implies that

$$L^2(\mathbb{R}) = \ldots \oplus W_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \ldots.$$

$\square$

3.6 *Corollary*

Every $f \in L^2(\mathbb{R})$ has a multi-resolution expansion

$$f = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \langle f, \psi_{jk}\rangle \psi_{jk},$$

converging in $L^2(\mathbb{R})$ norm.

WARNING: Note that $\int_{-\infty}^{\infty} \psi_{jk} = 0$ for all j and k. Hence if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\int_{-\infty}^{\infty} f \neq 0$ (e.g. $e^{-t^2/2}$), then the finite sums in Cor. 3.6 converge in L^2 norm but not in L^1 norm, since

$$\int_{-\infty}^{\infty} (f - \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{jk}\psi_{jk}) = \int_{-\infty}^{\infty} f \text{ (constant) } \neq 0,$$

whatever (c_{jk}) we take.

Later we construct similar wavelet bases in $L^2(\mathbb{R})$ consisting of smoother functions.

4 Band-limited functions and the sampling theorem

A function $f \in L^2(\mathbb{R})$ is said to be *time-limited*, if there is a $T > 0$ such that $f(t) = 0$ a.e. for all $|t| \geq T$. It is said to be *band-limited*, if there is a $b > 0$ such that $\hat{f}(w) = 0$ a.e. for all $|w| > b$.

4.1 *Definition*

Let $b > 0$. The *Paley–Wiener space*, $PW(b)$, is the space of all functions $f \in L^2(\mathbb{R})$ such that $\hat{f}(w) = 0$ for all $|w| > b$. Also written $PW(-b, b)$.

Note. $\hat{f}$ can be regarded as lying in $L^2(-b, b)$, and also

$$\int_{-b}^{b} |\hat{f}(w)|\, dw \leq \left(\int_{-b}^{b} |\hat{f}(w)|^2\, dw\right)^{1/2} \left(\int_{-b}^{b} 1^2\, dw\right)^{1/2} < \infty,$$

by the Cauchy–Schwarz inequality. So $\hat{f} \in L^1(-b, b)$. This actually implies that f is continuous and that $f(t) \to 0$ as $t \to \pm\infty$.

4.2 *Theorem*

If $f \in PW(b)$, then f is the restriction to $\mathbb{R}$ of a function analytic on the whole of $\mathbb{C}$ (which we also call f), satisfying

$$|f(z)| \leq \|\hat{f}\|_1 e^{2\pi b|y|}$$

for $z = x + iy \in \mathbb{C}$.

Proof: Define

$$f(z) = \int_{-b}^{b} \hat{f}(w) e^{2\pi i w z}\, dw,$$

and

$$g(z) = \int_{-b}^{b} (2\pi i w) \hat{f}(w) e^{2\pi i w z}\, dw.$$

Clearly,

$$\begin{aligned} |f(z)| &\leq \int_{-b}^{b} |\hat{f}(w)||e^{2\pi i w z}|\, dw \\ &\leq \|\hat{f}\|_{L^1(-b,b)} \|e^{2\pi i w z}\|_{L^\infty(-b,b)} \leq \|\hat{f}\|_1 e^{2\pi b |y|}. \end{aligned}$$

Thus the integral for f converges absolutely, and similarly for g. Also,

$$\int_0^v g(z)\, dz = \int_0^v \int_{-b}^{b} (2\pi i w) \hat{f}(w) e^{2\pi i w z}\, dw\, dz,$$

and, changing the order of integration and using Fubini's theorem, we get that this is

$$\int_{-b}^{b} \hat{f}(w) \left(e^{2\pi i w v} - 1 \right) dw = f(v) - f(0),$$

and hence $f'(v) = g(v)$, i.e., f is analytic on $\mathbb{C}$.

□

Functions satisfying such an inequality are said to be of *exponential type.* There is a converse, which we don't prove:

if f is analytic on $\mathbb{C}$ such that $f_{|\mathbb{R}}$ lies in $L^2(\mathbb{R})$ and, for some $A > 0$, we have $|f(z)| \leq A e^{2\pi b |y|}$ for all $z = x + iy \in \mathbb{C}$, then $f \in PW(b)$.

4.3 *Corollary*

A function cannot be both time-limited and band-limited, unless it is identically zero.

Proof: If f is analytic on $\mathbb{C}$ and $f(t) = 0$ for all real t with $|t| > T$, then $f \equiv 0$ by the principle of isolated zeroes.

□

LECTURE 6

We're now working towards the sampling theorem for band-limited functions.

4.4 *Lemma*

Let $f \in PW(b)$. Then

$$\hat{f}(w) = \frac{1}{2b}\sum_{n=-\infty}^{\infty} f\left(-\frac{m}{2b}\right)e^{m\pi iw/b} = \frac{1}{2b}\sum_{n=-\infty}^{\infty} f\left(\frac{n}{2b}\right)e^{-n\pi iw/b}, \tag{1}$$

with convergence in the $L^2(-b,b)$ norm (and hence in L^1 also).

Proof: Since $\hat{f} \in L^2(-b,b)$ it has an expansion in terms of the orthonormal basis functions $e_m(w) = \frac{1}{\sqrt{2b}}e^{m\pi iw/b}$ (cf. Theorem 2.2), namely

$$\hat{f} = \sum_{m=-\infty}^{\infty} \langle \hat{f}, e_m\rangle e_m,$$

and

$$\langle \hat{f}, e_m\rangle = \frac{1}{\sqrt{2b}}\int_{-b}^{b} \hat{f}(w)e^{-m\phi iw/b}\,dw = \frac{1}{\sqrt{2b}}f\left(-\frac{m}{2b}\right).$$

The other identity is obtained on writing $n = -m$.

□

4.5 *Theorem (Whittaker–Kotel'nikov–Shannon)*

Let $f \in PW(b)$, and write $t_n = \frac{n}{2b}$ for $n \in \mathbb{Z}$. Then

$$f(t) = \sum_{n=-\infty}^{\infty} f(t_n)\frac{\sin 2\pi b(t-t_n)}{2\pi b(t-t_n)},$$

converging in $L^2(\mathbb{R})$ (i.e., $PW(b)$) norm, as well as uniformly. (As usual, we give $\frac{\sin x}{x}$ the value 1 at $x = 0$.)

Proof: Since the Fourier transform preserves L^2 norms, we can invert (1) to get

$$\begin{aligned} f(t) &= \frac{1}{2b}\sum_{n=-\infty}^{\infty} f\left(\frac{n}{2b}\right)\int_{-b}^{b} e^{-n\pi iw/b}e^{2\pi itw}\,dw \\ &= \frac{1}{2b}\sum_{n=-\infty}^{\infty} f(t_n)\int_{-b}^{b} e^{-2\pi it_nw}e^{2\pi itw}\,dw, \\ &= \frac{1}{2b}\sum_{n=-\infty}^{\infty} f(t_n)\left[\frac{e^{2\pi i(t-t_n)w}}{2\pi i(t-t_n)}\right]_{-b}^{b}, \end{aligned}$$

which gives the result since

$$e^{2\pi i(t-t_n)b} - e^{-2\pi i(t-t_n)b} = 2i \sin 2\pi b(t-t_n).$$

Also, L^1 convergence in (1) implies uniform convergence of the series for f.

□

For $s \in \mathbb{R}$, we shall write k_s for the function given by

$$k_s(t) = \frac{\sin 2\pi b(t-s)}{\pi(t-s)} \qquad \text{for} \quad t \in \mathbb{R},$$

with the value $2b$ at $t = s$. Then

$$\hat{k}_s(w) = e^{-2\pi i s w} \chi_{[-b,b]}(w).$$

4.6 *Theorem*

Let $t_n = \frac{n}{2b}$ for $n \in \mathbb{Z}$. Then the functions $\frac{1}{\sqrt{2b}} k_{t_n}$, for $n \in \mathbb{Z}$, form an orthonormal basis for $PW(b)$. Moreover,

$$f(s) = \langle f, k_s \rangle \qquad \text{for} \quad f \in PW(b).$$

Proof: As in Lemma 4.4 we see that the functions

$$\frac{1}{\sqrt{2b}} e^{-n\pi i w/b} = \frac{1}{\sqrt{2b}} \hat{k}_{t_n}, \qquad n \in \mathbb{Z},$$

form an orthonormal basis for $L^2(-b,b)$. Now use the fact that Fourier transforms preserve norms and inner products. Also,

$$\langle f, k_s \rangle = \langle \hat{f}, \hat{k}_s \rangle = \int_{-b}^{b} \hat{f}(w) e^{2\pi i s w}\, dw = f(s).$$

□

Definition. A *reproducing kernel Hilbert space* is a Hilbert space H consisting of functions on a set S, such that for each $s \in S$ there is a $k_s \in H$ such that $\langle f, k_s \rangle = f(s)$ for all $f \in H$.

Examples.

1. $PW(b)$ with $k_s(t) = \frac{\sin 2\pi b(t-s)}{\pi(t-s)}$.

2. The Hardy space H^2 of all power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in the disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, with

$$\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty,$$

and inner product

$$\langle\sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n\rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}.$$

For $w \in \mathbb{D}$ we take $k_w(z) = \frac{1}{1-\overline{w}z}$, the *Cauchy–Szegö kernel.*

3. The finite-dimensional Hilbert space $\mathbb{C}^n$; let $S = \{1, 2, \ldots, n\}$, so that for $a = (a_1, \ldots, a_n)$ we have

$$a(j) = a_j = \langle a, e_j\rangle$$

for $j \in S$, with e_j the jth standard basis vector (all coordinates 0, except that the jth is 1).

4.7 *Proposition*

Let H be a reproducing kernel Hilbert space with r.k. $k_s \in H$ for $s \in S$. Then for $s, t \in S$ we have $k_s(t) = \overline{k_t(s)}$. Also $k_s(s) > 0$ unless $f(s) = 0$ for all $s \in H$.

Proof: We have

$$k_s(t) = \langle k_s, k_t\rangle = \overline{\langle k_t, k_s\rangle} = \overline{k_t(s)}.$$

Also $k_s(s) = \langle k_s, k_s\rangle = \|k_s\|^2$, and $k_s \neq 0$ unless $f(s) = \langle f, k_s\rangle = 0$ for all $f \in H$. □

Note in $PW(b)$ we have $k_s(t) = k_t(s) = k_0(t - s)$ (real and symmetric).

LECTURE 7

Remark. The sampling theorem gives a way of reconstructing f from values $f(t_n)$, with $t_n = \frac{n}{2b}$, but is not robust – small errors in measurements can easily add up. Oversampling helps avoid this problem: since $PW(b) \subset PW(c)$ for $c > b$, we can reconstruct f from values $f(\frac{n}{2c})$ if we wish.

Construction of a multi-resolution approximation and Littlewood–Paley wavelets

We take $b = \frac{1}{2}$ from now on, and write $V_0 = PW(1/2)$. Then V_0 has an orthonormal basis

$$t \mapsto \phi(t-k) = \frac{\sin \pi(t-k)}{\pi(t-k)}, \qquad (k \in \mathbb{Z}),$$

where $\phi(t) = k_0(t) = \frac{\sin \pi t}{\pi t}$.

4.8 *Theorem*

For $j \in \mathbb{Z}$, let $V_j = PW(2^{j-1})$. We have

$$\ldots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots.$$

Also $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$. Moreover, for each j, we have $f(t) \in V_0 \iff f(2^j t) \in V_j$. Finally, V_j has an orthonormal basis consisting of the functions $2^{j/2}\phi(2^j t - k)$, with $k \in \mathbb{Z}$.

Proof: Clearly, the V_j are increasing with j. If $f \in L^2(\mathbb{R})$, then $\hat{f} \in L^2(\mathbb{R})$, so given $\epsilon > 0$ we can pick j such that

$$\int_{|w|>2^{j-1}} |\hat{f}(w)|^2 \, dw < \epsilon^2.$$

Then define $g \in L^2(\mathbb{R})$ by

$$\hat{g}(w) = \hat{f}(w)\chi_{[-2^{j-1},2^{j-1}]}(w),$$

so that $g \in V_j$ and $\|f - g\| = \|\hat{f} - \hat{g}\| < \epsilon$. So the union of the V_j is dense.

Also if $f \in \bigcap V_j$, then $\hat{f} = 0$ a.e. for $|w| > 2^{j-1}$ for each j; letting $j \to -\infty$ we have $\hat{f} = 0$ a.e., so $f = 0$ a.e.

Now write $f_j(t) = f(2^j t)$, so that

$$\begin{aligned} \hat{f}_j(w) &= \int_{-\infty}^{\infty} f_j(t) e^{-2\pi i w t} \, dt \\ &= \int_{-\infty}^{\infty} f(s) e^{-2\pi i w s/2^j} \, \frac{ds}{2^j}, \qquad \text{with} \quad s = 2^j t, \\ &= \frac{1}{2^j}\hat{f}(w/2^j), \end{aligned}$$

so $f \in PW(b) \iff f_j \in PW(2^j b)$.

Finally, V_j has o.n.b.

$$\frac{1}{\sqrt{2b}} \frac{\sin 2\pi b(t - \frac{k}{2b})}{\pi(t - \frac{k}{2b})} = 2^{j/2}\phi(2^j t - k), \qquad (k \in \mathbb{Z}),$$

where $b = 2^{j-1}$ and $\phi(t) = \frac{\sin \pi t}{\pi t}$. □

Next, as for the Haar wavelet, we seek W_j such that $V_{j+1} = V_j \oplus W_j$ (orthogonal), and a function ψ such that $\psi(t-k)$, for $k \in \mathbb{Z}$ form an o.n.b. for W_0 (cf. Lemma 3.3).

In the transform space, we want $\hat{\psi}$ supported on $[-1,1]$ and orthogonal to all functions $\hat{f}$ in $L^2(-\frac{1}{2},\frac{1}{2})$. So we define

$$\hat{\psi}(w) = \begin{cases} 1 & \text{if } \frac{1}{2} \le |w| \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

This means that

$$\phi(t) = \frac{1}{\pi t}(\sin 2\pi t - \sin \pi t) \qquad \text{(exercise).}$$

4.9 *Lemma*

The functions $\psi_k(t) = \psi(t-k)$ form an orthonormal basis of a space W_0 such that $V_0 \oplus W_0 = V_1$ (an orthogonal direct sum).

Proof:

$$\begin{aligned} \hat{\psi}_k(w) &= \int_{-\infty}^{\infty} \psi(t-k) e^{-2\pi i w t}\, dt \\ &= \int_{-\infty}^{\infty} \psi(x) e^{-2\pi i w(x+k)}\, dx, \qquad \text{with} \quad x = t-k, \\ &= e^{-2\pi i w k}\hat{\psi}(w). \end{aligned}$$

Now

$$\begin{aligned} \langle \hat{\psi}_k, \hat{\psi}_\ell \rangle &= \left(\int_{-1}^{-1/2} + \int_{1/2}^{1} \right) e^{-2\pi i w k} e^{2\pi i w \ell}\, dw \\ &= \begin{cases} 1 & \text{if } k = \ell, \\ 0 & \text{if } k \ne \ell. \end{cases} \end{aligned}$$

Hence the $(\hat{\psi}_k)$, and so the (ψ_k), are orthonormal.
Also, $\langle \phi_k, \psi_\ell \rangle = 0$ for all k and ℓ since $\langle \hat{\phi}_k, \hat{\psi}_\ell \rangle = 0$, where as usual $\phi_k(t) = \phi(t-k)$.

LECTURE 8

Finally, since

$$L^2(-1,1) = L^2(-\frac{1}{2},\frac{1}{2}) \oplus L^2((-1,-\frac{1}{2}) \cup (\frac{1}{2},1)),$$

it is sufficient to verify that $(\hat{\psi}_k)$ is an o.n.b. for $L^2((-1,-\frac{1}{2}) \cup (\frac{1}{2},1))$ (we have already seen that they are orthonormal).
Note that $(e_k(w))_{k\in\mathbb{Z}} := (e^{2\pi i w k})_{k\in\mathbb{Z}}$ is an o.n.b. for $L^2(0,1)$.

Given $\gamma \in L^2((-1,-\frac{1}{2}) \cup (\frac{1}{2},1))$, define $\widetilde{\gamma} \in L^2(0,1)$ by

$$\widetilde{\gamma}(w) = \begin{cases} \gamma(w-1) & \text{if } 0 < w < \frac{1}{2}, \\ \gamma(w) & \text{if } \frac{1}{2} < w < 1. \end{cases}$$

Given $\epsilon > 0$ there is an N and $a_{-N}, \ldots, a_N \in \mathbb{C}$ such that

$$\int_0^1 \left| \widetilde{\gamma}(w) - \sum_{k=-N}^{N} a_k e^{-2\pi i w k} \right|^2 dw < \epsilon^2.$$

This equals

$$\left(\int_{-1}^{-1/2} + \int_{1/2}^{1} \right) \left| \gamma(w) - \sum_{k=-N}^{N} a_k e^{-2\pi i w k} \right|^2 dw,$$

and so $\|\gamma - \sum_{-N}^{N} a_k \hat{\psi}_k\|_2 < \epsilon$, as required. □

4.10 *Theorem*

The functions $\psi_{jk}(t) = 2^{j/2}\psi(2^j t - k), \quad (j,k \in \mathbb{Z})$, form an orthonormal basis for $L^2(\mathbb{R})$, the *Littlewood–Paley wavelets.*

Proof: We have done most of this. Note that

$$L^2(\mathbb{R}) = \ldots \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus \ldots$$

by Theorem 4.8, where W_j has orthonormal basis $(2^{j/2}\psi(2^j t - k))$, for $k \in \mathbb{Z}$: just repeat the proof of Theorem 3.5. □

5 Riesz bases and frames

5.1 *Definition*

A sequence (ϕ_k) is a Hilbert space H is called a *Riesz basis* for H if
(i) H is the closed linear span of the ϕ_k;
(ii) there are constants $A, B > 0$ such that

$$A\left(\sum |a_k|^2\right) \le \left\|\sum a_k \phi_k\right\|^2 \le B\left(\sum |a_k|^2\right)$$

for all sequences $(a_k) \in \ell^2$. A similar definition can be made in finite-dimensional spaces, taking finite sums only, but we are mostly concerned with the infinite-dimensional case.

Examples.
1. Any orthonormal basis (ϕ_k), and then we can take $A = B = 1$.
2. In $\mathbb{C}^n$, any basis $\{\phi_1, \ldots, \phi_n\}$.

5.2 *Theorem*

The sequence (ϕ_k) is a Riesz basis for H if and only if there is a bounded invertible linear operator $U : H \to H$ such that $(U\phi_k)$ is an orthonormal basis for H.

Proof: If $(U\phi_k)$ is an orthonormal basis, then

$$\begin{aligned} \sum a_k\phi_k &= U^{-1}\left(\sum a_k U(\phi_k)\right), \qquad \text{and so} \\ \left\|\sum a_k\phi_k\right\| &\le \|U^{-1}\| \left\|\sum a_k U(\phi_k)\right\| = \|U^{-1}\| \left(\sum |a_k|^2\right)^{1/2}. \end{aligned}$$

Also,

$$\begin{aligned} \left\|\sum a_k(U\phi_k)\right\| &\le \|U\| \left\|\sum a_k\phi_k\right\|, \qquad \text{and so} \\ \left\|\sum a_k\phi_k\right\| &\le \|U\|^{-1} \left(\sum |a_k|^2\right)^{1/2}. \end{aligned}$$

Conversely, if (ϕ_k) is a Riesz basis, then let (ψ_k) be any o.n.b. for H. Define $U : H \to H$ by

$$U \sum a_k\phi_k = \sum a_k\psi_k$$

for finite sums. Then

$$\begin{aligned} \left\|\sum a_k\psi_k\right\| &\le A^{-1} \left\|\sum a_k\phi_k\right\|, \qquad \text{and} \\ \left\|\sum a_k\phi_k\right\| &\le B \left\|\sum a_k\psi_k\right\|, \end{aligned}$$

so that U and U^{-1} are bounded on finite sums. Since the closed linear span of the (ϕ_k) and (ψ_k) both equal H, we can extend U to an isomorphism from H to H. □

5.3 *Definition*

A sequence (ϕ_k) in a Hilbert space H is called a *frame*, if there are constants $A, B > 0$ such that

$$A\|\phi\|^2 \le \sum |\langle \phi, \phi_k\rangle|^2 \le B\|\phi\|^2 \qquad \forall \phi \in H. \tag{2}$$

Examples.
1. Orthonormal bases, with $A = B = 1$.
2. Riesz bases, since

$$\sum |\langle \phi, \phi_k\rangle|^2 = \sum |\langle \phi, U^{-1}U\phi_k\rangle|^2 = \sum |\langle (U^{-1})^*\phi, \psi_k\rangle|^2 = \|(U^{-1})^*\phi\|^2,$$

which can be bounded above by $\|(U^{-1})^*\|^2 \, \|\phi\|^2$, and below by $\|U^*\|^{-2} \, \|\phi\|^2$.

However, frames needn't be linearly independent. If (ϕ_k) is a frame, then adding in the zero vector, or repeating each vector a variable number $\leq N$ times, does not stop it being a frame. Indeed, any finite set spanning $\mathbb{C}^n$ is a frame in $\mathbb{C}^n$ (exercise).

We call the numbers $\langle \phi, \phi_k \rangle$ the *frame coefficients* of ϕ and would like to use them to reconstruct ϕ.

Example 1. (Daubechies–Morlet) Take

$$\psi(x) = \frac{2}{\sqrt{3}} \pi^{-1/4} (1 - x^2) e^{-x^2/2},$$

the normalized Mexican hat function. Then the functions

$$\psi_{jk}(x) = 2^{j/2} \psi(2^j x - k), \qquad (j, k \in \mathbb{Z}),$$

form a frame for $L^2(\mathbb{R})$ with $B/A \approx 1.116$. Used in vision analysis, and processing seismic data.

Example 2. (Landau) In $PW(b)$ any set of reproducing kernels k_{t_j}, with $\delta < t_{j+1} - t_j < \frac{1}{2b}$ ("uniformly discrete") forms a frame.

A frame is *tight*, if $A = B$, e.g., any orthonormal basis. There are other examples; e.g., in $\mathbb{C}^2$, the set

$$\{\phi_1, \phi_2, \phi_3\} = \left\{ (1,0), (-\frac{1}{2}, \frac{\sqrt{3}}{2}), (-\frac{1}{2}, -\frac{\sqrt{3}}{2}) \right\}$$

satisfies

$$\sum_{k=1}^{3} |\langle \phi, \phi_k \rangle|^2 = \frac{3}{2} \|\phi\|^2$$

for all $\phi \in \mathbb{C}^2$.

LECTURE 9

From now on we are looking at countably infinite frames in a separable Hilbert space.

5.4 *Theorem*

A sequence (ϕ_k) in a Hilbert space H is a frame for H if and only if there is a *surjective* bounded linear operator $T : \ell^2 \to H$ such that

$$T((a_k)) = \sum a_k \phi_k \qquad \text{for all} \quad (a_k) \in \ell^2.$$

Proof: Suppose that T exists, and consider $T^* : H \to \ell^2$. Since

$$\langle T^*\psi, (a_k)\rangle = \langle \psi, T((a_k))\rangle = \langle \psi, \sum a_k\phi_k\rangle = \sum \overline{a_k}\langle \psi, \phi_k\rangle,$$

we have $(T^*\psi)_k = \langle \psi, \phi_k\rangle$.
Hence,

$$\sum |\langle \psi, \phi_k\rangle|^2 = \|T^*\psi\|^2 \leq \|T^*\|^2\|\psi\|^2,$$

i.e., the upper bound in (2).
Also, T is onto, so by Banach's open mapping theorem $\exists C > 0$ such that for each $\psi \in H$ there is an (a_k) such that $T((a_k)) = \psi$, and $\|(a_k)\| \leq C\|\psi\|$.
Hence,

$$\langle T^*\psi, (a_k)\rangle = \langle \psi, T((a_k))\rangle = \|\psi\|^2 \geq \|(a_k)\|\,\|\psi\|/C,$$

so that $\|T^*\psi\| \geq \|\psi\|/C$, and hence

$$\sum |\langle \psi, \phi_k\rangle|^2 \geq \|\psi\|^2/C^2.$$

Thus (ϕ_k) is a frame.

Conversely, if we have a frame (ϕ_k), let $U : H \to \ell^2$ be defined by $(U\psi)_k = \langle \psi, \phi_k\rangle$, and take $T : \ell^2 \to H$ to equal U^*. Now the image of T is H (check) and

$$\langle T((a_k)), \psi\rangle = \langle (a_k), U\psi\rangle = \sum a_k\langle \phi_k, \psi\rangle,$$

so $T((a_k)) = \sum a_k\phi_k$. □

Recovery. We want a formula recovering ψ from its frame coefficients $\langle \psi, \phi_k\rangle$. Consider the operator $S = TT^* : H \to H$. Note that $S^* = (TT^*)^* = T^{**}T^* = S$, i.e., S is self-adjoint.
Also

$$S\psi = T(\langle \psi, \phi_k\rangle) = \sum \langle \psi, \phi_k\rangle\phi_k,$$

and

$$\langle S\psi, \psi\rangle = \langle TT^*\psi, \psi\rangle = \langle T^*\psi, T^*\psi\rangle = \sum |\langle \psi, \phi_k\rangle|^2 \geq A\|\psi\|^2,$$

so that $\|S\psi\| \geq A\|\psi\|$, and S is invertible.
Now we have

$$\psi = S^{-1}S\psi = S^{-1}\sum \langle \psi, \phi_k\rangle\phi_k = \sum \langle \psi, \phi_k\rangle\widetilde{\phi}_k,$$

where $\widetilde{\phi}_k = S^{-1}\phi_k$.

5.5 *Theorem*

The sequence $(\widetilde{\phi}_k) = (S^{-1}\phi_k)$ forms a frame in H, called the *dual frame* to (ϕ_k). Moreover, the dual frame to $(\widetilde{\phi}_k)$ is (ϕ_k) again.

Proof: We have to calculate

$$\begin{aligned}\sum |\langle \psi, \widetilde{\phi}_k\rangle|^2 &= \sum |\langle S^{-1}\psi, \phi_k\rangle|^2 = \|T^*S^{-1}\psi\|^2 = \langle T^*S^{-1}\psi, T^*S^{-1}\psi\rangle \\ &= \langle S^{-1}\psi, TT^*S^{-1}\psi\rangle = \langle S^{-1}\psi, \psi\rangle.\end{aligned}$$

Since S is self-adjoint and invertible, we can say that $\exists A', B' > 0$ such that

$$A'\|\psi\|^2 \leq |\langle S^{-1}\psi, \psi\rangle| \leq B'\|\psi\|^2,$$

which implies that $(\widetilde{\phi}_k)$ forms a frame.
So why is $(\widetilde{\widetilde{\phi}}_k)$ the same as $(\widetilde{\phi}_k)$? Well, if $\widetilde{T}$ is the frame operator corresponding to $(\widetilde{\phi}_k)$, then we have

$$(\widetilde{T}^*\psi)_k = \langle \psi, \widetilde{\phi}_k\rangle = \langle \psi, S^{-1}\phi_k\rangle = \langle S^{-1}\psi, \phi_k\rangle,$$

so that $\widetilde{T}^* = T^*S^{-1}$, i.e., $\widetilde{T} = S^{-1}T$ and

$$\widetilde{S} = \widetilde{T}\widetilde{T}^* = S^{-1}TT^*S^{-1} = S^{-1}SS^{-1} = S^{-1}.$$

Thus $\widetilde{\widetilde{\phi}}_k = \widetilde{S}^{-1}\widetilde{\phi}_k = S\widetilde{\phi}_k = \phi_k$. □

5.6 *Theorem*

We have

$$\left\| \mathrm{I} - \frac{2}{B+A}S \right\| \leq \frac{B-A}{B+A} < 1, \tag{3}$$

and so

$$S^{-1} = \frac{2}{B+A}\sum_{n=0}^{\infty}\left(I - \frac{2}{B+A}S\right)^n. \tag{4}$$

Proof: Let C denote $(A+B)/2$. Then

$$\begin{aligned}\langle (S - C\mathrm{I})\psi, \psi\rangle &\leq \langle S\psi, \psi\rangle - C\|\psi\|^2 \\ &= \langle T\psi, T\psi\rangle - C\|\psi\|^2 \\ &\leq (B-C)\|\psi\|^2 = \frac{B-A}{2}\|\psi\|^2.\end{aligned}$$

But $S - C\mathrm{I}$ is self-adjoint, and hence $\|S - C\mathrm{I}\| \leq \frac{B-A}{2}$, or, equivalently (3).
Now write $U = \mathrm{I} - S/C$, so that $\|U\| < 1$, and $(\mathrm{I} - U)^{-1} = \sum_{n=0}^{\infty} U^n$, which gives (4). □

Hence, if $\left\|\mathrm{I} - \frac{2}{B+A} S\right\|$ is small, we have a rapidly converging series for S^{-1}. Then we can use this, since

$$\psi = \sum \langle \psi, \phi_k \rangle (S^{-1}\phi_k) = S^{-1} \sum \langle \psi, \phi_k \rangle \phi_k,$$

giving an inversion formula to recover ψ from its frame coefficients. Here's an estimate for the error.

LECTURE 10

5.7 *Corollary*

Let $R = \mathrm{I} - \frac{2}{B+A} S$, and $\widetilde{\phi}_k^{(N)} = \frac{2}{B+A} \sum_{n=0}^{N} R^n \phi_k$ for $k = 1, 2, \ldots$. Then

$$\left\|\psi - \sum \langle \psi, \phi_k \rangle \widetilde{\phi}_k^{(N)}\right\| \le \left(\frac{B-A}{B+A}\right)^{N+1} \|\psi\|.$$

Proof:

$$\widetilde{\phi}_k = \frac{2}{B+A} \sum_{n=0}^{\infty} R^n \phi_k = \widetilde{\phi}_k^{(N)} + R^{N+1} \widetilde{\phi}_k.$$

Hence

$$\left\|\psi - \sum \langle \psi, \phi_k \rangle \widetilde{\phi}_k^{(N)}\right\| = \left\|\sum \langle \psi, \phi_k \rangle R^{N+1} \widetilde{\phi}_k\right\| = \|R^{N+1}\psi\|.$$

But $\|R\| \le (B-A)/(B+A)$, by Theorem 5.6, and hence the result.

6 Windowed Fourier transforms

Let $g(t)$ be a well-localised function. We'll assume that $g \in L^2(\mathbb{R})$, and w.l.o.g. $\|g\|_2 = 1$. Usually g is real and non-negative. "Well-localised" means that $\int_{|t|>T} |g|^2$ is small for some small $T > 0$.

Examples. (i) $g(t) = \pi^{-1/4} \exp(-t^2/2)$.

(ii) $g(t) = \begin{cases} \sqrt{\frac{3}{2}}(1-|t|) & \text{if } |t| \le 1, \\ 0 & \text{otherwise.} \end{cases}$

Let $f \in L^2(\mathbb{R})$, and write $f_t(u) = f(u)\overline{g(u-t)}$. Then we define

$$\widetilde{f}(w,t) = \hat{f}_t(w) = \int_{-\infty}^{\infty} e^{-2\pi i w u} f(u) \overline{g(u-t)} \, du.$$

This describes the behaviour of f near time t at frequency w.

6.1 *Theorem (Time-frequency duality)*

$\widetilde{f}(w,t) = e^{-2\pi iwt}\widetilde{\hat{f}}(-t,w)$, where the windowed transform of $\hat{f}$ is formed using the window $\hat{g}$.

Proof: We have

$$\widetilde{f}(w,t) = \langle f, g_{w,t}\rangle = \langle \hat{f}, \hat{g}_{w,t}\rangle,$$

where $g_{w,t}(u) = e^{2\pi iwu}g(u-t)$.
Now,

$$\begin{aligned} \hat{g}_{w,t}(\nu) &= \int_{-\infty}^{\infty} e^{-2\pi i\nu u}e^{2\pi iwu}g(u-t)\,du \\ &= \int_{-\infty}^{\infty} e^{-2\pi i\nu(x+t)}e^{2\pi iw(x+t)}g(x)\,dx, \qquad \text{with} \quad x = u-t, \\ &= e^{-2\pi i\nu t}e^{2\pi iwt}\hat{g}(\nu - w). \end{aligned}$$

So

$$\widetilde{f}(w,t) = e^{-2\pi iwt}\int_{-\infty}^{\infty} \hat{f}(\nu)e^{2\pi i\nu t}\overline{\hat{g}(\nu-w)}\,d\nu,$$

as required. □

Hence if g and $\hat{g}$ are both well-localised (e.g. $g(t) = 2^{1/4}e^{-\pi t^2}$ and $\hat{g}(w) = 2^{1/4}e^{-\pi w^2}$), then the windowed Fourier transform gives localised behaviour of both f and $\hat{f}$ simultaneously. But there is a limitation as to how well we can do.

6.2 *Theorem (Heisenberg's inequality)*

For every $f \in L^2(\mathbb{R})$ we have

$$\left(\int_{-\infty}^{\infty} t^2|f(t)|^2\,dt\right)^{1/2}\left(\int_{-\infty}^{\infty} w^2|\hat{f}(w)|^2\,dw\right)^{1/2} \geq \frac{1}{4\pi}\|f\|_2^2.$$

Comments. In probability and statistics, if f has L^2 norm equal to 1, then $|f|^2$ and $|\hat{f}|^2$ are probability density functions whose standard deviations satisfy

$$\sigma_f\sigma_{\hat{f}} \geq \frac{1}{\sqrt{4\pi}}.$$

In quantum mechanics the uncertainties in measured position and momentum satisfy

$$(\Delta x)(\Delta p) \geq \hbar/2,$$

where $\hbar$ is Planck's constant.

We get equality if f (and hence $\hat{f}$) is a Gaussian function.

More generally, for all real t_0 and w_0 we have, for $g \in L^2(\mathbb{R})$, that

$$\left(\int_{-\infty}^{\infty} (t-t_0)^2 |g(t)|^2 \, dt\right)^{1/2} \left(\int_{-\infty}^{\infty} (w-w_0)^2 |\hat{g}(w)|^2 \, dw\right)^{1/2} \geq \frac{1}{4\pi} \|g\|_2^2.$$

This can be deduced from the theorem by putting

$$f(t) \;=\; g(t+t_0) e^{-2\pi i w_0 t},$$

and

$$\hat{f}(w) \;=\; g(w+w_0) e^{2\pi i w t_0} e^{2\pi i w_0 t_0}.$$

LECTURE 11

Proof of Theorem 6.2: Without loss of generality we may assume that f is C^1 and that $|f(t)| = O(1/t^2)$ at $\pm\infty$ and $|\hat{f}(w)| = O(1/w^2)$ at $\pm\infty$. This is a dense set of functions, and the general case follows by approximation.
Now

$$\begin{aligned} \|f\|_2^2 &= \int_{-\infty}^{\infty} 1.f\overline{f} \\ &= \Big[t f(t) \overline{f(t)} \Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} t[f'(t)\overline{f(t)} + f(t)\overline{f'(t)}] \, dt, \end{aligned}$$

on integrating by parts. The first term vanishes, and so

$$\begin{aligned} \|f\|_2^2 &\leq 2\int_{-\infty}^{\infty} |tf(t)| \, |f'(t)| \, dt \\ &\leq 2\|tf(t)\|_2 \|f'\|_2, \end{aligned}$$

using Cauchy–Schwarz. Now

$$\int_{-\infty}^{\infty} f'(t) e^{-2\pi i w t} \, dt = \left[f(t) e^{-2\pi i w t} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} f(t) 2\pi i w e^{-2\pi i w t} \, dt,$$

again integrating by parts, so that $\|f'\|_2 = \|\mathcal{F}f'\|_2 = 2\pi \|w\hat{f}(w)\|_2$.
Hence

$$\|f\|_2^2 \leq 4\pi \|tf(t)\|_2 \|w\hat{f}(w)\|_2.$$

$\square$

The inversion formula.
We write C for $\|g\|_2$.

6.3 *Theorem*

$$f(u) = \frac{1}{C^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widetilde{f}(w,t) g(u-t) e^{2\pi i w u} \, dw \, dt,$$

assuming that the integrals converge absolutely.

Proof: Recall that

$$f_t(u) = f(u)\overline{g(u-t)} = \int_{-\infty}^{\infty} \widetilde{f}(w,t) e^{2\pi i w u} \, dw.$$

Hence

$$\int_{-\infty}^{\infty} f(u) |g(u-t)|^2 \, dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widetilde{f}(w,t) g(u-t) e^{2\pi i w u} \, dw \, dt. \tag{5}$$

Now the LHS of (5) is $C^2 f(u)$, and hence the result.

□

Remark 1. By Plancherel's formula,

$$\begin{aligned} \|\widetilde{f}(w,t)\|_{L^2(\mathbb{R}^2)} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f_t(u)|^2 \, du \, dt \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(u)|^2 |g(u-t)|^2 \, du \, dt = \|f\|_2^2 \|g\|_2^2, \end{aligned}$$

so the windowed Fourier transform gives a linear operator

$$W : L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$$

(which is **not** surjective), satisfying $\|Wf\|_2 = C\|f\|_2$.

Remark 2. The W.F.T. can be discretized – compute $\widetilde{f}(mw_0, nt_0)$ for $m, n \in \mathbb{Z}$, where w_0 and t_0 are fixed positive real numbers. Then

$$\widetilde{f}(mw_0, nt_0) = \langle f, g_{mn} \rangle,$$

where $g_{mn}(u) = e^{2\pi i m w_0 u} g(u - nt_0)$, so by the results of Section 5, we get a stable inversion formula if and only if the (g_{mn}) form a frame.

It turns out to be necessary, but not sufficient in general, that $w_0 t_0 < 1$ (so sampling rapidly enough); for the Gaussian window this is also sufficient (Lyubarskii, 1989).

7 The wavelet transform

We saw in Sections 3 and 4 that for certain functions $\psi \in L^2(\mathbb{R})$ the family $\psi_{jk}(t) = 2^{j/2}\psi(2^j t - k)$ with $j, k \in \mathbb{Z}$ forms an o.n.b. of $L^2(\mathbb{R})$. Here j controls scaling (frequency) and k controls position (time). We now generalize this, starting with a continuous version of it.

7.1 *Definitions*

Let $\psi \in L^2(\mathbb{R})$ satisfy the *admissibility condition*

$$C_\psi := \int_{-\infty}^{\infty} \frac{|\hat{\psi}(w)|^2}{|w|}\, dw < \infty. \tag{6}$$

We write

$$\psi^{x,y}(t) = |x|^{-1/2} \psi\left(\frac{t-y}{x}\right),$$

for $x, y \in \mathbb{R}$ with $x \neq 0$.
For example, take $\psi(t)$ to be the Haar wavelet, so that

$$\hat{\psi}(w) = \frac{(1 - e^{-i\pi w t})^2}{2\pi i w} = O(w) \qquad \text{as} \quad w \to 0.$$

Then $\psi^{x,y}$ (for $x > 0$ at least) takes values $\pm 1/\sqrt{x}$ on the interval between y and $x + y$.

Now define

$$\begin{aligned} \overset{\circ}{f}(x,y) &= \langle f, \psi^{x,y} \rangle \\ &= \int_{-\infty}^{\infty} f(t) |x|^{-1/2} \overline{\psi\left(\frac{t-y}{x}\right)}, \end{aligned}$$

the *Wavelet Transform* of f.
Note that if $\hat{\psi}$ is continuous, then by (6) it must satisfy $\hat{\psi}(0) = 0$, i.e.,

$$\int_{-\infty}^{\infty} \psi(t)\, dt = 0.$$

So ψ cannot be a positive function like the Gaussian, but it could be the Mexican hat function.

LECTURE 12

7.2 *Theorem*

For all $f, g \in L^2(\mathbb{R})$, we have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overset{\circ}{f}(x,y) \overline{\overset{\circ}{g}(x,y)}\, \frac{dx\, dy}{x^2} = C_\psi \langle f, g \rangle.$$

Proof: (Fuller calculations given in lectures.) The LHS above equals

$$A := \int_{-\infty}^{\infty} \langle f, \psi^{x,y} \rangle \, \langle \psi^{x,y}, g \rangle \, \frac{dx\, dy}{x^2}.$$

Now
$$\widehat{\psi^{x,y}}(w) = |x|^{1/2} e^{-2\pi i w y}\widehat{\psi}(xw),$$
so
$$A = \int_{-\infty}^{\infty} \langle \widehat{f}, \widehat{\psi^{x,y}}\rangle \, \langle \widehat{\psi^{x,y}}, \widehat{g}\rangle \, \frac{dx\,dy}{|x|} = \int_{-\infty}^{\infty} \langle \check{F}_x, \check{G}_x\rangle \, \frac{dx}{|x|},$$
where $\check{F}_x(y)$ denotes the inverse Fourier transform of $F_x(w) = \widehat{f}(w)\overline{\widehat{\psi}(xw)}$. Thus
$$\begin{aligned} A &= \int_{-\infty}^{\infty} \langle F_x, G_x\rangle \, \frac{dx\,dy}{|x|} = \int_{-\infty}^{\infty} \widehat{f}(w)\overline{\widehat{\psi}(xw)}\,\overline{\widehat{g}(w)}\widehat{\psi}(xw) \, \frac{dx\,dw}{|w|} \\ &= C_\psi \langle \widehat{f}, \widehat{g}\rangle = C_\psi \langle f, g\rangle. \end{aligned}$$

□

Remark. We can interpret this as saying that
$$f = C_\psi^{-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overset{\circ}{f}\,(x,y)\frac{\psi^{x,y}}{x^2}\,dx\,dy,$$
where the integral is taken in a weak sense (i.e., take the inner product of both sides with any $g \in L^2(\mathbb{R})$.

In fact, it converges in norm as a vector-valued integral, too, in the sense that
$$\left\| f - C_\psi^{-1} \int_{|y|\le c}\int_{a\le |x|\le b} \overset{\circ}{f}\,(x,y)\frac{\psi^{x,y}}{x^2}\,dx\,dy \right\| \to 0$$
as $a \to 0$, and $b, c \to \infty$. Proof omitted.

7.3 *Corollary*

For $f \in L^2(\mathbb{R})$,
$$\|f\|_2^2 = C_\psi^{-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\,\overset{\circ}{f}\,(x,y)|^2 \, \frac{dx\,dy}{x^2},$$
i.e., the wavelet transform maps $L^2(\mathbb{R})$ isometrically into a closed subspace H of $L^2(\mathbb{R}^2, d\mu)$, where the measure on $\mathbb{R}^2$ is given by $d\mu = \frac{dx\,dy}{C_\psi x^2}$.

Further, H is a reproducing kernel Hilbert space with kernel
$$k_{(s,t)}(x,y) = \langle \psi^{s,t}, \psi^{x,y}\rangle.$$

Proof: The first part follows directly from Theorem 7.2, on putting $f = g$. For the reproducing kernel properties, note that
$$\begin{aligned} \overset{\circ}{f}\,(s,t) &= \langle f, \psi^{s,t}\rangle = \langle \overset{\circ}{f}, \overset{\circ}{\psi}{}^{\,s,t}\rangle_{L^2(\mathbb{R}^2,d\mu)} \\ &= C_\psi^{-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overset{\circ}{f}\,(x,y)\overline{\overset{\circ}{\psi}{}^{\,s,t}(x,y)} \, \frac{dx\,dy}{x^2} \\ &= \langle \overset{\circ}{f}, k_{(s,t)}\rangle_{L^2(\mathbb{R}^2,d\mu)}, \end{aligned}$$

where

$$k_{(s,t)}(x,y) = \overset{\circ}{\psi}{}^{s,t}(x,y) = \langle \psi^{s,t}, \psi^{x,y} \rangle.$$

$\square$

Remark. This transform has found applications in the analysis of radar and sonar data; there are also connections with quantum mechanics and group representations.
It is considered superior to the windowed Fourier transform (WFT) because the WFT cannot deal efficiently with several scalings/resolutions at once (the "window" is a fixed size). So the wavelet transform tends to be more efficient.

Discretising the wavelet transform.
For simplicity, assume ψ is an even function, so that $\psi^{x,y} = \psi^{-x,y}$ and so we only need to consider $x > 0$.

Fix $a > 1$ and $b > 0$. For $j, k \in \mathbb{Z}$ set $x = a^{-j}$ and $y = ka^{-j}b$. Write

$$\psi_{j,k}(t) = \psi^{x,y}(t) = a^{j/2}\psi(a^j t - kb),$$

and consider the products $\langle f, \psi_{j,k} \rangle$.
By the results of Section 5 we know that we can recover f in a robust manner if and only if the $(\psi_{j,k})$ form a frame.
It is necessary for ψ to be *admissible*, i.e., (6) holds. If this is true and ψ and $\widehat{\psi}$ are "well-behaved" (a complicated condition to do with decay at infinity), then for a, b sufficiently small we do get a frame. Often, indeed, $a = 2$ and $b = 1$ will work (Daubechies, very technical). To go further, i.e., to get orthonormal bases, would be nicer still, and we do this in the next section.

LECTURE 13

8 Multi-resolution analysis (MRA)

By a *multi-resolution analysis*, we mean a chain of subspaces of $L^2(\mathbb{R})$,

$$\dots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \dots.$$

such that $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R})$ and $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$.
We also ask that for $j, k \in \mathbb{Z}$

- $f \in V_0 \iff$ the function $t \mapsto f(2^j t)$ lies in V_j;
- $f \in V_0 \iff$ the function $t \mapsto f(t-k)$ lies in V_0; and

- there is a function $\phi \in V_0$ such that $(\phi_k)_{k\in\mathbb{Z}}$ form an o.n.b. of V_0, where $\phi_k(t) = \phi(t-k)$.

Then for $j \in \mathbb{Z}$ the space V_j has an orthonormal basis consisting of the functions

$$\phi_{jk}(t) = 2^{j/2}\phi(2^j t - k), \qquad k \in \mathbb{Z}.$$

Writing P_j for the orthogonal projection onto V_j, we have that $P_j f \to f$ as $j \to \infty$, and $P_j f \to 0$ as $j \to -\infty$.

We are seeking a "wavelet" $\psi \in V_1$ such that

$$\psi_{jk}(t) = 2^{j/2}\psi(2^j t - k), \qquad j, k \in \mathbb{Z}. \tag{7}$$

form an o.n.b. of $L^2(\mathbb{R})$.

As in Sections 3 and 4, we write $V_j \oplus W_j = V_{j+1}$, an orthogonal direct sum, i.e.,

$$\begin{aligned} W_j &= \{f \in V_{j+1} : \langle f, g\rangle = 0 \quad \forall g \in V_j\} \\ &= \{f \in V_{j+1} : \langle f, \phi_{jk}\rangle = 0 \quad \forall k \in \mathbb{Z}\}, \end{aligned}$$

since $(\phi_{jk})_{k\in\mathbb{Z}}$ is an o.n.b. of V_j.

By the same arguments as before, once we have a MRA we can deduce that

$$L^2(\mathbb{R}) = \ldots \oplus W_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \ldots,$$

and also $f \in W_0$ if and only if $t \mapsto f(2^j t)$ lies in W_j.
Finally, we seek a $\psi \in W_0$ such that $(\psi_k)_{k\in\mathbb{Z}}$ form an o.n.b. for W_0, where $\psi_k(t) = \psi(t-k)$. Then we can conclude that the family $(\psi_{jk})_{j,k\in\mathbb{Z}}$, as in (7), forms an o.n.b. for $L^2(\mathbb{R})$.

8.1 *Example (Meyer wavelets)*

Fix an integer $d \geq 0$ and let $\theta : [0,1] \to \mathbb{R}$ be a non-negative C^d function satisfying

$$\theta(u) = \begin{cases} 0 & \text{if} \quad u \in [0, \frac{1}{3}], \\ \pi/2 & \text{if} \quad u \in [\frac{2}{3}, 1], \end{cases}$$

and

$$\theta(u) + \theta(1-u) = \frac{\pi}{2}$$

for all $u \in [0,1]$. The way we shall do this is to let P_k be an odd polynomial of degree $2d+1$ satisfying $P_k(1) = 1$ and $P_k^{(m)} = 0$ for $m = 1, 2, \ldots, d$ (no condition here if $d = 0$).
Thus we can take $P_0(x) = x$, and $P_1(x) = \frac{3}{2}x - \frac{1}{2}x^3$.

Define

$$\theta(u) = \frac{\pi}{4}(1 + P_k(6u-3)) \qquad \text{for} \quad \theta \in \left[\frac{1}{3}, \frac{2}{3}\right].$$

Finally we define the "father wavelet" by means of its Fourier transform, taking

$$\widehat{\phi}(w) = \cos(\theta(|w|)) \qquad \text{on} \quad [-1,1]$$

and 0 outside that interval. Clearly $\widehat{\phi} \in C^d$ and has compact support. This implies that ϕ is smooth and rapidly-decreasing, in that $t^d\phi(t)$ lies in $L^2(\mathbb{R})$.

8.2 *Proposition*

For any $\phi \in L^2(\mathbb{R})$ the orthonormality property

$$\langle \phi_k, \phi_\ell \rangle = \delta_{k,\ell}$$

is equivalent to the condition that

$$K(w) := \sum_{k=-\infty}^{\infty} |\widehat{\phi}(w+k)|^2 = 1 \qquad a.e.$$

Proof: Note that K, if finite, is 1-periodic. Now

$$\begin{aligned} \|\widehat{\phi}\|_2^2 &= \int_{-\infty}^{\infty} |\widehat{\phi}(w)|^2\,dw = \sum_{k=-\infty}^{\infty} \int_k^{k+1} |\widehat{\phi}(w)|^2\,dw \\ &= \sum_{k=-\infty}^{\infty} \int_0^1 |\widehat{\phi}(w+k)|^2\,dw = \int_0^1 K(w)\,dw, \end{aligned}$$

(exchanging the integral sum is O.K. by the monotone convergence theorem, for example). So $K \in L^1(0,1)$ and

$$\begin{aligned} \langle \phi_k, \phi_\ell \rangle &= \langle \widehat{\phi}_k, \widehat{\phi}_\ell \rangle = \int_{-\infty}^{\infty} \widehat{\phi}(w)e^{-2\pi ikw}\overline{\widehat{\phi}(w)e^{2\pi i\ell w}}\,dw \\ &= \int_{-\infty}^{\infty} |\widehat{\phi}(w)|^2 e^{-2\pi i(k-\ell)w}\,dw = \int_0^1 K(w)e^{-2\pi i(k-\ell)w}\,dw, \end{aligned}$$

which is the $(k-\ell)$th Fourier coefficient of the 1-periodic function K. We have orthonormality if and only if this coefficient is $\delta_{k,\ell}$; this is equivalent to the condition that $K = 1$ a.e.

□

Note that for the Meyer wavelet, we have $\sum_{k=-\infty}^{\infty} |\widehat{\phi}(w+k)|^2 = 1$; this is obvious for $w \in [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, as all terms are 0 except for one which is 1. Also, for $w \in [\frac{1}{3}, \frac{2}{3}]$ we get

$$\begin{aligned} K(w) &= \cos^2\theta(w) + \cos^2\theta(1-w) \\ &= \cos^2\theta(w) + \cos^2\Big(\frac{\pi}{2} - \theta(w)\Big) = \cos^2\theta(w) + \sin^2\theta(w) = 1. \end{aligned}$$

LECTURE 14

We can also identify V_0, or at least its transform $\widehat{V_0}$.

$$\begin{aligned}\widehat{f} \in \widehat{V_0} &\iff \widehat{f} = \sum a_k \widehat{\phi_k}, \qquad \text{with} \quad \sum |a_k|^2 < \infty \\ &\iff \widehat{f}(w) = \sum a_k e^{-2\pi i k w} \widehat{\phi}(w) \qquad (L_2 \text{ convergence}) \\ &\iff \widehat{f}(w) = \widehat{\phi}(w) g(w),\end{aligned}$$

where g is 1-periodic and $g_{|[0,1]} \in L^2(0,1)$.

Likewise, $\widehat{f} \in \widehat{V_j} \iff f = \widehat{\phi}_j \widehat{g}_j$, where g_j has period 2^j and is locally L^2.

Using this, we can see for the Meyer MRA that $\bigcap_j \widehat{V_j} = \{0\}$ and $\overline{\bigcup_j \widehat{V_j}} = L^2(\mathbb{R})$.

We next need to construct ψ, and we work towards a general construction of this.

8.3 *Proposition*

Let (V_n) and ϕ give a MRA of $L^2(\mathbb{R})$. Then
(i) $\exists (h_k)_k \in \ell^2(\mathbb{Z})$ with $\sum |h_k|^2 = 1$ such that

$$\phi = \sum h_k \phi_{1,k}.$$

(ii) There is a 1-periodic function $h(t) = \sum_{k=-\infty}^{\infty} h_k e^{2\pi i k t}$ (convergence in L^2), such that $\|h\|_{L^2(0,1)} = 1$ and

$$\sum_{k=-\infty}^{\infty} \overline{h_k} h_{k+2\ell} = \delta_{\ell,0}, \qquad (\ell \in \mathbb{Z}).$$

(iii) A function $\sum g_k \phi_{1,k}$ lies in W_0 if and only if $(g_k) \in \ell^2$ and

$$\sum_{k=-\infty}^{\infty} \overline{g_k} h_{k+2\ell} = 0, \qquad (\ell \in \mathbb{Z}).$$

Proof: (i) Since $\phi \in V_0 \subset V_1$ we may express ϕ in terms of the orthonormal basis $(\phi_{1,k})_{k\in\mathbb{Z}}$ of V_1.
(ii) The existence and properties of h come straight from Parseval's identity. Also

$$\phi(t) = \sum h_k 2^{1/2} \phi(2t - k)$$

and

$$\phi(t + \ell) = \sum h_j 2^{1/2} \phi(2t + 2\ell - j) = \sum_n h_{n+2\ell} 2^{1/2} \phi(2t - n),$$

with $n = j - 2\ell$. Now the result follows since

$$\langle \phi_{-\ell}, \phi \rangle = \sum_n h_{n+2\ell} \overline{h_n} = \delta_{\ell,0}.$$

(iii) A function $\chi = \sum g_k \phi_{1,k}$ lies in W_0 if and only if $\langle \phi_{-\ell}, \chi \rangle = 0$ for all $\ell \in \mathbb{Z}$; the calculation is now similar to (ii) above.

□

The Haar wavelet revisited

We have

$$\phi = \frac{1}{\sqrt{2}} \phi_{1,0} + \frac{1}{\sqrt{2}} \phi_{1,1},$$

and so $h(t) = \frac{1}{\sqrt{2}}(1 + e^{2\pi i t})$.
In this case we defined

$$\psi = \frac{1}{\sqrt{2}} \phi_{1,0} - \frac{1}{\sqrt{2}} \phi_{1,1},$$

so our two sequences are

$$\begin{aligned} (h_j) &= (\ldots, 0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, \ldots), \\ (g_j) &= (\ldots, 0, 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0, \ldots), \end{aligned} \qquad (8)$$

so that (g_j) is orthogonal to (h_j) and all its translates by an even number of steps.

LECTURE 15

Back to the general case

8.4 *Proposition*

Let h be as above, and consider it as a function in $L^2(0,1)$. then
(i)

$$\int_0^1 |h(t)|^2 e^{4\pi i \ell t} \, dt = \delta_{0,\ell}, \qquad (\ell \in \mathbb{Z}); \qquad \text{and} \quad |h(t)|^2 + |h(t + \frac{1}{2})|^2 = 2 \quad \text{a.e.}$$

(ii) a function $g \in L^2(0,1)$ corresponds to $\sum g_k \phi_{1,k}$ in W_0 if and only if

$$\int_0^1 h(t) \overline{g(t)} e^{4\pi i \ell t} \, dt = 0 \qquad (\ell \in \mathbb{Z}).$$

Equivalently,

$$h(t)\overline{g(t)} + h(t + \frac{1}{2})\overline{g(t + \frac{1}{2})} = 0 \qquad \text{a.e.}$$

We may equivalently write $g(t) = \lambda(t)\overline{h(t+\frac{1}{2})}$, with λ a 1-periodic function such that $\lambda(t) + \lambda(t+\frac{1}{2}) = 0$ a.e., or, equivalently

$$g(t) = e^{2\pi it}\mu(2t)\overline{h(t+\frac{1}{2})},$$

where μ is 1-periodic.

Proof: (i) Note that

$$\begin{aligned}\int_0^1 |h(t)|^2 e^{4\pi i\ell t}\,dt &= \langle \sum_k h_k e^{2\pi i(k+2\ell)t}, \sum_k h_k e^{2\pi ikt}\rangle \\ &= \sum_k \overline{h_k} h_{k-2\ell} = \delta_{\ell,0},\end{aligned}$$

and so $|h(t)|^2 - 1$ is orthogonal to $e^{4\pi i\ell t}$ for each $\ell \in \mathbb{Z}$, i.e., the even-numbered Fourier coefficients are all zero. Hence $h(t)e^{2\pi it}$ has all odd-numbered coefficients zero, so is $\frac{1}{2}$-periodic, that is,

$$(|h(t)|^2 - 1)e^{2\pi it} = (|h(t+\frac{1}{2})|^2 - 1)e^{2\pi i(t+\frac{1}{2})},$$

or $|h(t)|^2 - 1 = -(|h(t+\frac{1}{2})|^2 - 1)$, as required.
(ii) Similarly, $h(t)\overline{g(t)}e^{2\pi it}$ is $\frac{1}{2}$-periodic, i.e.,

$$h(t)\overline{g(t)}e^{2\pi it} = h(t+\frac{1}{2})\overline{g(t+\frac{1}{2})}e^{2\pi i(t+\frac{1}{2})}.$$

Now if $h(t+\frac{1}{2}) = 0$ then $h(t) \neq 0$, so $g(t) = 0$, and we can write $g(t) = \lambda(t)\overline{h(t+\frac{1}{2})}$ a.e., with λ 1-periodic.
Now

$$h(t)\overline{g(t)} + h(t+\frac{1}{2})\overline{g(t+\frac{1}{2})} = h(t)\overline{\lambda(t)}h(t+\frac{1}{2}) + h(t+\frac{1}{2})\overline{\lambda(t+\frac{1}{2})}h(t) = 0,$$

and so we have $\lambda(t) + \lambda(t+\frac{1}{2}) = 0$ (if $h(t+\frac{1}{2}) = 0$, then we can define $\lambda(t)$ to ensure this).
This means we can write $\lambda(t) = e^{2\pi it}\mu(2t)$ with μ 1-periodic.

□

Example. Take $\mu \equiv 1$. So $g(t) = \overline{h(t+\frac{1}{2})}$.
Thus

$$\sum_n g_n e^{2\pi int} = \sum_k \overline{h_k} e^{-2\pi ik(t+\frac{1}{2})} e^{2\pi it},$$

or $g_n = (-1)^n \overline{h_{1-n}}$. This fits in with the example (8) above.

8.5 *Lemma*

With ϕ and h as above we have

$$\widehat{\phi}(w) = \frac{1}{\sqrt{2}} h(w/2)\widehat{\phi}(w/2).$$

Letting $\psi = \sum g_n \phi_{1,n}$ where $g_n = (-1)^n \overline{h_{1-n}}$, so that $g(w) = e^{2\pi i w}\overline{h(w+1/2)}$, we have

$$\widehat{\psi}(w) = \frac{1}{\sqrt{2}} g(w/2)\widehat{\phi}(w/2) = \frac{1}{\sqrt{2}} e^{\pi i w}\overline{h((w+1)/2)}\widehat{\phi}(w/2).$$

Proof: With $\phi = \sum h_k \phi_{1,k}$ we have

$$\begin{aligned} \widehat{\phi_{1,k}}(w) &= \int_{-\infty}^{\infty} \sqrt{2}\phi(2t-k)e^{-2\pi i w t}\, dt \\ &= \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \phi(x) e^{-2\pi i w(x+k)/2}\, dx \qquad (x = 2t-k) \\ &= \frac{1}{\sqrt{2}} \widehat{\phi}(w/2) e^{-\pi i w k}. \end{aligned}$$

So $\widehat{\phi}(w) = \frac{1}{\sqrt{2}}\widehat{\phi}(w/2)\sum_k h_k e^{-\pi i w k}$ with L^2 convergence, which equals $\frac{1}{\sqrt{2}} h(w/2)\widehat{\phi}(w/2)$. Similarly $\widehat{\psi}(w) = \frac{1}{\sqrt{2}} g(w/2)\widehat{\phi}(w/2)$.

$\square$

Another example.
For the Littlewood–Paley wavelet we had $\widehat{\phi} = \chi_{(-1/2,1/2)}$, and $h(w) = \sqrt{2}\widehat{\phi}(2w)/\widehat{\phi}(w)$, which is 1-periodic.
We get that $\widehat{\psi}(w) = e^{\pi i w}\chi_{(-1,-1/2)\cup(1/2,1)}(w)$. By choosing μ differently we can obtain the ψ we obtained before.

8.6 *Theorem*

In the MRA situation of this chapter, let $\nu(w)$ be 1-periodic and satisfy $|\nu(w)| = 1$ a.e. Define ψ by

$$\widehat{\psi}(w) = \frac{1}{\sqrt{2}} \nu(w)\overline{h((w+1)/2)}\widehat{\phi}(w/2).$$

Then $(\psi_k)_{k\in\mathbb{Z}}$, given by $\psi_k(t) = \psi(t-k)$, generate an o.n.b. for W_0; hence $(\psi_{j,k})_{j,k\in\mathbb{Z}}$, given by $\psi_{j,k}(t) = 2^{j/2}\psi(2^j - k)$, form an orthonormal basis for $L^2(\mathbb{R})$.

LECTURE 16

9 Spline wavelets and compactly-supported wavelets

9.1 *Definition*

Suppose that $a_0 < a_1 < \ldots < a_N$ are real and that $n \geq 1$. A *spline function* of degree n, with *knots* at $a_0, \ldots, a_N$ is a function $f : [a_0, a_N] \to \mathbb{R}$ such that $f \in C^{n-1}[a_0, a_N]$ and $f_{|[a_j, a_{j+1}]}$ is a polynomial of degree n for each $0 \leq j \leq N-1$.

So for $n = 1$ these functions are piecewise linear, for $n = 2$ piecewise quadratic with continuous derivative, and so on.
We also allow $n = 0$, when we simply have a step function that can only jump at the knots.
These are of use in approximation and interpolation schemes. Certain basis functions are known as *B-splines.* In particular for $n = 0$ we see an obvious link with the space V_0 used for constructing Haar wavelets.

To correspond to the case $n = 1$ we define

$$\phi(x) = \begin{cases} 1 - |x| & \text{for } 0 \leq |x| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now if f is a degree-1 spline with knots in $\mathbb{Z}$, it can be written in terms of ϕ_k, where as usual $\phi_k(t) = \phi(t - k)$.
The Battle–Lemarié construction produces wavelets using this ϕ, and similar basis functions.

9.2 *Proposition*

With ϕ as above we have:
(i)

$$\widehat{\phi}(w) = \left(\frac{\sin \pi w}{\pi w} \right)^2;$$

(ii)

$$\phi(x) = \frac{1}{2}\phi(2x + 1) + \phi(2x) + \frac{1}{2}\phi(2x - 1),$$

i.e.,
$\phi = \sum h_k \phi_{1,k}$, where $h_{-1} = h_1 = \frac{1}{2\sqrt{2}}$, and $h_0 = \frac{1}{\sqrt{2}}$, and the rest 0; as usual $\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k)$.
(iii)

$$K(w) := \sum_{k=-\infty}^{\infty} |\widehat{\phi}(w + k)|^2 = \frac{2}{3} + \frac{1}{3}\cos 2\pi w.$$

Proof: We leave parts (i) and (ii) as exercises. For (iii), note that K is a period-1 function whose mth Fourier coefficient is $\langle \phi, \phi_m \rangle$, as in the proof of (8.2).
Now

$$\langle \phi, \phi_0 \rangle = 2 \int_0^1 (1-x)^2 \, dx = \frac{2}{3},$$

and

$$\langle \phi, \phi_1 \rangle = \langle \phi, \phi_{-1} \rangle = \int_0^1 (1-x)x \, dx = \frac{1}{6},$$

with all other inner products zero, so

$$K(w) = \frac{2}{3} + \frac{1}{6} e^{2\pi i w} + \frac{1}{6} e^{-2\pi i w} = \frac{2}{3} + \frac{1}{3} \cos 2\pi w.$$

□

In general, it is possible to construct similar smoother ϕ that is a finite linear combination on translates, and with

$$\widehat{\phi}(w) = e^{-i\pi r w} \left(\frac{\sin \pi w}{\pi w} \right)^{n+1},$$

where $r = 0$ for N odd and $r = 1$ for N even.
Problem: in general ϕ is not a “father wavelet”, as the translates of ϕ are not orthonormal, except in the case $n = 0$.

9.3 *Theorem (the orthonormalization theorem)*

Let $\phi \in L^2(\mathbb{R})$ be such that for some constants $A, B > 0$ the function

$$K(w) := \sum_{k=-\infty}^{\infty} |\widehat{\phi}(w+k)|^2$$

satisfies $0 < A \le K(w) \le B$ for almost all $w \in \mathbb{R}$. Then the family $(\phi_k)_{k \in \mathbb{Z}}$ forms a Riesz basis for its closed linear span V_0, and an orthonormal basis can be obtained by taking the functions $(\phi_k^\sharp)_{k \in \mathbb{Z}}$ defined by

$$\widehat{\phi^\sharp}(w) = \widehat{\phi}(w) / K(w)^{1/2},$$

and $\phi_k^\sharp(t) = \phi^\sharp(t-k)$.

Proof: We calculate

$$\begin{aligned}
\left\|\sum_l c_l \phi_l\right\|_2^2 &= \left\|\sum_l c_l \widehat{\phi_l}\right\|_2^2 = \left\|\sum_l c_l e^{-2\pi i l w}\widehat{\phi}(w)\right\|^2 \\
&= \int_{-\infty}^{\infty}\left|\sum_l c_l e^{-2\pi i l w}\right|^2 |\widehat{\phi}(w)|^2\, dw \\
&= \sum_{k=-\infty}^{\infty}\int_k^{k+1}\left|\sum_l c_l e^{-2\pi i l w}\right|^2 \left|\widehat{\phi}(w)\right|^2\, dw \\
&= \sum_{k=-\infty}^{\infty}\int_0^{1}\left|\sum_l c_l e^{-2\pi i l w}\right|^2 \left|\widehat{\phi}(w+k)\right|^2\, dw \\
&= \int_0^{1}\left|\sum_l c_l e^{-2\pi i l w}\right|^2 K(w)\, dw.
\end{aligned}$$

Thus

$$A\sum_l |c_l|^2 \le \left\|\sum_l c_l \phi_l\right\|_2^2 \le B\sum_l |c_l|^2,$$

and it's a Riesz basis.
Now, repeating the calculation with $\phi^\sharp$ we get an orthonormal sequence (i.e., the same with $A = B = 1$), since

$$K^\sharp(w) := \sum_{k=-\infty}^{\infty} |\widehat{\phi^\sharp}(w+k)|^2 = 1.$$

We should remark that (ϕ_k) and $(\phi_k^\sharp)$ have the same closed linear span, since

$$\begin{aligned}
V_0 &:= \{f \in L^2(\mathbb{R}) : \widehat{f} = \nu\widehat{\phi},\ \nu \in L^2(0,1),\ \text{extended 1-periodically}\} \\
&= \{f \in L^2(\mathbb{R}) : \widehat{f} = \nu_1\widehat{\phi^\sharp},\ \nu_1 \in L^2(0,1),\ \text{extended 1-periodically}\},
\end{aligned}$$

with $\nu_1 = K\nu$. See also (8.6).

□

For the Battle–Lemarié example, with $n = 1$ we have

$$\widehat{\phi^\sharp}(w) = \left(\frac{\sin \pi w}{\pi w}\right)^2 \frac{1}{(\frac{2}{3} + \frac{1}{3}\cos 2\pi w)^{1/2}}.$$

Now $\phi^\sharp$ is still a spline, but no longer compactly supported. It does have exponential decay, however. V_0 consists of all continuous L^2 functions that are piecewise linear on every $[n, n+1]$. We could now work out ψ as in Section 8, but it gets very complicated.

LECTURE 17

Compactly-supported wavelets

For good localization we would like wavelets which decrease rapidly as $|t| \to \infty$. The following result links the behaviour of f and $\widehat{f}$.

9.4 *Proposition*

(i) Suppose that $f, f', \ldots, f^{(n)}$ are all $L^1(\mathbb{R})$ functions (so that in fact $f, f', \ldots, f^{(n-1)}$ are all absolutely continuous as well). Then $\widehat{f}(w) = O(|w|^{-n})$ as $|w| \to \infty$.
(ii) Suppose that $\widehat{f}(w) = O(|w|^{-n})$ as $|w| \to \infty$. Then f is C^{n-2}.
(iii) Parts (i) and (ii) hold if we swap the roles of f and $\widehat{f}$.

Proof: (i)

$$\begin{aligned} \widehat{f}(w) &= \int_{-\infty}^{\infty} f(t) e^{-2\pi i w t}\, dt \\ &= \frac{1}{2\pi i w} \int_{-\infty}^{\infty} f'(t) e^{-2\pi i w t}\, dt \\ &= \ldots \\ &= \frac{1}{(2\pi i w)^n} \int_{-\infty}^{\infty} f^{(n)}(t) e^{-2\pi i w t}\, dt, \end{aligned}$$

integrating by parts repeatedly. Then since $f^{(n)} \in L^1$, the final integral is bounded independently of w so $|f(w)| \le \|f^{(n)}\|_1/(2\pi|w|)^n$.
(ii)

$$f(t) = \int_{-\infty}^{\infty} \widehat{f}(w) e^{2\pi i w t}\, dw.$$

We may differentiate under the integral sign as everything is absolutely convergent, and so

$$f^{(m)}(t) = \int_{-\infty}^{\infty} \widehat{f}(w)(2\pi i w)^m e^{2\pi i w t}\, dw,$$

at least for $1 \le m \le n-2$. Moreover, $\widehat{f}(w)(2\pi i w)^{n-2}$ lies in $L^1(\mathbb{R})$, and so $f^{(n-2)}$ is continuous.
(iii) Easy, since $\widehat{\widehat{f}}(t) = f(-t)$.

□

9.5 *Corollary*

If f is C^∞ with all derivatives in $L^1(\mathbb{R})$, then $\widehat{f}$ decreases more rapidly than any power of w. Likewise, if $\widehat{f}$ decreases rapidly, then $f \in C^\infty$.

Properties of particular wavelets

Wavelet	Properties of $\psi(t)$	Properties of $\hat{\psi}(w)$
Haar	Compact support, discontinuous	$O(1/w)$, C^∞
Littlewood–Paley	$O(1/t)$, C^∞	Compact support, discontinuous
Meyer	Rapidly-decreasing, C^∞	Compact support, can be C^∞
Battle–Lemarié	Rapidly-decreasing, C^k	$O(1/w^k)$, C^∞
Daubechies	Compact support, C^k	$O(1/w^k)$, C^∞

Here k is arbitrarily large, but finite.
What is not possible is for both ψ and $\widehat{\psi}$ to have compact support, since a function f and its transform $\widehat{f}$ cannot both be time-limited.
Also we can't have ψ being C^∞ with exponential decay if the $(\psi_{j,k})$ form an orthonormal basis. We can if they form a frame (e.g., Mexican hat).

Construction of Daubechies wavelets (sketch)

Recall that we had, in Section 8, the notation

$$\begin{aligned}
\phi &= \sum h_k \phi_{1,k}, \\
h(t) &= \sum h_k e^{2\pi i k t}, \\
\widehat{\phi}(w) &= \frac{1}{\sqrt{2}} h(w/2) \widehat{\phi}(w/2), \\
\psi &= \sum g_k \phi_{1,k}, \qquad \text{with} \\
g_k &= \overline{h_{1-k}}(-1)^k, \\
|h(t)|^2 + |h(t+1/2)|^2 &= 2 \qquad \text{(orthonormality of } (\phi_k)).
\end{aligned}$$

9.6 *Proposition*

Suppose that ϕ is a compactly-supported function, $\widehat{\phi}(0) = 0$, that ϕ generates a MRA (V_k), and (ϕ_k) is an o.n.b. for V_0. Then $h(t)$ is a trigonometric polynomial in t, $h(0) = \sqrt{2}$, and ψ is also compactly-supported.

Proof: Since ϕ and $\phi_{1,k}$ are compactly-supported, it's clear that only finitely-many $\phi_{1,k}$ have supports meeting $\operatorname{supp}\phi$, so $h_k = 0$ for all but finitely-many k.
Also $\widehat{\phi}(w) = \frac{1}{\sqrt{2}} h(w/2) \widehat{\phi}(w/2)$; put $w = 0$ and we see that $h(0) = \sqrt{2}$.
Finally, $g_k = \overline{h_{1-k}}(-1)^k \neq 0$ for only finitely-many k and so $\psi = \sum g_k \phi_{1,k}$ is compactly-supported as each $\phi_{1,k}$ is. □

LECTURE 18

9.7 *Proposition*

The scaling function h determines ϕ (up to a constant) by the identity

$$\widehat{\phi}(w) = \widehat{\phi}(0) \prod_{m=1}^{\infty} \left(\frac{1}{\sqrt{2}} h(w/2^m) \right).$$

Proof: Since $h(0) = \sqrt{2}$ and h is a trigonometric polynomial, we have

$$h(w/2^m) = h(0) + \frac{w}{2^m} h'(\xi),$$

for some $0 < \xi < w/2^m$. Thus there is a constant $C > 0$ such that

$$\left| \frac{1}{\sqrt{2}} h(w/2^m) - 1 \right| \leq Cw/2^m$$

for all $|w| \leq 1$.
Now an infinite product $\prod a_m$ "converges absolutely" when $\sum |\log a_m| < \infty$, or equivalently $\sum(1 - |a_m|) < \infty$. We apply this idea with $a_m = \frac{1}{\sqrt{2}} h(w/2^m)$.
So

$$\begin{aligned} \widehat{\phi}(w) &= \frac{1}{\sqrt{2}} h(w/2) \widehat{\phi}(w/2) \\ &= \ldots \\ &= \left[\prod_{m=1}^{n} \frac{1}{\sqrt{2}} h(w/2^m) \right] \widehat{\phi}(w/2^n) \qquad \text{for} \quad n = 1, 2, \ldots. \end{aligned}$$

Letting $n \to \infty$ and noting that $\widehat{\phi}(w/2^n) \to \widehat{\phi}(0)$, we have the result. $\square$

9.8 *Construction*

We take h to be given by $h(w) = \sqrt{2} P(e^{2\pi i w})$, with P a polynomial of degree $2N - 1$; here $N \geq 1$ is a fixed integer.
If we have $|P(z)|^2 + |P(-z)|^2 = 1$ whenever $|z| = 1$, then, with $z = \exp(2\pi i w)$, we will have $h(w) = \sqrt{2} P(z)$ and $h(w + 1/2) = \sqrt{2} P(-z)$, so we will obtain

$$|h(w)|^2 + |h(w + 1/2)|^2 = 2,$$

which is one of the necessary conditions.
We shall choose

$$P(z) = \left(\frac{1+z}{2} \right)^N W(z),$$

with $\deg W = N - 1$. Thus $P(-1) = 0$ and we will take $P(1) = 1$ by making $W(1) = 1$.
This property leads to smoothness of ϕ (details omitted).

9.9 *Proposition*

For $N = 1$ the construction produces $P(z) = (1+z)/2$, and $|P(z)|^2 + |P(-z)|^2 = 1$ for $|z| = 1$.
Then $h(w) = \frac{1}{\sqrt{2}}(1 + e^{2\pi i w})$, the scaling function for the Haar wavelet.

Proof: W is degree 0, hence constant, so $W = W(1) = 1$. It is easy to verify that

$$\begin{aligned} \left|\frac{1+z}{2}\right| &= |\cos \pi w|, \qquad \text{and} \\ \left|\frac{1-z}{2}\right| &= |\sin \pi w|, \end{aligned}$$

for $z = \exp(2\pi i w)$. Thus $|P(z)|^2 + |P(-z)|^2 = 1$, and the remainder of the calculation is straightforward. ☐

9.10 *Theorem (Strichartz, 1993)*

Let $C(z) = \cos \pi w = (z^{1/2} + \overline{z}^{1/2})/2$ and $S(z) = \sin \pi w = (z^{1/2} - \overline{z}^{1/2})/2i$. Then

$$1 = \sum_{k=0}^{2N-1} \binom{2N-1}{k} C^{4N-2-2k} S^{2k},$$

and if $|P(z)|^2$ equals the sum of the first N terms, then P satisfies the conditions of Construction 9.8.

Proof: The first identity is just what we get on expanding $1 = (C^2 + S^2)^{2N-1}$ by the binomial theorem.
Now if

$$|P(z)|^2 = \sum_{k=0}^{N-1} \binom{2N-1}{k} C^{4N-2-2k} S^{2k},$$

which is non-negative since all its terms are non-negative, then the transformation $z \mapsto -z$ corresponds to $w \mapsto w + 1/2$ (modulo 1); thus $C(-z) = \cos \pi(w + 1/2) = -S(z)$ and $S(-z) = \sin \pi(w + 1/2) = C(z)$.
Thus, with $\ell = 2N - 1 - k$ below, we get

$$\begin{aligned} |P(-z)|^2 &= \sum_{k=0}^{N-1} \binom{2N-1}{k} S^{4N-2-2k} C^{2k} \\ &= \sum_{\ell=N}^{2N-1} \binom{2N-1}{\ell} S^{2\ell} C^{4N-2-2\ell}, \end{aligned}$$

and indeed

$$|P(z)|^2 + |P(-z)|^2 = \sum_{k=0}^{2N-1} \binom{2N-1}{k} C^{4N-2-2k} S^{2k} = 1.$$

□

Note that $z = -1$ is when $w = 1/2$ and $C = 0$, and $|P(z)|^2$ has a zero of degree $4N - 2 - 2(N-1) = 2N$ there, so P will have a zero of degree N.

Example. $N = 2$.

$$\begin{aligned} |P(z)|^2 &= \sum_{k=0}^{N-1} \binom{2N-1}{k} C^{4N-2-2k} S^{2k} \\ &= \sum_{k=0}^{1} \binom{3}{k} C^{6-2k} S^{2k} \\ &= C^6 + 3C^4 S^2 \\ &= \left| \frac{1+z}{2} \right|^4 (C^2 + 3S^2). \end{aligned}$$

We now need to do a *spectral factorization*, choose W a polynomial of degree 1 with $W(1) = 1$ such that

$$|W(e^{2\pi i w})|^2 = C^2 + 3S^2 = 1 + 2S^2 = 2 - \cos 2\pi w,$$

or

$$|W(z)|^2 = 2 - \frac{z + \overline{z}}{2}$$

when $|z| = 1$.
We try $W(z) = a + bz$ with a, b real, so that $|W(z)|^2 = (a + bz)(a + b\overline{z})$, and hence $2 = a^2 + b^2$ and $-\frac{1}{2} = ab$. Also $a + b = W(1) = 1$.
The solution to these equations is

$$P(z) = \frac{1}{2} \left(\frac{1+z}{2} \right)^2 \left[(1 \pm \sqrt{3}) + (1 \mp \sqrt{3}) z \right].$$

9.11 *Final details*

P determines h, and h determines $\widehat{\phi}$ and hence ϕ, by Prop. 9.7. Using the coefficients of h we can now obtain ψ too.

Details that we have omitted.

1. ϕ is compactly-supported – the idea is to show that the infinite product defining $\widehat{\phi}$ produces an entire function lying in a suitable Paley–Wiener space, and hence ϕ itself has compact support contained in $[0, 2N-1]$.

2. The continuity and differentiability properties of ϕ and ψ are also to do with the properties of $\widehat{\phi}$.

3. We need to do a spectral factorization to obtain the higher-order wavelets. This requires the Fejér–Riesz theorem which says they exist. Can be done algebraically for $N \leq 3$, and numerically/iteratively beyond that.

THE END