Interpolation by vector-valued analytic functions, with applications to controllability

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Abstract

In this paper, norm estimates are obtained for the problem of minimal-norm tangential interpolation by vector-valued analytic functions, expressed in terms of the Carleson constants of related scalar measures. Applications are given to the controllability properties of linear semigroup systems with a Riesz basis of eigenvectors.

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1 Introduction and notation

The main theme of this paper is to obtain certain vector-valued generalizations of the Shapiro–Shields interpolation theory for the Hardy space \( H^2 \) of the right-hand complex half-plane \( \mathbb{C}_+ \) as developed in [12, 22, 23], of which a good treatment can be found in the book of Koosis [10]. Specifically, given a Hilbert space \( \mathcal{H} \), bounded linear operators \( G_k \) on \( \mathcal{H} \), vectors \( a_1, \ldots, a_n \) in \( \mathcal{H} \), and pairwise distinct points \( z_1, \ldots, z_n \) in \( \mathbb{C} \) we estimate the minimal norm of a function \( f \in H^2(\mathbb{C}_+, \mathcal{H}) \), satisfying the interpolation conditions

\[
G_k f(z_k) = a_k \quad (k = 1, \ldots, n)
\]  

(1)

where the \( G_k \) are suitable bounded linear operators on \( \mathcal{H} \) (all necessary notation is explained below). If the \( G_k \) are invertible, the situation is not very different from the scalar case. So the typical situation which will interest us here is that the \( G_k \) are not invertible, particularly the case that \( \text{rank} G_k = 1 \) for all \( k \). This can be regarded as a problem of tangential interpolation in the sense of [2] (see

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Our main result, Theorem 2.7, is an interpolation criterion in terms of boundedness of the embedding

\[ H^2(\mathbb{C}_+, \mathcal{H}) \to L^2(\mathbb{C}_+, \mathcal{H}, \mu) \]

for a certain operator-valued measure \( \mu \) which depends only on the points \( (z_k) \) and the operators \( (G_k) \). We shall see that in many cases a sharp estimate can be given in terms of the Carleson constants of appropriate scalar measures.

A dual problem to this which appears in the literature is that of the existence of \( f \in H^2(\mathbb{C}_+, \mathcal{H}) \) such that

\[ f(z_k) = G_k a_k \]

for a given \( \ell^2 \) sequence \( (a_k) \) in \( \mathcal{H} \). For \( G_k \) with closed range, this can quite easily be brought into the framework of our interpolation question (1).

Apart from its intrinsic importance, this interpolation problem arises naturally in systems and control theory, where questions of controllability and observability can be studied by these methods – see [4, 8, 15] for more on this, particularly in the 1-dimensional case, which corresponds to scalar inputs or outputs. We will give applications of our vector interpolation results to controllability in diagonal systems with vector inputs in the second part of this paper.

In the remainder of this section we establish the necessary notation and definitions. In Section 2 we give norm estimates for the minimum-norm interpolation problem, which are then applied to controllability problems in Section 3.

Let \( (z_k)_{k \in \mathbb{N}} \) be a Blaschke sequence of pairwise distinct elements in the right half plane \( \mathbb{C}_+ = \{ z \in \mathbb{C} : \text{Re } z > 0 \} \). Let

\[ b_k(z) = \frac{z - z_k}{z + \overline{z}_k} \]

be the Blaschke factor for \( z_k \). For \( n \in \mathbb{N}, 1 \leq k \leq n \), let

\[ B_n(z) = \prod_{j=1}^{n} b_j(z), \quad B_{n,k}(z) = \prod_{j=1, j \neq k}^{n} b_j(z), \quad b_{n,k} = B_{n,k}(z_k), \]

and

\[ b_{\infty,k} = \lim_{n \to \infty} B_{n,k}(z_k). \]

Also \( k_{z_k} \) denotes the reproducing kernel at \( z_k \), so that

\[ k_{z_k}(z) = \frac{1}{2\pi} \frac{1}{z + \overline{z}_k}, \]

and \( (f, k_{z_k}) = f(z_k) \) for all \( f \in H^2(\mathbb{C}_+) \). The angle between two non-zero vectors \( v_1, v_2 \) in a Hilbert space \( V \) is given by

\[ \angle(v_1, v_2) := \arccos \left( \frac{\langle v_1, v_2 \rangle}{\|v_1\| \|v_2\|} \right). \]

The angle between two subspaces \( V_1 \) and \( V_2 \) of \( V \) is then defined by

\[ \angle(V_1, V_2) := \inf_{v_1 \in V_1 \setminus \{0\}, v_2 \in V_2 \setminus \{0\}} \angle(v_1, v_2). \]
2 Interpolation

2.1 Carleson-type embedding theorems for matrix measures

We first recall the classical Carleson embedding theorem [3]:

**Theorem 2.1** Let $\mu$ be a nonnegative Borel measure on the right half plane $\mathbb{C}_+$. Then the following are equivalent:

1. The embedding $H^2(\mathbb{C}_+) \to L^2(\mathbb{C}_+, \mu)$ is bounded.

2. There exists a constant $C > 0$ such that

$$\mu(Q_I) \leq C |I|$$

for all intervals $I \subset \mathbb{R}$, \hspace{1cm} (2)

where $Q_I = \{z = x + iy \in \mathbb{C}_+ : y \in I, 0 < x < |I|\}$.

In this case, $\mu$ is called a Carleson measure.

Next we state a well-known easy matrix analogue of this theorem. We include a proof for the convenience of the reader. Let us use the following notation:

For a finite or infinite-dimensional Hilbert space $\mathcal{H}$,

$$H^2(\mathbb{C}_+, \mathcal{H}) = \left\{ f : \mathbb{C}_+ \to \mathcal{H} \text{ analytic: } \sup_{\varepsilon > 0} \int_{\mathbb{R}} \|f(it + \varepsilon)\|^2 dt < \infty \right\}.$$

**Theorem 2.2** Let $\mu$ be a nonnegative $N \times N$ matrix-valued Borel measure on the right half plane $\mathbb{C}_+$, and let

$$L^2(\mathbb{C}_+, \mathbb{C}^N, \mu) = \left\{ f : \mathbb{C}_+ \to \mathbb{C}^N \text{ measurable: } \int_{\mathbb{C}_+} \langle d\mu(z)f(z), f(z) \rangle < \infty \right\},$$

with the usual identification of functions which agree a.e. Then the following are equivalent:

1. The embedding $H^2(\mathbb{C}_+, \mathbb{C}^N) \to L^2(\mathbb{C}_+, \mathbb{C}^N, \mu)$ is bounded.

2. There exists a constant $C > 0$ such that

$$\int_{\mathbb{C}_+} |k_\lambda(z)|^2 d\mu(z) \leq C \|k_\lambda\|^2_{H^2} 1 \quad \text{for all } \lambda \in \mathbb{C}_+.$$

Here $1$ denotes the identity matrix in $\mathbb{C}^{N \times N}$.

3. There exists a constant $C > 0$ such that

$$\mu(Q_I) \leq C |I| 1 \quad \text{for all intervals } I \subset \mathbb{R}, \hspace{1cm} (3)$$

where $Q_I = \{z = x + iy \in \mathbb{C}_+ : y \in I, 0 < x < |I|\}$. 

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4. \(\text{tr}\ \mu\), the trace of \(\mu\), is a scalar Carleson measure, i.e.

\[ H^2(\mathbb{C}_+) \rightarrow L^2(\mathbb{C}_+, \text{tr}\ \mu) \]

is bounded.

In this case, \(\mu\) is called a matrix Carleson measure, and we denote the smallest constant \(C\) such that (3) holds by \(\text{Carl}(\mu)\).

Proof \((4) \Rightarrow (1)\) follows trivially from

\[
\int_{\mathbb{C}_+} \langle d\mu(z) f(z), f(z) \rangle \leq \int_{\mathbb{C}_+} \|f(z)\|^2 \text{tr}\, d\mu(z).
\]

\((1) \Rightarrow (2)\) is immediate by choosing \(f = k\lambda e, e \in \mathbb{C}^N\).

\((2) \Rightarrow (3)\) is easily obtained as in the scalar case by choosing \(\lambda = |I|/2 + ic(I)\), where \(c(I)\) denotes the centre of \(I\), and observing that

\[
|k_{\lambda_I}(z)|^2 = \frac{1}{4\pi^2} \left| \frac{1}{z + |I|/2 - ic(I)} \right|^2 \geq \frac{1}{12\pi^2} |I|^{-2} \quad \text{for } z \in Q_I,
\]

\[
\|k_{\lambda_I}\|^2_{H^2} = \frac{1}{2\pi} |I|^{-1}.
\]

\((3) \Rightarrow (4)\) This is immediate from the monotonicity of the trace and the scalar case.

Theorem 2.2 does not generalise to the infinite-dimensional case ([13]), but the following still holds as a consequence of the scalar case:

**Theorem 2.3** Let \(\mu\) be a nonnegative operator-valued Borel measure on the right half plane \(\mathbb{C}_+\), and let

\[ L^2(\mathbb{C}_+, \mathcal{H}, \mu) = \left\{ f : \mathbb{C}_+ \rightarrow \mathcal{H} \text{ strongly measurable} : \int_{\mathbb{C}_+} \langle d\mu(z) f(z), f(z) \rangle < \infty \right\}. \]

Let \(\|\mu\|\) be the total variation of \(\mu\),

\[
\|\mu\|(A) = \sup \left\{ \sum_{i=1}^n \|\mu(A_i)\| : A_1, \ldots, A_n \text{ pairwise disjoint}, \ A = A_1 \cup \cdots \cup A_n, n \in \mathbb{N} \right\}
\]

Suppose that \(\|\mu\|\) is a scalar Carleson measure. Then the embedding

\[ H^2(\mathbb{C}_+, \mathcal{H}) \rightarrow L^2(\mathbb{C}_+, \mathcal{H}, \mu) \]

is bounded.

Proof Follows immediate from the scalar case.
2.2 Description of certain shift-invariant subspaces of $H^2(\mathbb{C}_+, \mathcal{H})$

Let $\mathcal{H}$ be a separable Hilbert space. In this subsection we describe certain shift-invariant subspaces of $H^2(\mathbb{C}_+, \mathcal{H})$ by means of Blaschke–Potapov products (see e.g. [20, p. 76]).

**Lemma 2.4** Let $(z_k)_{k \in \mathbb{N}}$ be a sequence of pairwise distinct elements of $\mathbb{C}_+$. For each $k \in \mathbb{N}$, let $L_k \subseteq \mathcal{H}$ be a closed linear subspace of $\mathcal{H}$. Let

$$L_n = \{ f \in H^2(\mathbb{C}_+, \mathcal{H}) : f(z_k) \in L_k \text{ for } 1 \leq k \leq n \}.$$ 

Then $L_n = \Theta^L_n H^2(\mathbb{C}_+, \mathcal{H})$, where $\Theta^L_n$ denotes the matrix-valued Blaschke–Potapov product

$$\Theta^L_n(z) = \left( b_1(z)P_{L_1}^+ + P_{L_1} \right) \cdots \left( b_n(z)P_{L_n}^+ + P_{L_n} \right) \quad (z \in \mathbb{C}_+),$$

where

$$\tilde{L}_1 = L_1, \quad \tilde{L}_k = \Theta^L_{k-1}(z_k)^{-1}L_k, \quad 2 \leq k \leq n,$$

and $P_{L_k}$ is the orthogonal projection $\mathcal{H} \to \tilde{L}_k$.

Furthermore, $P_{L_k}^+ \Theta^L_n(z_k) = 0$ for $k = 1, \ldots, n$. We write

$$\Theta^L_{n,k} = \left( b_1(z_k)P_{L_1}^+ + P_{L_1} \right) \cdots P_{L_k}^+ \cdots \left( b_n(z_k)P_{L_n}^+ + P_{L_n} \right)$$

(4)

which is the continuous extension of $P_{L_k}^+ b_k(z)^{-1} \Theta^L_n(z)$ to $z_k$. One verifies that the right inverse $(\Theta^L_{n,k})^{-1} : L_k^+ \to \mathcal{H}$ is well-defined.

If $(z_k)$ is a Blaschke sequence, then $\Theta^L_{n,k}$ converges normally (uniformly on compact subsets of $\mathbb{C}_+$) to an operator-valued inner function $\Theta^L$ with $\Theta^L H^2(\mathbb{C}_+, \mathcal{H}) = \bigcap_{n \in \mathbb{N}} L_n$.

**Proof** One verifies easily that $\Theta^L_1$ can be chosen as

$$\Theta^L_1(z) = (b_1(z)P_{L_1}^+ + P_{L_1})$$

by choosing a suitable orthonormal basis of $\mathcal{H}$ and applying the classical theorem of Beurling (see e.g. [6], page 114) for the scalar case componentwise.

Now suppose that the lemma holds for some $n \in \mathbb{N}$. Let $f \in L_{n+1}$. Notice that $\Theta^L_n(z_{n+1})$ is an invertible bounded linear operator.

Since $L_{n+1} \subset L_n$, we can write $f = \Theta^L_n \tilde{f}$ with $\tilde{f} \in H^2(\mathbb{C}_+, \mathcal{H})$, $f(z_{n+1}) = \Theta^L_n(z_{n+1}) \tilde{f}(z_{n+1}) \in L_{n+1}$ and therefore $f(z_{n+1}) \in \tilde{L}_{n+1}$. Applying the argument for the case $n = 1$ again, we obtain $\tilde{f} = (b_{n+1}(z)P_{L_{n+1}}^+ + P_{L_{n+1}}) \tilde{f}$ for some $\tilde{f} \in H^2(\mathbb{C}_+, \mathcal{H})$.

Conversely, one verifies easily that each function $f \in \Theta^L_n H^2(\mathbb{C}_+, \mathcal{H})$, $\Theta^L_n$ defined as above, is in $L_n$.

The uniform convergence of Blaschke products in operator norm on compact subsets of $\mathbb{C}_+$ is shown similarly to the scalar case (see e.g. [14, p. 281]), and a weak* compactness argument shows easily that $\Theta^L H^2(\mathbb{C}_+, \mathcal{H}) = \bigcap_{n \in \mathbb{N}} L_n$. 

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Clearly \( \cap_{n \in \mathbb{N}} \mathcal{L}_n \supseteq \Theta_n^L H^2(\mathbb{C}_+, \mathcal{H}) \). Conversely, for each \( f \in \cap_{n \in \mathbb{N}} \mathcal{L}_n \), write \( f = \Theta_n^L g_n \) for each \( n \in \mathbb{N} \) and find a weak* convergent subsequence \( g_{n_k} \to g \). Then \( f(z) = w^* - \lim_{k \to \infty} \Theta_{n_k}^L(z)g_{n_k}(z) = \Theta^L(z)g(z) \) for \( z \in \mathbb{C}_+ \), and \( f \in \Theta^L H^2(\mathbb{C}_+, \mathcal{H}) \).

We require some further notation at this point. For the subspace \( \mathcal{L}_n \) as above, we temporarily write \( \mathcal{L}_n^L \) to indicate its dependency on the subspaces \( L_1, \ldots, L_n \).

We think of the functions in the Hardy space \( H^2(\mathbb{C}_+, \mathcal{H}) \), where \( \mathbb{C}_- = \{ z \in \mathbb{C} : \text{Re} z < 0 \} \), and of the matrix inner functions on \( \mathbb{C}_- \) as anti-analytic functions on the right half plane.

In this sense, we write
\[
\mathcal{L}_n^L = \{ f \in H^2(\mathbb{C}_-, \mathcal{H}) : f(z_k) \in L_k \text{ for } 1 \leq k \leq n \},
\]
\[
\mathcal{L}_n^{L\perp} = \{ f \in H^2(\mathbb{C}_-, \mathcal{H}) : f(z_k) \in L_k^\perp \text{ for } 1 \leq k \leq n \}
\]

for the “flipped” subspace and \( \mathcal{L}_n^{L}, \mathcal{L}_n^{L\perp} \) for the corresponding operator-valued inner functions on \( \mathbb{C}_- \), constructed as in Lemma 2.4.

**Lemma 2.5** \( \Theta_n^L \mathcal{B}_n \) is an inner function on \( \mathbb{C}_- \), and
\[
\Theta_n^L \mathcal{B}_n H^2(\mathbb{C}_-, \mathcal{H}) = \mathcal{L}_n^{L\perp},
\]
and \( \Theta_n^L \mathcal{B}_n = \mathcal{L}_n^{L\perp} U_n \) for some constant unitary \( U_n \).

**Proof** Suppose first that \( L_1, \ldots, L_n \) are finite-dimensional. For \( z \in i\mathbb{R} \),
\[
\Theta_n^L(z) \mathcal{B}_n(z) = (P_{L_1}^\perp + \bar{b}_1(z)P_{L_1}) \cdots (P_{L_n}^\perp + \bar{b}_n(z)P_{L_n}).
\]

This is obviously an inner function on \( \mathbb{C}_- \), since each \( \bar{b}_k \) is an inner function on \( \mathbb{C}_- \). Recall from the construction in Lemma 2.4 that \( \bar{L}_1 = L_1 \). It follows that \( (\Theta_n^L \mathcal{B}_n f)(z_1) \in L_1^\perp \) for all \( f \in H^2(\mathbb{C}_-, \mathcal{H}) \). But by the uniqueness property in the Beurling–Lax Theorem, the order of the \( (z_1) \) does not matter, and we deduce that \( (\Theta_n^L \mathcal{B}_n f)(z_k) \in L_k^\perp \) for all \( f \in H^2(\mathbb{C}_-, \mathcal{H}), k = 1, \ldots, n \).

Thus \( \Theta_n^L \mathcal{B}_n H^2(\mathbb{C}_-, \mathcal{H}) \subseteq \mathcal{L}_n^{L\perp} \). Since \( \Theta_n^L \mathcal{B}_n H^2(\mathbb{C}_-, \mathcal{H}) \) is finite-codimensional in \( H^2(\mathbb{C}_-, \mathcal{H}) \), equality follows from a dimension argument. The case of general \( L_1, \ldots, L_n \) follows now from an approximation argument.

The last part of the Lemma follows again from the uniqueness property in the Beurling–Lax Theorem.

The following consequence of Lemma 2.5 will be required in the proof of the main theorem of the section, 2.7.

**Corollary 2.6** Let \( k \in \{1, \ldots, n\} \) and let \( \Theta_n^{L,k} (\Theta_n^{L})^{-1} \) be as defined in Lemma 2.4,
\[
\Theta_n^{L,k} = (b_1(z_k)P_{L_1}^\perp + P_{L_1}) \cdots (b_k(z_k)P_{L_k}^\perp + P_{L_k}) \cdots (b_n(z_k)P_{L_n}^\perp + P_{L_n}).
\]

Then
\[
P_{L_k}^\perp (\Theta_n^{L,k})^{-1} f = \frac{1}{b_{n,k}} \mathcal{L}_n^{L\perp}(z_k) U_n,
\]
where \( U_n \) is a constant unitary independent of \( k \).
**Proof** By definition,

\[
P_{L_k}^\perp (\Theta_n L_{k,n})^{-1*} = \left( \frac{1}{b_{n,k}} \right) \left( b_1(z_k)^{-1} P_{L_1}^\perp + P_{L_1} \right) \cdots P_{L_k}^\perp \left( b_n(z_k)^{-1} P_{L_n}^\perp + P_{L_n} \right)
\]

\[
= \frac{1}{b_{n,k}} \left( P_{L_1}^\perp + b_1(z_k) P_{L_1} \right) \cdots P_{L_k}^\perp \left( P_{L_n}^\perp + b_n(z_k) P_{L_n} \right)
\]

\[
= \frac{1}{b_{n,k}} \sum_{\ell=1}^{\infty} L_{n,k} \Theta_n(z_k) = \frac{1}{b_{n,k}} \sum_{\ell=1}^{\infty} \Theta_n(z_k) U_n,
\]

where the last equality follows from Lemma 2.5.

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### 2.3 Interpolation Theorems

Let \( \mathcal{H} \) be a separable Hilbert space, and let \( (G_k)_{k \in \mathbb{N}} \) be a sequence of non-zero bounded linear operators on \( \mathcal{H} \) with closed range. We will be particularly interested in the case of finite-dimensional \( \mathcal{H} \) and of \( G_k \) being of finite rank, specifically of rank 1. We write

\[
I_k = (\ker G_k) ^\perp, \quad J_k = \text{range} G_k, \quad \text{for } k \in \mathbb{N},
\]

and denote \( \dim I_k = \dim J_k \) by \( d_k \) in the case that \( G_k \) has finite rank. In the finite rank case, we write, slightly abusing notation, \( G_k ^{-1} : \mathbb{C}^N \to I_k \subseteq \mathbb{C}^N \) for the linear operator defined by \( (G_k |_{I_k \to J_k})^{-1} P_{I_k} \). We fix a Blaschke sequence \( (z_k)_{k \in \mathbb{N}} \) of pairwise distinct elements of \( \mathbb{C}^* \) and the sequence \( (G_k)_{k \in \mathbb{N}} \).

Using matrix-valued Blaschke–Potapov products, we can formulate our generalizations of McPhail’s result [12, Theorem 2 (B)] (see also [23]) for vector-valued \( H^2 \) spaces. For \( n \in \mathbb{N} \) let

\[
m_n = \sup_{a \in \mathcal{B}_{n-1} \cap J_k, \|a\|_2 \leq 1} \inf \{ \|f\|_2 : f \in H^2(\mathbb{C}^+, \mathcal{H}), G_k f(z_k) = a_k, k = 1, \ldots, n \},
\]

and

\[
m = \sup_{a \in \mathcal{B}_{n-1} \cap J_k, \|a\|_2 \leq 1} \inf \{ \|f\|_2 : f \in H^2(\mathbb{C}^+, \mathcal{H}), G_k f(z_k) = a_k, k \in \mathbb{N} \}.
\]

A weak* compactness argument shows that \( m = \sup_{n \in \mathbb{N}} m_n \). Here is our interpolation result for \( H^2(\mathbb{C}^+, \mathcal{H}) \).

**Theorem 2.7** Let \( (G_k)_{k \in \mathbb{N}}, (I_k)_{k \in \mathbb{N}}, (z_k)_{k \in \mathbb{N}}, (m_n)_{n \in \mathbb{N}} \) be defined as above. Let \( \Theta^L \) be the inner functions associated to the sequence \( (z_k)_{k \in \mathbb{N}} \) and the sequence of subspaces \( (I_k^L)_{k \in \mathbb{N}} \), see Lemma 2.4 for the definition. Then

\[
m = \| \mathcal{J}_\mu \|_{H^2(\mathbb{C}^+, \mathcal{H})}
\]
Lemma 2.9

We require the following intermediate result.

\[ \mu = 2\pi \sum_{k=1}^{\infty} \frac{|2 \text{Re} z_k|^2}{|b_{\infty,k}|^2} \Theta^I(z_k)^* G_k^{-1} (G_k^{-1})^* \Theta^I(z_k) \delta_{z_k} \]

and \( J_\mu \) is the embedding

\[ J_\mu : H^2(\mathbb{C}_+, \mathcal{H}) \to L^2(\mathbb{C}_+, \mathcal{H}, \mu). \]

**Proof**

Let \( \Theta_n^I \) be the inner function associated to \( z_1, \ldots, z_n \) and the sequence of subspaces \( I_1, \ldots, I_n \), and \( \Theta_n^{I^\perp} \) be the inner function associated to \( z_1, \ldots, z_n \) and the sequence of subspaces \( I_1^\perp, \ldots, I_n^\perp \), see Lemma 2.4 for the definition. Using the uniform convergence of \( \Theta_n^I, \Theta_n^{I^\perp} \) on compact subsets of \( \mathbb{C}_+ \), the theorem is a consequence the following finite interpolation result.

**Lemma 2.8** Let \( z_1, \ldots, z_n \) be pairwise distinct points in \( \mathbb{C}_+ \) and let \( G_1, \ldots, G_n \), \( m_n \) be as defined above. Then

\[ m_n = \| J_{\mu_n} \|_{H^2(\mathbb{C}_+, \mathcal{H})} \quad (n \in \mathbb{N}), \]

where

\[ \mu_n = 2\pi \sum_{k=1}^{n} \frac{|2 \text{Re} z_k|^2}{|b_{n,k}|^2} \Theta_n^I(z_k)^* G_k^{-1} (G_k^{-1})^* \Theta_n^I(z_k) \delta_{z_k}, \]

and \( J_{\mu_n} \) is the embedding

\[ J_{\mu_n} : H^2(\mathbb{C}_+, \mathcal{H}) \to L^2(\mathbb{C}_+, \mathcal{H}, \mu_n). \]

**Proof** of Lemma 2.8. For \( a \in \Theta_{k=1}^n J_k \), let

\[ \Phi_a(z) = \sum_{k=1}^{n} b_k(z)^{-1} \frac{z_k + 1}{z_k - 1} (\Theta_{n,k}^{I^\perp})^{-1} G_k^{-1} a_k \quad (z \in \mathbb{C}_+ \setminus \{z_1, \ldots, z_n\}). \]

Recall that \( G_k^{-1} : J_k \to I_k \) and \( (\Theta_{n,k}^{I^\perp})^{-1} : I_k \to \mathcal{H} \). Obviously \( \Phi_a \in L^2(i\mathbb{R}, \mathcal{L}(\mathcal{H})) \).

Let \( F_a = \Theta_n^{I^\perp} \Phi_a \).

We require the following intermediate result.

**Lemma 2.9** \( F_a \in H^2(\mathbb{C}_+, \mathcal{H}), \) and \( G_k F_a(z_k) = a_k \) for \( k = 1, \ldots, n \).

**Proof** of Lemma 2.9. First, we have to show that \( F_a \) extends to a holomorphic function on \( \mathbb{C}_+ \). For this, it is sufficient to show that \( F_a \) extends continuously to \( z_1, \ldots, z_n \). Fix \( k \in \{1, \ldots, n\} \). Recall that \( P_k \Theta_n^{I^\perp}(z_k) = 0 \) and that \( P_k \Theta_n^{I^\perp} \) is analytic on \( \mathbb{C}_+ \), so \( P_k \Theta_n^{I^\perp}(z) = (z - z_k) A_k(z) \), where \( A_k \) is an analytic function on \( \mathbb{C}_+ \) taking values in \( \mathcal{L}(\mathcal{H}, I_k) \).

One verifies with Lemma 2.4 that \( A_k(z_k) = \frac{1}{2 \text{Re} z_k} \Theta_n^{I^\perp}_{n,k} \). So in a sufficiently small neighbourhood of \( z_k \),

\[ P_k \Theta_n^{I^\perp}(z_k) b_k(z) = (z - z_k) A_k(z) b_k(z)^{-1} = (z + z_k) A_k(z), \]
which extends continuously to $z_k$. Since $b_j(z)^{-1}$ is holomorphic in a neighbourhood of $z_k$ for $k \neq j$, it follows that $F_a$ extends continuously to $z_k$. Therefore, $F_a$ defines a holomorphic function on $\mathbb{C}_+$. Since $\Phi_a|_{\mathbb{R}} \in L^2(\mathbb{R}, \mathcal{L}(\mathcal{H}))$ and $\Theta_n^I$ is inner, $F_a \in H^2(\mathbb{C}_+, \mathbb{C}^N)$.

It remains to show that $G_k F_a(z_k) = a_k$ for $k = 1, \ldots, n$.

Notice that

$$G_k F_a(z_k) = G_k P_k F_a(z_k) = G_k A(z_k) \left(2 \Re z_k \frac{1 + z_k}{1 + \bar{z}_k} (\Theta_n^{I^*})^{-1} G_k^{-1} a_k \right) = a_k,$$

by taking the continuous extension of $(z - z_k)A_k(z)\Phi_a$ to $z_k$. This finishes the proof of 2.9.

We can now proceed with the proof of Lemma 2.8. For all $g \in H^2(\mathbb{C}_+, \mathcal{H})$ with $G_k g(z_k) = a_k$, we have $g(z_k) - F_a(z_k) \in I_+^k$. As before, we think of the elements in $H^2(\mathbb{C}_+, \mathcal{H})$ as anti-analytic functions on $\mathbb{C}_+$.

By Lemma 2.4,

$$m_n = \sup_{a \in \oplus_{k=1}^n J_k, ||a||=1} \inf_{f \in H^2(\mathbb{C}_+, \mathcal{H})} \| F_a - \Theta_n^I f \|_2$$

$$= \sup_{a \in \oplus_{k=1}^n J_k, ||a||=1} \inf_{f \in H^2(\mathbb{C}_+, \mathcal{H})} \| \Phi_a - f \|_2$$

$$= \sup_{a \in \oplus_{k=1}^n J_k, ||a||=1} \| \Phi_a \|_2 \| \Phi_a \|_{L^2(\mathbb{R}, \mathcal{H})/H^2(\mathbb{C}_+, \mathcal{H})}$$

$$= \sup_{f \in H^2(\mathbb{C}_+, \mathcal{H}), ||f||=1} \left| \langle \Phi_a, f \rangle \right|$$

$$= \sup_{f \in H^2(\mathbb{C}_+, \mathcal{H}), ||f||=1} \left| \sum_{k=1}^n b_k(z) \frac{1 + z_k}{1 + \bar{z}_k} (\Theta_n^{I^*})^{-1} G_k^{-1} a_k, f \right|$$

Here, we have used Corollary 2.6 in the ante-penultimate equation. This finishes the proof of 2.8.
We can instead consider a Carleson embedding for a simpler measure, restricted to an invariant subspace of the shift operator:

**Corollary 2.10** We have

\[
m = \|\mathcal{J}_\mu \Theta^H(\mathbb{C}_+, \mathcal{H})\|,
\]

where

\[
\mu = 2\pi \sum_{k=1}^{\infty} \frac{|2 \text{Re} z_k|^2}{|b_{\infty,k}|^2} G_k^{-1}(G_k^{-1})^* \delta_{z_k},
\]

**Proof** This follows immediately from Theorem 2.7.

We have thus reduced the interpolation problem to boundedness of an operator-weighted Carleson embedding. In the finite-dimensional case, this reduces to a Carleson condition:

**Theorem 2.11** Let \( \mathcal{H} = \mathbb{C}^N \), let \((G_k)_{k \in \mathbb{N}}, (I_k)_{k \in \mathbb{N}}, (z_k)_{k \in \mathbb{N}}, \) be defined as above, and let \( E \) be the evaluation operator

\[
f \mapsto E(f) = (G_k f(z_k))_{k \in \mathbb{N}}.
\]

Then \( E(H^2(\mathbb{C}_+, \mathbb{C}^N)) \supseteq \ell^2(I_k) \), if and only if the scalar measure

\[
\sum_{k=1}^{\infty} \frac{|2 \text{Re} z_k|^2}{|b_{\infty,k}|^2} \|(G_k^{-1})^* \Theta^I(z_k)\|^2 \delta_{z_k}
\]

is a Carleson measure.

**Proof** A weak* compactness argument shows that \( E(H^2(\mathbb{C}_+, \mathbb{C}^N)) \supseteq \ell^2(I_k) \) if and only if \((m_n)\) is bounded. The remainder follows directly from the comparison of the norm and the trace of a positive matrix, Theorem 2.7 and Theorem 2.2.

With Theorem 2.3, we obtain

**Corollary 2.12** If \( \mathcal{H} \) is a separable Hilbert space and

\[
\sum_{k=1}^{\infty} \frac{|2 \text{Re} z_k|^2}{|b_{\infty,k}|^2} \|(G_k^{-1})^* \Theta^I(z_k)\|^2 \delta_{z_k},
\]

is a scalar Carleson measure, then \( E(H^2(\mathbb{C}_+, \mathcal{H})) \supseteq \ell^2(I_k) \).

### 2.4 Some estimates for Blaschke products

The difficulty in the application of Theorem 2.11 is the computation of the values of the Blaschke–Potapov products \( \Theta^I(z_k) \) and of \( \|G_k^{-1} \Theta^I(z_k)\| \). Corollary 2.10 avoids this and therefore yields at once simple upper and lower estimates for \((m_n, n \in \mathbb{N})\). Although the following estimate is rather coarse, it will be applied to controllability questions in Section 3 to obtain quite precise estimates. Let

\[
\gamma_n = \inf \{\|F\|_\infty : F \in H^\infty(\mathbb{C}_+, \mathcal{H}), \|F(z_k)\| = 1, F(z_k) \in I_k, k = 1, \ldots, n\}.
\]
Corollary 2.13

\[ m_n \lesssim \text{Carl}(\sum_{k=1}^{n} \delta_{z_k} \frac{|2 \Re z_k|^2 \|G_k^{-1}\|^2}{|b_{n,k}|^2})^{1/2}, \quad \text{and} \]

\[ m_n \gtrsim \gamma_n^{-1} \text{Carl}(\sum_{k=1}^{n} \delta_{z_k} \frac{|2 \Re z_k|^2}{|b_{n,k}|^2 \|G_k\|^2})^{1/2}. \]

Proof The first estimate follows directly from Theorem 2.7. For the second, let \( \varepsilon > 0 \), and choose \( F \in H^\infty(\mathbb{C}_+, \mathcal{H}) \) with \( \|F(z_k)\| = 1 \) and \( F(z_k) \in I_k \), \( k = 1, \ldots, n \), \( \|F\|_{\infty} \leq \gamma_n + \varepsilon \). Then

\[ \{(\gamma_n + \varepsilon)^{-1}F\hat{f} : \hat{f} \in H^2(\mathbb{C}_+), \|\hat{f}\| \leq 1\} \subseteq \{f \in \Theta_n^j H^2(\mathbb{C}_+, \mathbb{C}^N) : \|f\| \leq 1\}. \]

Taking the limit as \( \varepsilon \to 0 \), we obtain \( \|J_{\mu_n} |\Theta_n^j H^2(\mathbb{C}_+, \mathbb{C}^N)\| \geq \gamma_n^{-1} \|J_{\nu_n}\| \), where

\[ \nu_n = \sum_{k=1}^{n} \delta_{z_k} \frac{|2 \Re z_k|^2}{|b_{n,k}|^2 \|G_k\|^2} \]

and \( J_{\nu_n} : H^2(\mathbb{C}_+) \to L^2(\mathbb{C}_+, \nu_n) \) is the scalar Carleson embedding; now the result follows from the scalar Carleson embedding theorem 2.1.

In the case that \( (z_k) \) is the union of \( K \) Carleson sequences, and for an estimate up to constants of the \( m_n \), \( \Theta_n^j(z_k) \) can be replaced by a Blaschke–Potapov product with at most \( K \) factors, \( \Theta_n^{j,r}(z_k) \) (where the factors correspond to the \( z_j \) in a suitably small hyperbolic \( r \)-neighbourhood of \( z_k \)), see also Corollary 2.20 and Theorem 2.24.

Corollary 2.14 Let \( (z_k) \) be the union of \( K \) (not necessarily disjoint) Carleson sequences and let \( r > 0 \) be such that each of the Carleson sequences is \( r \)-separated in the hyperbolic metric. For \( k \in \mathbb{N} \), define \( \Theta_{n,z_k,r}^j \) as the Blaschke–Potapov product associated to the shift-invariant subspace

\[ \mathcal{L}_{n,z_k,r} = \{f \in H^2(\mathbb{C}_+, \mathbb{C}^N) : f(z_j) \in I_j \text{ for all } z_j \text{ with } d(z_j, z_k) < r/2, j \leq n\}. \]

Then

\[ m_n \approx \|J_{\mu_{n},i,r} |H^2(\mathbb{C}_+, \mathbb{C}^N) \to L^2(\mathbb{C}_+, \mathcal{H}, \mu_{n},i,r)\| \quad (n \in \mathbb{N}), \]

where

\[ \mu_{n,i,r} = \sum_{k=1}^{n} \frac{|2 \Re z_k|^2}{|b_{n,k}|^2 \|G_k^{-1}\| \Theta_{n,z_k,r}^j(z_k)} \delta_{z_k} \]

and \( J_{\mu_{n,i,r}} \) is the associated Carleson embedding.

Proof Fix \( k \in \mathbb{N} \). Note that because of the uniqueness property of the inner function in the Beurling–Lax Theorem (see e.g. [21], page 99), \( \Theta_{n}^j(z_k) \Theta_{n}^j(z_k)^* \) and therefore \( \|G_k^{-1} \Theta_{n}^j(z_k)\| \) does not depend on the order in which the Blaschke factors are chosen in the construction. We can therefore start the construction from Lemma 2.4 with the term associated to \( z_k \), then continue with the terms
associated to the $z_j, d(z_j, z_k) < r/2$, and finally add the terms corresponding to the remaining $z_j$. In this way, we can write
\[
(\Theta^I_n(z_k))^* \Theta^I_n(z_k) = (\Lambda^I_{n,r}(z_k))^* (\Theta^I_{n,z_k,r}(z_k))^* \Theta^I_{n,z_k,r}(z_k) \Lambda^I_{n,r}(z_k)
\]
where $\Lambda^I_{n,r}(z_k)$ is a contraction which is bounded below with a lower bound depending only on $r$ and $K$. Therefore,
\[
\|G_k^{-1} \Theta^I_n(z_k)\|^2 = \| (\Lambda^I_{n,r}(z_k))^* (\Theta^I_{n,z_k,r}(z_k))^* G_k^{-1} \Theta^I_{n,z_k,r}(z_k) \Lambda^I_{n,r}(z_k) \|
\approx \|G_k^{-1} \Theta^I_n(z_k)\|^2.
\]
The remainder of the corollary follows from Theorems 2.7 and 2.11. ■

2.5 Angles between subspaces

In the case that $G_k^* G_k$ is a scalar multiple of $P_k$ (e.g. the rank 1 case), we have
\[
\|G_k^{-1} \Theta^I_n(z_k)\|^2 = \frac{1}{\|G_k\|^2} \|P_k \Theta^I_n(z_k)\|^2 = \frac{1}{\|G_k\|^2} \|\Theta^I_n(z_k)\|^2,
\]
whereas, in general, we just have the inequality
\[
\frac{1}{\|G_k\|^2} \|\Theta^I_n(z_k)\|^2 \leq \|G_k^{-1} \Theta^I_n(z_k)\|^2 \leq \|G_k^{-1}\|^2 \|\Theta^I_n(z_k)\|^2. \tag{5}
\]
In case dim $\mathcal{H} < \infty$, it is therefore sufficient to compute $\|\Theta^I_n(z_k)\|^2$ in the above setting. Since the Blaschke–Potapov product is not easy to compute, the following expression in terms of angles between certain subspaces and of the Gramian of the system $(k_{z_k} I_k)$ is sometimes useful. Indeed, the controllability results in Section 3 will use this form.

Let $n \in \mathbb{N}$. For $k = 1, \ldots, n$, we write
\[
K_{k,I} = ((b_k P_k + P_{k-1}) H^2(\mathbb{C}_+, \mathbb{C}^N))^\perp = k_{z_k} I_k,
K'_{k,I,n} = \text{span}\{k_{z_j} I_j : j = 1, \ldots, n, j \neq k\} = (\Theta^I_{k,n})^* H^2(\mathbb{C}_+, \mathbb{C}^N))^\perp
\]
and
\[
K'_{k,I} = \text{span}\{k_{z_j} I_j : j \in \mathbb{N}, j \neq k\} = (\Theta^I_{k,n})^* H^2(\mathbb{C}_+, \mathbb{C}^N))^\perp.
\]
where $\Theta^I_{k,n}$ is the Blaschke–Potapov product as in Lemma 2.4 corresponding to $\{z_j, j = 1, \ldots, n, j \neq k\}$, and $\Theta^I_{k,n}$ is the infinite Blaschke–Potapov product corresponding to $\{z_j, j \in \mathbb{N}, j \neq k\}$.

If the subspaces $(I_k)$ are all finite-dimensional, we have for the system of subspaces $(k_{z_k} I_k)_{k \in \mathbb{N}}$ the Gramian
\[
\mathcal{G}_n = (b_i(z_j) P_{j,i} P_{i,j})_{1 \leq i,j \leq n} : \oplus_{j=1}^n I_j \rightarrow \oplus_{i=1}^n I_i.
\]
Invertibility of the Gramian $\mathcal{G}_n$ is equivalent to unconditionality of the system of subspaces $(k_{z_j} I_j)_{j \leq n}$, see [14], page 140.
Now \(G_n\) and \(G_{n,k}\) are \(d^{(n)}\) respectively \(d_k^{(n)}\) dimensional square matrices, \(d^{(n)} = d_1 + \cdots + d_n\), \(d_k^{(n)} = d_k + \cdots + d_{k-1} + d_{k+1} + \cdots + d_n\), and therefore have well-defined determinants. Since the \(z_j\) are pairwise distinct, the spaces \((k_j, I_j)_{j \leq n}\) are linearly independent and therefore unconditional in \(H^2(\mathbb{C}_+,\mathcal{H})\) for any finite \(n\), and the determinants of \(G_n\) and \(G_{n,k}\) are nonzero. Here is our identity for \(\|\Theta_n^I(z_k)\|^2\).

**Lemma 2.15** For \(1 \leq k \leq n\),

\[
\frac{1}{(\sin \angle(\mathcal{K}_{k,I}, \mathcal{K}_{I,n}))^2} = \frac{||\Theta_n^I(z_k)||^2}{b_{nk}^2}.
\]

If the space \(I_k\) is one-dimensional, then we also have

\[
\frac{||\Theta_n^I(z_k)||^2}{b_{nk}^2} = \frac{\det \mathcal{G}_{n,k}}{\det \mathcal{G}_n}.
\]

**Proof** For the first equality, we start with the same argument as in the scalar case in [14, p. 239].

\[
\begin{align*}
\sin \angle(\mathcal{K}_{k,I}, \mathcal{K}_{I,n}))^2 &= \inf_{f \in \mathcal{K}_{k,I}: ||f|| = 1} ||P_{\mathcal{K}_{k,I,n}, f}||^2 \\
&= \inf_{e \in \mathcal{I}_{k,I}: ||e|| = 1} ||P_{+} \Theta_{k,n}^I e^* k_{z_k}||^2 \\
&= \inf_{e \in \mathcal{I}_{k,I}: ||e|| = 1} \sup_{g \in H^2(\mathbb{C}_+, \mathcal{H}), ||g|| = 1} |\langle \Theta_{k,n}^I e^* k_{z_k}, g \rangle|^2 \\
&= \inf_{e \in \mathcal{I}_{k,I}: ||e|| = 1} \sup_{g \in H^2(\mathbb{C}_+, \mathcal{H}), ||g|| = 1} \Re z_k |\langle e, \Theta_{k,n}^I(z_k)g(z_k) \rangle|^2 \\
&= \inf_{e \in \mathcal{I}_{k,I}: ||e|| = 1} ||\Theta_{k,n}^I(z_k)^{-1} (z_k)e||^2 b_{nk}^2.
\end{align*}
\]

We can think of the term associated with \(z_k\) as being the last in the Blaschke–Potapov product \(\Theta_n^I\). Then \(\Theta_n^I(z_k) = \Theta_{n,k}^I(z_k)P_{I_k}\) with \(I_k = \Theta_{n,k}^I(z_k)^{-1} I_k\), and we obtain

\[
\frac{b_{nk}^2}{(\sin \angle(\mathcal{K}_{k,I}, \mathcal{K}_{I,n}))^2} = \sup_{e \in \mathcal{I}_k} \frac{||e||^2}{||\Theta_{k,n}^I(z_k)e||^2} = \sup_{\tilde{e} \in \mathcal{I}_k} \frac{||\Theta_{k,n}^I(z_k)\tilde{e}||^2}{||\tilde{e}||^2} = \|\Theta_{k,n}^I(z_k)\|^2 P_{I_k} = ||\Theta_n^I(z_k)||^2.
\]

For the second equality, recall that \(\mathcal{G}_n = J_n^* \mathcal{J}_n\), where \(\mathcal{J}_n\) is the embedding map

\[J_n : \oplus_{j=1}^n I_j \to H^2(\mathbb{C}_+, \mathbb{C}_N), (v_1, \ldots, v_n) \mapsto \sum_{j=1}^n v_j k_{z_j}.\]
Choosing an orthonormal basis \(e_{1,j}, \ldots, e_{d_j,j}\) of each of the \(I_j\), we find that

\[
\det G_n = (\text{vol}_{d(n)}(\tilde{k}_{z_1}e_{1,1}, \ldots, \tilde{k}_{z_1}e_{d_1,1}, \ldots, \tilde{k}_{z_n}e_{1,n}, \ldots, \tilde{k}_{z_n}e_{d,n}))^2
\]

and

\[
\det G_{n,k} = (\text{vol}_{d(n)}(\tilde{k}_{z_1}e_{1,1}, \ldots, \tilde{k}_{z_{k-1}}e_{1,k-1}, \tilde{k}_{z_{k-1}}e_{d_{k-1},k-1}, 
\tilde{k}_{z_{k+1}}e_{1,k+1}, \ldots, \tilde{k}_{z_n}e_{d_n,n}))^2.
\]

If \(\dim I_k = 1\), then an elementary geometric argument shows that

\[
\text{vol}_{d(n)}(\tilde{k}_{z_1}e_{1,1}, \ldots, \tilde{k}_{z_1}e_{d_1,1}, \ldots, \tilde{k}_{z_n}e_{1,n}, \ldots, \tilde{k}_{z_n}e_{d_n,n})
= \text{vol}_{d(n)}(\tilde{k}_{z_1}e_{1,1}, \ldots, \tilde{k}_{z_{k-1}}e_{1,k-1}, \tilde{k}_{z_{k-1}}e_{d_{k-1},k-1}, 
\tilde{k}_{z_{k+1}}e_{1,k+1}, \ldots, \tilde{k}_{z_n}e_{d_n,n})
\times |\sin \angle(k_{z_1}e_{1,k}, K_{k,I}, I)|,
\]

which yields the required result.

\[\blacksquare\]

**Corollary 2.16** If \(N = \dim \mathcal{H} < \infty\) and \((z_k)\) is not a Blaschke sequence, then there exists \(k \in \mathbb{N}\) such that \(\angle(K_{k,I}, K'_{k,I}) = 0\).

**Proof** We will use the following facts:

(1) (Beurling–Lax Theorem, see e.g. [19], Thm 3.1.7. For uniqueness, see e.g. [21], page 99.) If \(\mathcal{L} \subseteq H^2(\mathbb{C}_+, \mathbb{C}^N)\) is a shift-invariant closed subspace, then there exists \(0 \leq r \leq N\) and a matrix-valued inner function \(\Theta : \mathbb{C}_+ \to \mathcal{L}(\mathbb{C}^r, \mathbb{C}^N)\) such that \(\mathcal{L} = \Theta H^2(\mathbb{C}_+, \mathbb{C}^r)\). The inner function \(\Theta\) is unique up to a constant unitary matrix factor \(U : \mathbb{C}^r \to \mathbb{C}^r\).

(2) If \(\mathcal{L} \subseteq H^2(\mathbb{C}_+, \mathbb{C}^N)\) is a shift-invariant closed subspace and \(\Theta\) is the corresponding matrix-valued inner function as in Part 1, then the set \(\{z \in \mathbb{C}_+ : \text{rank } \Theta(z) < r\}\) forms a Blaschke sequence.

This can easily be seen as follows: Since \(\Theta(i\omega)\) has rank \(r\) almost everywhere on the imaginary axis \(i\mathbb{R}\), there exists a \(r \times r\) submatrix \(\Theta_r\) such that \(\Theta_r(i\omega)\) has rank \(r\) on a subset of positive measure of \(i\mathbb{R}\). It follows that \(\text{det}(\Theta_r)\) is a nontrivial function in \(H^\infty(\mathbb{C}_+)\), the zero set of which is a Blaschke sequence. Thus for all \(z \in \mathbb{C}_+\) which do not appear as terms of this Blaschke sequence, one has \(\text{rank } \Theta(z) \geq \text{rank } \Theta_r(z) = r\).

To prove the corollary, suppose that \((z_j)\) is not a Blaschke sequence. Let \(\mathcal{L} = \{f \in H^2(\mathbb{C}_+, \mathbb{C}^N) : f(z_j) \in I_j^+\\text{ for all } j \in \mathbb{N}\}\), and let \(\Theta : \mathbb{C}_+ \to \mathcal{L}(\mathbb{C}^r, \mathbb{C}^N)\) be the corresponding matrix-valued inner function. By (2), there exists \(k \in \mathbb{N}\) such that \(\text{rank } \Theta(z_k) = r\).

Now let \(D_k(z) = b_k(z)P_k + P_k^+\), \(I_j^+ = (b_k(z)P_k + P_k^+)^{-1}I_j^+\), \(\tilde{I}_j = (\tilde{I}_j^+)^{-1}\) for all \(j \neq k\). Let \(\tilde{L}_k = \{g \in H^2 : g(z_j) \in \tilde{I}_j^+\\text{ for all } j \neq k\}\), and let \(\Theta : \mathbb{C}_+ \to \mathcal{L}(\mathbb{C}^r, \mathbb{C}^N)\) be the corresponding matrix-valued inner function. Then one sees
easily that \( D_k \mathcal{L}_k = D_k \Theta' H^2(\mathbb{C}^r) = \Theta H^2(\mathbb{C}_+, \mathbb{C}^r) = \mathcal{L} \), and it follows that \( r = r' \), \( D_k \Theta' = \Theta U \) for some fixed unitary matrix \( U : \mathbb{C}^r \to \mathbb{C}^r \). Redefining \( \Theta \), we can assume that \( D_k \Theta' = \Theta \). In particular, \( \Theta(z_k) = P_{I_k}^r \Theta'(z_k) \). Since \( \text{rank } \Theta(z_k) = r \) and \( \text{rank } \Theta'(z_k) \leq r \), it follows that \( \text{rank } \Theta'(z_k) = r \). From \( \mathcal{L}_k \supseteq \mathcal{L} \) we obtain that \( \text{range } \Theta'(z_k) \supseteq \text{range } \Theta(z_k) \), therefore the ranges are equal. Thus \( P_{I_k}^r \Theta'(z_k) = 0 \). It follows that for each \( f \in \mathcal{L}_k \), \( P_{I_k}^r f = 0 \), and \( \angle(\mathcal{K}_k, I_k) = 0 \). 

We return to our description of \( \lVert G_k^{-1*} \Theta_n^I(z_k) \rVert^2 \) in terms of angles between subspaces. The case of more general \( G_k \) can be described in the same framework as above.

We will represent the \( G_k \) in a form in which they frequently appear in applications. Let \( G_k : \mathbb{C}^N \to \mathbb{C}^N \) be given by \( G_k x = \sum_{i=1}^{d_k} \langle x, g_{k,i} \rangle j_{k,i} \), where the \((j_{k,i})_{i=1, \ldots, d_k}\) form an orthonormal basis of \( J_k \).

Since we are only interested in the norm of \( G_k^{-1*} \Theta^I(z_k) \), we can multiply by a unitary matrix from the left and assume that \( j_{k,i} = e_i \) for \( i = 1, \ldots, d_k \), \( k \in \mathbb{N} \), where \((e_i)_{i=1, \ldots, N}\) denotes the standard basis of \( \mathbb{C}^N \).

That means, for each \( k \in \mathbb{N} \), the operator \( G_k \) is given as

\[
G_k = \begin{pmatrix}
g_{k,1}^* \\
\vdots \\
g_{k,d_k}^* \\
0 \\
\vdots \\
0
\end{pmatrix},
\]  

(8)

and, recalling that \( d_k = \dim I_k \) is the rank of \( G_k \), \((g_{k,i})_{i=1, \ldots, d_k}\) is a basis of \( I_k \).

Let \( k \in \mathbb{N} \) be fixed. Let \((v_{k,i})_{i=1, \ldots, d_k}\) be the dual basis of \( I_k \) to \((g_{k,i})_{i=1, \ldots, d_k}\), that means, \( \langle g_{k,i}, v_{k,l} \rangle = \delta_{il} \) for \( i, l = 1, \ldots, d_k \).

It is easy to see that

\[
G_k^{-1*} x = \sum_{i=1}^{d_k} \langle x, v_{k,i} \rangle e_i \text{ for } x \in \mathbb{C}^N.
\]

Writing \( G_{k,i}^{-1*} = \langle \cdot, v_{k,i} \rangle e_i \), \( G_k^{-1*} = \sum_{i=1}^{d_k} G_{k,i}^{-1*} \), and using that the \( G_{k,i}^{-1*} \) have orthogonal ranges, one obtains (for example by passing to the Hilbert–Schmidt norm) that

\[
\lVert G_k^{-1*} \Theta_n^I(z_k) \rVert^2 \approx \sum_{i=1}^{d_k} \lVert G_{k,i}^{-1*} \Theta_n^I(z_k) \rVert^2,
\]  

(9)

with an equivalence constant depending only on the dimension \( N \).

We can again appeal to the uniqueness property of the inner function in the Beurling–Lax Theorem to see that \( \Theta_n^I(z_k) \Theta_n^I(z_k)^* \) does not depend on the order in which the Blaschke factors are chosen in the construction, and therefore assume that the factor associated to \( z_k \) is the first one appearing in the construction of \( \Theta_n^I \).
Thus we are left to determine

\[ \| G_{k,i}^{-1} \left( b_n(z_k) P_{t_i} + P_{l_i} \right) \tilde{\Theta}_n^f(z_k) \| , \]

where \( \tilde{\Theta}_n^f(z_k) \) stands for the remaining Blaschke–Potapov factors in \( \Theta_n^f(z_k) \).

Writing \( P_{v_{k,i}} \) for the orthogonal projection on \( \mathbb{C} v_{k,i} \), we obtain

\[
\begin{align*}
\| G_{k,i}^{-1} & (b_n(z_k) P_{t_i} + P_{l_i}) \tilde{\Theta}_n^f(z_k) \| \\
& = \| v_{k,i} \| P_{v_{k,i}} P_{l_i} \tilde{\Theta}_n^f(z_k) \| \\
& = \| G_{k,i}^{-1} v_i \| \| (b_n(z_k)P_{v_{k,i}} + P_{v_{k,i}}) \tilde{\Theta}_n^f(z_k) \| \\
& = \| G_{k,i}^{-1} v_i \| \| \Theta_n^{I(k,i)}(z_k) \| ,
\end{align*}
\]

where

\[ I^{(k,i)}_j = I_j \text{ for } j \neq k \text{ and } I^{(k,i)}_k = \mathbb{C} v_{k,i} = \text{span}\{I_k^\perp \cup g_{k,i}^\perp \text{ s.t. } 1 \leq j \leq d_k, j \neq i \}^\perp. \]

Applying Lemma 2.15 to \( \Theta_n^{I(k,i)} \), we obtain

\[
\frac{\| G_{k,i}^{-1} \Theta_n^f(z_k) \|^2}{|b_{n,k}|^2} = \frac{\| G_{k,i}^{-1} v_i \|^2}{(\sin (\angle (k_z, C v_{k,i}, K_{k,n}'))^2)} \approx \frac{\| G_{k,i}^{-1} v_i \|^2}{(\sin (\angle (k_z, \text{span}\{I_k^\perp \cup g_{k,j}^\perp \text{ s.t. } 1 \leq j \leq d_k, j \neq i \}^\perp, K_{k,I,n}'))^2)}.
\]

(10)

Altogether, we have proven

**Lemma 2.17** Suppose that \( N = \dim \mathcal{H} < \infty \), and suppose that for each \( k \in \mathbb{N} \), the operator \( G_k : \mathbb{C}^N \to \mathbb{C}^N \) is given by (8). Then

\[
\frac{\| (G_k^{-1})^* \Theta_n^f(z_k) \|^2}{|b_{n,k}|^2} \approx \sum_{i=1}^{d_k} \frac{\| G_{k,i}^{-1} v_i \|^2}{|\angle (k_z, V_{k,i}^G, K_{k,I,n}'))^2|},
\]

where

\[ V_{k,i}^G = \text{span}\{g_{k,i}, \ldots, g_{k,d_k}\}^\perp \cup \text{span}\{g_{k,j}^\perp \text{ s.t. } 1 \leq j \leq d_k, j \neq i \}^\perp \]

for \( 1 \leq i \leq d_k \). The equivalence constant depends only on \( N \).

We obtain some interesting consequences of Lemmas 2.15 and 2.17.

**Theorem 2.18** Suppose that \( N = \dim \mathcal{H} < \infty \). Suppose that there is a sequence of positive real numbers \((\alpha_k)\) such that with the above notation, \( G_k^* G_k = \alpha_k^2 P_{l_k} \) for all \( k \in \mathbb{N} \). Then

\[
m = \sup \{ m_n \approx \text{Carl}(\sum_{k=1}^{\infty} \frac{(\text{Re} z_k)^2}{\alpha_k^2 |\angle (k_z, K_{k,I}, K_{k,I})|^2} \delta_{z_k})^{1/2} \}
\]

with equivalence constant depending only on \( N \).
Remark 2.19 This theorem implies in particular that uniform minimality and unconditionality are the same for the system of subspaces \((k_z I_k)_{k \in \mathbb{N}}\) in \(H^2(\mathbb{C}+)\).

Namely, given a uniformly minimal system \((k_z I_k)_{k \in \mathbb{N}}\), the sequence \((k_z)\) is a finite union of uniformly minimal sequences in \(H^2(\mathbb{C}+)\); therefore \((z_k)\) is a finite union of Carleson sequences, and \(\sum_{k=1}^{\infty} \text{Re} z_k \delta_{z_k}\) is a Carleson measure (see e.g. [14], Lecture VII). Hence the evaluation map \(E : H^2(\mathbb{C}+, \mathbb{C}^N) \to \ell^2(I_k), f \mapsto (P_{I_k}(\text{Re} z_k)^{1/2} f(z_k))\), is bounded and surjective by Theorem 2.18 and the classical Carleson Embedding Theorem, and its adjoint \(E^* : \ell^2(I_k) \to H^2(\mathbb{C}+, \mathbb{C}^N), (x_k) \mapsto \sum_{k=1}^{\infty} x_k(\text{Re} z_k)^{1/2} z_k\), is bounded and bounded below, which means that the system \(k_z I_k\) is unconditional.

This fact also follows from a more general result, due to Treil [24], [25]; namely, that a system of invariant subspaces of the backward shift operator on the vector-valued Hardy space \(H^2(\mathbb{C}+, \mathcal{H})\) is a Riesz system if and only if it is uniformly minimal. This result even holds for systems of invariant subspaces of completely a completely non-unitary Hilbert space contraction \(T\) with finite defect and co-defect [26].

Note that in the half-plane any reproducing kernel \(k_z\) is an eigenvector of a backward shift \(S^*\) on \(H^2(\mathbb{C}+)\); where \(S\), the operator of multiplication by \((1-z)/(1+z)\), is a shift of finite multiplicity.

Proof of Theorem 2.18 If \(G_k^* G_k = \alpha_k^2 P_{I_k}\), then \(G_k^{-1} G_k^{-1*} = \alpha_k^{-2} P_{I_k}\) and \(\|G_k^{-1*} \Theta_n^i(z_k)\|^2 = \frac{1}{|\alpha_k|^2} \|\Theta_n^j(z_k)\|^2\). So by the matrix Carleson embedding theorem, it is sufficient to use the identity

\[
\frac{1}{(\sin|\angle(K_{k,I}, K_{k,I,n}'))|^2} = \frac{\|\Theta_n^j(z_k)\|^2}{|b_{nk}|^2},
\]

from Lemma 2.15.

For the second part of the theorem, it is sufficient to prove a continuity property of the angle:

\[
\sin \angle(K_{k,I}, K_{k,I}') = \lim_{n \to \infty} \sin \angle(K_{k,I}, K_{k,I,n}'),
\]

This follows again from the argument in [14, p. 239] and from the normal convergence of the \(\Theta_n^{1,2}\) to an inner function \(\Theta_n^{1,2} = (\Theta_n^{1,2} H^2(\mathbb{C}+, \mathcal{H}))^\perp = K_{k,I}'\).

The Carleson Embedding Theorem and the Monotone Convergence Theorem now imply the result.

For the case of \((z_k)\) being the union of \(K\) Carleson sequences, we get the following application of Corollary 2.14.

Corollary 2.20 Let \(G_k^* G_k = \alpha_k^2 P_{I_k}\) for all \(k\), let \((z_k)\) be the union of \(K\) Carleson sequences, and let \(r > 0\) be such that each of the Carleson sequences is \(r\)-separated in the hyperbolic metric. For \(k \in \mathbb{N}\), define

\[
K_{k,I,r,n} = \text{span}\{k_{z_j} I_j : j = 1, \ldots, n, j \neq k, d(z_j, z_k) < r/2\}.
\]
Then
\[ m = \sup_{n \in \mathbb{N}} m_n \approx \mathrm{Carl}(\sum_{k=1}^{\infty} \frac{(\text{Re} z_k)^2}{\alpha_k^2 | \angle (K_{k,i}, K_{k,i,r}) |^2} \delta_{z_k})^{1/2} \]
with equivalence constant depending only on \( N, r, K \). Here, we define
\[ K_{k,i,r} = \text{span}\{ k_j I_j : j \in \mathbb{N}, j \neq k, d(z_j, z_k) < r/2 \} \quad (11) \]
and \( |\angle (K_{k,i}, K_{k,i,r}) | = \pi/2, \) if \( \{ z_j : j = 1, \ldots, n, j \neq k, d(z_j, z_k) < r/2 \} = \emptyset \).

**Proof** This follows from the proof of Theorem 2.18, Corollary 2.14, the Carleson Embedding Theorem, and the Monotone Convergence Theorem. \( \blacksquare \)

In general, computation of the sin \( \angle (K_{k,i}, K_{k,i,r}) \) requires again the computation of the Blaschke–Potapov product as in Lemma 2.4 for the subspace \( \{ f \in H^2(\mathbb{C}, \mathbb{C}^N) : f(z_j) \in I_j \text{ for all } j \text{ with } 0 < d(z_j, z_k) < r/2 \} \). However, at least for the case \( K = 2 \), we get a workable description up to a constant of \( m \):

**Example 2.21** Let \( N = \dim \mathcal{H} < \infty \), let \( (z_k) \cup (z_k') \) be the union of two Carleson sequences and let \( r > 0 \) such that each of the Carleson sequences is \( r \)-separated in hyperbolic metric. For \( k \in \mathbb{N} \), define
\[ \beta_k = \begin{cases} 1 & \text{if } \{ z_j : 0 < d(z_j, z_k) < r/2 \} = \emptyset, \\ \left( |b_k(z_k')|^2 + |\sin \angle (I_k, I_k')|^2 \right)^{1/2} & \text{if } \exists k' \in \mathbb{N} \text{ with } 0 < d(z_k', z_k) < r/2. \end{cases} \]
Then
\[ m \approx \mathrm{Carl}(\sum_{k=1}^{\infty} \frac{(\text{Re} z_k)^2}{\alpha_k^2 \beta_k^2} \delta_{z_k})^{1/2} \]
with equivalence constant depending only on \( N, r \). \( \blacksquare \)

Let us now consider the case where \( G_k^{-1} G_k^{-1*} \) is not a multiple of a projection.

**Corollary 2.22** Suppose that \( N = \dim \mathcal{H} < \infty \), and suppose that for each \( k \in \mathbb{N} \), the operator \( G_k : \mathbb{C}^N \to \mathbb{C}^N \) is given by (8). Then
\[ m = \sup_{n \in \mathbb{N}} m_n \approx \mathrm{Carl}(\sum_{k=1}^{\infty} \sum_{\ell=1}^{d_k} \frac{(\text{Re} z_k)^2 \| G_k^{-1} c_\ell \|^2}{\| z_k V_{k,i}^G \|_{K_{k,i}, K_{k,i}'}^2} \delta_{z_k})^{1/2} \]
with equivalence constant depending only on \( N \). As above,
\[ V_{k,i}^G = \text{span}\{ \text{span}\{ g_{k,1}, \ldots, g_{k,d_k} \}, \text{span}\{ g_{k,j} : 1 \leq j \leq d_k, j \neq i \} \} \]
for \( 1 \leq i \leq d_k \).

**Proof** It only remains to apply (10) and (9) to Theorem 2.11, and to use continuity of the angle. \( \blacksquare \)

In an infinite-dimensional version, the Hilbert–Schmidt estimates employed above do not work any more, and we only have an upper bound for \( m \) from Theorem 2.3.
Corollary 2.23 With the notation as above, \( \mathcal{H} \) a separable Hilbert space,

\[
m \lesssim \text{Carl} \left( \sum_{k=1}^{\infty} (\text{Re } z_k)^2 \| G_k^{-1} \|^2 \frac{\angle(\mathcal{K}_{k,I}, \mathcal{K}_{k,I})}{\| \delta z_k \|^{1/2}} \right).
\]

Proof By Theorem 2.7, we have to estimate the norm of the vector Carleson embeddings

\[
\mathcal{J}_{\mu_n} : H^2(\mathbb{C}_+, \mathcal{H}) \to L^2(\mathbb{C}_+, \mathcal{H}, \mu_n),
\]
where \( \mu_n = \sum_{k=1}^{n} \frac{|2 \text{Re } z_k|^2}{|b_{n,k}|^2} \Theta_n^{I}(z_k)^* G_k^{-1}(G_k^{-1})^* \Theta_n^{I}(z_k) \delta z_k \). By Theorem 2.3, it is sufficient to estimate

\[
\text{Carl} \left( \sum_{k=1}^{n} \frac{|2 \text{Re } z_k|^2}{|b_{n,k}|^2} \| G_k^{-1} \|^2 \| \Theta_n^{I}(z_k) \|^2 \delta z_k \right)
\]

\[
\leq \text{Carl} \left( \sum_{k=1}^{n} \frac{|2 \text{Re } z_k|^2}{|b_{n,k}|^2} \| G_k^{-1} \|^2 \| \Theta_n^{I}(z_k) \|^2 \delta z_k \right)
\]

\[
= \text{Carl} \left( \sum_{k=1}^{n} \frac{(\text{Re } z_k)^2 \| G_k^{-1} \|^2}{\sin(\angle(\mathcal{K}_{k,I}, \mathcal{K}_{k,I,n})^2)} \delta z_k \right)
\]

\[
\approx \text{Carl} \left( \sum_{k=1}^{n} \frac{(\text{Re } z_k)^2 \| G_k^{-1} \|^2}{\| (\mathcal{K}_{k,I}, \mathcal{K}_{k,I,n})^2 \|} \delta z_k \right)
\]

where the last equality follows from (6). The continuity property of the angle for the estimate on \( m \) follows as in the proof of Corollary 2.18.

Up till now, we have characterised \( \ell^2(J_k) \subset E(H^2(\mathbb{C}_+, \mathbb{C}^N)) \), where \( E \) is the evaluation operator defined in Theorem 2.11. We conclude this section with equivalent conditions for \( E(H^2(\mathbb{C}_+, \mathbb{C}^N)) = \ell^2(J_k) \) and \( E(H^2(\mathbb{C}_+, \mathbb{C}^N)) \subset \ell^2(J_k) \).

Theorem 2.24 Let \( \mathcal{H} = \mathbb{C}^N \) and for each \( k \in \mathbb{N} \), let \( G_k : \mathbb{C}^N \to \mathbb{C}^N \) with the notation as above.

1. The following are equivalent

   (a) \( E(H^2(\mathbb{C}_+, \mathbb{C}^N)) = \ell^2(J_k) \).

   (b) \( (z_k)_{k \in \mathbb{N}} \) is the union of at most \( N \) Carleson sequences, the linear maps

\[
\frac{1}{(\text{Re } z_k)^{1/2}} G_k^* : J_k \to I_k
\]

   are uniformly bounded above and below, and there exists a constant \( r > 0 \) such that, with the same notation as in (11),

\[
\inf_{k \in \mathbb{N}} \angle(\mathcal{K}_{k,I,r}, \mathcal{K}_{k,I}) > 0.
\]

2. The following are equivalent

   (a) \( E(H^2(\mathbb{C}_+, \mathbb{C}^N)) \subset \ell^2(J_k) \).
There exists a constant \( M > 0 \) such that

\[
\sum_{z_k \in Q_I} \|G_k\|^2 \leq M|I|, \quad \text{whenever} \quad I \subset \mathbb{R}
\]

whenever \( I \subset \mathbb{R} \) is a bounded interval.

**Proof** 1. We first show (a) \( \Rightarrow \) (b). If \( E : H^2(\mathbb{C}_+, \mathbb{C}^N) \to \ell^2(J_k) \) is bounded and surjective, then

\[
E^*: \ell^2(J_k) \to H^2(\mathbb{C}_+, \mathbb{C}^N), \quad (x_k) \mapsto \sum_{k \in \mathbb{N}} k_{z_k} G_k^* x_k
\]

is bounded and bounded below. Applying \( E^* \) to \( (0, \ldots, 0, x_k, 0, \ldots) \), we see that \( \| k_{z_k} G_k^* x_k \| \approx \| x_k \| \) for all \( k \in \mathbb{N} \), \( x_k \in J_k \) and that the linear maps

\[
\frac{1}{(\text{Re} \ z_k)^{1/2}} G_k^* : J_k \to I_k
\]

are uniformly bounded above and below. In other words, the map

\[
\ell^2(J_k) \to \ell^2(I_k), \quad (x_k) \mapsto \left( \frac{1}{(\text{Re} \ z_k)^{1/2}} G_k^* x_k \right)
\]

is an isomorphism of Hilbert spaces. Now the fact that \( E^* \) is bounded and bounded below implies that the system of subspaces \( \{ k_{z_k} I_k \}_{k \in \mathbb{N}} \) is unconditional in \( H^2(\mathbb{C}_+, \mathbb{C}^N) \), or equivalently, uniformly minimal, which means that \( (z_k) \) is a union of at most \( N \) Carleson sequences and \( \inf_{k \in \mathbb{N}} \angle(K_{k,I}^r, K_{k,I}) > 0 \) for some \( r > 0 \).

(b) \( \Rightarrow \) (a) This follows by a simple reversal of the above argument.

2. This follows from

\[
\|E(f)\|^2 = \sum_{k=1}^{\infty} \|G_k f(z_k)\|^2 = \int_{\mathbb{C}_+} \langle d\mu f(z), f(z) \rangle,
\]

where \( \mu = \sum_{k=1}^{\infty} G_k^* G_k \delta_{z_k} \), the matrix Carleson embedding theorem 2.2, and a comparison of trace and norm.

**Remark 2.25** For the special case when the \( G_k \) all have rank 1, an alternative proof of Theorem 2.24 part 1 can be found in [9] and Theorem 2.24 part 2 was proved in [7] and [28]. In this situation condition (12) reads

\[
\inf_{n \in \mathbb{N}} \frac{\|g_{n,1}\|^2}{\text{Re}(z_n)} > 0,
\]

and condition (13) simplifies to

\[
\inf_{m \in \mathbb{N}} \min_{s_n \in \Lambda_m(r)} \angle(g_{j,1} e^{-z_n t}, \text{span}_{z_j \in \Lambda_m(r)} \{ g_{n,1} e^{-z_j t} \}) > 0,
\]

where

\[
\Lambda_m(r) := \left\{ z_n : \frac{|z_n - z_m|}{|z_n + z_m|} < r \right\}.
\]
3 Controllability

In this section we apply the results on interpolation by vector-valued analytic functions to controllability problems of infinite-dimensional linear systems. We study a system of the form

\[ \dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0, \]

\[ x(0) = x_0. \]

Here we assume that \( A \) is the generator of an exponentially stable \( C_0 \)-semigroup \((T(t))_{t \geq 0}\) on a Hilbert space \( H \) such that the eigenvectors of \( A \) form an orthonormal basis \((\phi_n)_{n \in \mathbb{N}}\) of \( H \) and the corresponding eigenvalues \((\lambda_n)_{n \in \mathbb{N}}\) are pairwise distinct. The eigenvalues \((\lambda_n)_{n \in \mathbb{N}}\) then lie in the open left half plane uniformly bounded away from the imaginary axis. For every \( \alpha \in \mathbb{R} \) we introduce the interpolation space

\[ H_\alpha = \left\{ \sum_{n=1}^{\infty} x_n \phi_n : \{x_n|\lambda_n|^\alpha\}_{n \in \mathbb{N}} \in \ell^2 \right\}, \]

equipped with the scalar product

\[ \langle x, y \rangle_\alpha := \sum_{n \in \mathbb{N}} \langle x, \phi_n \rangle \overline{\langle y, \phi_n \rangle} |\lambda_n|^{2\alpha}. \]

The spaces \( H_\alpha \) are Hilbert spaces with \( H_0 = H \) and \( H_1 = D(A) \). We denote the dual pairing between \( H_\alpha \) and \( H_{-\alpha} \) by \( \langle \cdot, \cdot \rangle_{H_\alpha \times H_{-\alpha}} \).

In the sequel let \( \alpha \geq 0, B \in \mathcal{L}(\mathbb{C}^N, H_{-\alpha}) \) and \( u \in L^2(0, \infty; \mathbb{C}^N) \). Thus \( B \) can be represented by

\[ Bv = \sum_{n=1}^{\infty} \langle v, b_n \rangle \phi_n, \quad v \in \mathbb{C}^N, \]

where the sequence \( (b_n)_{n \in \mathbb{N}} \subseteq \mathbb{C}^N \) satisfies

\[ b_n := B^* \phi_n \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\|b_n\|^2}{|\lambda_n|^{2\alpha}} < \infty. \]

For more information on the spaces \( H_{-\alpha} \) see for example [27]. One important feature of these interpolation spaces \( H_{-\alpha} \) is that the semigroup \((T(t))_{t \geq 0}\) can be extended to a \( C_0 \)-semigroup on \( H_{-\alpha} \), which we denote by \((T_{-\alpha}(t))_{t \geq 0}\), and the generator of this extended semigroup, denoted by \( A_{-\alpha} \), is an extension of \( A \). By a solution of the system (14) we mean the so-called mild solution given by

\[ x(t) = T(t)x_0 + \int_0^t T_{-\alpha}(t-s)Bu(s) \, ds, \]

which is a continuous function with values in the interpolation space \( H_{-\alpha} \). We introduce the operator \( B_\infty \in \mathcal{L}(L^2(0, \infty; \mathbb{C}^N), H_{-\alpha}) \) by

\[ B_\infty u := \int_0^\infty T_{-\alpha}(s)Bu(s) \, ds. \]

We shall discuss the following controllability concepts.
\textbf{Definition 3.1} Let $\tau > 0$. We say that the system (14) is

1. \textit{null-controllable in time} $\tau$, if $R(T(\tau)) \subseteq R(B_\infty)$;
2. \textit{approximately controllable}, if $R(B_\infty) \cap \mathcal{H}$ is dense in $\mathcal{H}$;
3. \textit{exactly controllable}, if $\mathcal{H} \subset R(B_\infty)$.

Here $R(\cdot)$ denotes the range of an operator. It is easy to see that every exactly controllable system is approximately controllable and null-controllable in any time $\tau > 0$. Some of these properties (but with bounded control operators $B$) have been studied for diagonal systems in [15].

\section{3.1 Conditions for exact controllability}

Concerning exact controllability we obtain the following equivalent conditions. A sequence $(f_n)$ in a Hilbert space $\mathcal{H}$ is called a \textit{Bessel sequence}, if there exists a constant $\beta > 0$ such that for every $N \in \mathbb{N}$ and $a_1, \ldots, a_N \in \mathbb{C}$ we have

$$\beta \sum_{n=1}^{N} |a_n|^2 \leq \left\| \sum_{n=1}^{N} a_n f_n \right\|^2.$$

\textbf{Theorem 3.2} The following statements are equivalent:

1. System (14) is exactly controllable.
2. There exists a constant $m > 0$ such that for all intervals $I \subset \mathbb{R}$:

$$\sum_{-\lambda_n \in Q_I} \frac{(\text{Re} \lambda_n)^2}{\|b_n\|^2} \angle (e^{\lambda n} b_n, \text{span}_{j \neq n, j \in \mathbb{N}} \{e^{\lambda j} b_j\})^2 \leq m |I|,$$

where $Q_I$ is defined in (2).
3. $\{b_n e^{\lambda n}\}_{n \in \mathbb{N}}$ is a Bessel sequence in $L^2(0, \infty; \mathbb{C}^N)$.

The proof of Theorem 3.2 will be given at the end of this subsection.

\textbf{Remark 3.3} (16) implies immediately that all $b_n$ have to be nonzero. Moreover, a necessary condition for exact controllability of system (21) is

$$\sup_{n \in \mathbb{N}} \frac{|\text{Re} \lambda_n|}{\|b_n\|^2} < \infty.$$  \hfill (17)

This follows directly from the fact that $|\angle (e^{\lambda n} b_n, \text{span}_{j \neq n, j \in \mathbb{N}} \{e^{\lambda j} b_j\})| \leq \pi/2$.

\textbf{Remark 3.4} Using [1, Theorem II.2.4 on page 65], exact controllability implies that the sequence $(-\lambda_n)_{n \in \mathbb{N}}$ has to be a Blaschke sequence, that is,

$$\sum_{n \in \mathbb{N}} \frac{-\text{Re} \lambda_n}{1 + |\lambda_n|^2} < \infty.$$  \hfill (18)
Proof This can be seen directly from Corollary 2.16.

Remark 3.5 One can also prove a higher-rank version of Theorem 3.2, using Corollary 2.23 and the proof below, but shall not require it.

Unfortunately, condition (16) is not easy to verify. Thus we give in the following theorem sufficient and necessary conditions for (16). For a sequence \( s = (s_k) \), finite or infinite, in \( \mathbb{C}_+ \) we define

\[
\delta(s) = \inf_k \prod_{j \neq k} \frac{|s_j - s_k|}{|s_j + s_k|}.
\]

Thus \( 0 \leq \delta(s) \leq 1 \) and \( \delta(s) > 0 \) if and only if either (i) \( s \) is a finite sequence of distinct points, or (ii) \( s \) is an infinite Carleson sequence.

Theorem 3.6 We have

1. If the system (14) is exactly controllable, then there exists a constant \( m > 0 \) such that for all \( n \in \mathbb{N} \), and all intervals \( I \subset \mathbb{R} \),

\[
\frac{\delta(-\lambda_1, \ldots, -\lambda_n)^2}{\log^2 \delta(-\lambda_1, \ldots, -\lambda_n)} \sum_{-\lambda_k \in Q_I} \frac{(\text{Re} \lambda_k)^2}{\|b_k\|^2} \prod_{j=1, j \neq k}^n \left| \frac{\lambda_j + \lambda_k}{\lambda_j - \lambda_k} \right|^2 \leq m|I|. \quad (18)
\]

2. If there exists a constant \( m > 0 \) such that for all \( n \in \mathbb{N} \), all intervals \( I \subset \mathbb{R} \),

\[
\sum_{-\lambda_k \in Q_I} \frac{(\text{Re} \lambda_k)^2}{\|b_k\|^2} \prod_{j=1, j \neq k}^n \left| \frac{\lambda_j + \lambda_k}{\lambda_j - \lambda_k} \right|^2 \leq m|I|,
\]

then the system (14) is exactly controllable.

The proof of Theorem 3.6 will be given at the end of this subsection. In order to prove Theorem 3.2 and 3.6 we reduce the question of exact controllability to an interpolation problem. Using the special representation of \( A \) and \( B \) we see that

\[
\int_0^\infty T(t)Bu(t) \, dt = \sum_{n=1}^{\infty} \int_0^\infty e^{\lambda_n t} \langle u(t), b_n \rangle dt \phi_n = \sum_{n=1}^{\infty} \langle \hat{u}(-\lambda_n), b_n \rangle \phi_n,
\]

for every \( u \in L^2(0, \infty; \mathbb{C}_N) \). Using the fact that the Laplace transform is a constant multiple of an isometric isomorphism between \( L^2(0, \infty; \mathbb{C}_N) \) and \( H^2(\mathbb{C}_+, \mathbb{C}_N) \), the system (14) is exactly controllable if and only if \( \ell^2(\mathbb{N}) \subseteq E(H^2(\mathbb{C}_+, \mathbb{C}_N)) \). Here \( E : H^2(\mathbb{C}_+, \mathbb{C}_N) \to \{ x : \mathbb{N} \to \mathbb{C} \} \) is defined by

\[
Eg := (\langle g(-\lambda_n), b_n \rangle)_n.
\]

We have the following two useful propositions.
Proposition 3.7  Let \( w = (w_k) \) be a sequence (finite or infinite) in \( \mathbb{C}^N \), and \( s = (s_k) \) a corresponding sequence in \( \mathbb{C}_+ \). Define
\[
\gamma(s, w) = \inf \{ \|F\|_\infty : F \in H^\infty(\mathbb{C}_+, \mathbb{C}^N), F(s_k) = w_k \ \forall k \}.
\]
Then if \( \delta(s) < \infty \), we have
\[
\| (w_k) \|_\infty \leq \gamma(s, w) \leq C \delta(s)^{-1} \log \delta(s)^{-1} \| (w_k) \|_\infty,
\]
where \( C \) is a constant that depends only on \( N \).

Proof  This follows immediately from the scalar case of the Carleson interpolation theorem, where a constant growing as \( \delta^{-1} \log \delta^{-1} \) is known to apply (see [10, p. 274]). All that is required is to find \( N \) scalar functions interpolating the coordinates of the \( w_k \) and then combine them as a vector-valued function. \( \blacksquare \)

Proposition 3.8  The following statements are equivalent
1. The system (14) is exactly controllable.
2. There exists a constant \( c > 0 \) such that
\[
\int_0^\infty \|B^* T_\alpha^*(t) x\|^2 dt \geq c \|x\|^2, \quad x \in \mathcal{H}_\alpha.
\]

Proof  Let \( B_\infty \in \mathcal{L}(L^2(0, \infty; \mathbb{C}^N), \mathcal{H}_{-\alpha}) \) be defined by (15). An easy calculation shows that the dual of \( B_\infty \) is given by
\[
(B_\infty^* x)(t) = B^* T_\alpha^*(t) x, \quad x \in \mathcal{H}_\alpha.
\]
The statement of the proposition now follows from the relation of images and kernels of linear operators, see e.g. [29]. \( \blacksquare \)

Proof of Theorem 3.2 and 3.6  We first prove the equivalence of Parts 1 and 3 in Theorem 3.2. Taking \( N \in \mathbb{N} \), we have for \( x := \sum_{n=1}^N a_n \phi_n \)
\[
B^* T_\alpha^*(t) x = \sum_{n=1}^N a_n b_n e^{-\lambda_n t}.
\]
If the system is exactly controllable then by Proposition 3.8 there exists a constant \( c > 0 \) such that
\[
c \sum_{n=1}^N |a_n|^2 = c \|x\|^2 \leq \int_0^\infty \|B^* T_\alpha^*(t) x\|^2 dt = \int_0^\infty \left\| \sum_{n=1}^N a_n b_n e^{-\lambda_n t} \right\|^2 dt.
\]
This implies Part 3 and the converse direction follows by reversing the arguments.

We choose \( \mathcal{H} := \mathbb{C}^N \) and we define \( G_k \in \mathbb{C}^{N \times N} \) by \( G_k^* := (b_k \ 0 \cdots 0) \), \( k \in \mathbb{N} \). Note that system (14) is exactly controllable if and only if \( \ell^2(\mathbb{N}) \subseteq E(H^2(\mathbb{C}_+, \mathbb{C}^N)) \).
A weak* compactness argument shows that the latter holds if and only if
\[
\sup_{n \in \mathbb{N}} \sup_{x \in \ell^2(\mathbb{C}^N)} \inf \ \{ \|f\|_2 : f \in H^2(\mathbb{C}_+, \mathcal{H}), G_k f(-\lambda_k) = (x_k \ 0 \cdots 0)^T, k = 1, \cdots, n \}
\]

\[ 24 \]
is finite. Thus we have reduced the question of exact controllability to an interpolation problem treated in Section 2. Using the notation of Section 2 we have
\[ \angle(K_{k,I}, K'_{k,I}) = \angle(e^{\lambda_k t} b_k, \text{span}_{j \neq k, j \in \mathbb{N}} \{e^{\lambda_j t} b_j\}). \]

Theorem 3.2 now follows from Corollary 2.18. Further, using Proposition 3.7, Corollary 2.13 implies Theorem 3.6, since \(\|G^{-1}_k\| = \|G_k\|^{-1} = \|b_k\|^{-1}.\)

### 3.2 Conditions for null controllability

We now turn our attention to null controllability. We start with the following useful proposition, which is of independent interest.

**Proposition 3.9** The following statements are equivalent.

1. The system (14) is null controllable in time \(\tau\).
2. \(\{(e^{\lambda_n \tau} x_n)_n : (x_n)_n \in \ell^2(\mathbb{N})\} \subseteq E(H^2(\mathbb{C}_+, \mathbb{C}^N)),\) where the operator \(E\) is defined in (20).

**Proof** The system (14) is null-controllable in time \(\tau\) if and only if the operator \(B_\infty\), defined by
\[ B_\infty u := \int_0^\infty T(t)Bu(t) \, dt, \]
(21)
satisfies \(R(T(\tau)) \subseteq B_\infty(L^2(0, \infty; \mathbb{C}^N)).\) A calculation similar to the one in Subsection 3.1 shows that this is equivalent to the fact that for every \((x_n)_k \in \ell^2(\mathbb{N})\) there exists a function \(g \in H^2(\mathbb{C}_+, \mathbb{C}^N)\) such that \(\langle g(-\lambda_k), b_k \rangle = e^{\lambda_k \tau} x_k, k \in \mathbb{N}.\)

Note, that the system (14) is exactly controllable if and only if \(\ell^2(\mathbb{N})\) is a subset of \(E(H^2(\mathbb{C}_+, \mathbb{C}^N))\). Replacing \(b_k\) by \(e^{-\lambda_k \tau} b_k\) in the previous subsection, we obtain the following two theorems (Theorem 3.10 and Theorem 3.12). The sequence \((e^{-\lambda_k \tau} b_k)_{k \in \mathbb{N}}\) does not in general satisfy the condition
\[ \sum_{n=1}^{\infty} e^{-2 \text{Re} \lambda_n \tau} \|b_n\|^2 |\lambda_n|^{2\alpha} < \infty. \]
However, this condition is not needed for the proof of Theorem 3.2 and Theorem 3.6.

**Theorem 3.10** The following statements are equivalent:

1. System (14) is null-controllable in time \(\tau\).
2. There exists a constant \(m > 0\) such that for all intervals \(I \subset \mathbb{R}\)
\[ \sum_{-\lambda_n \in Q_I} \|b_n\|^2 \angle(e^{\lambda_n (t-\tau)} b_n, \text{span}_{j \neq n, j \in \mathbb{N}} \{e^{\lambda_j (t-\tau)} b_j\})^2 \leq m|I|, \]
where \(Q_I\) is defined in (2).
Remark 3.11 (22) implies immediately that all \( b_n \) have to be nonzero. A necessary condition for null-controllability in time \( \tau \) of system (21) is

\[
\sup_{n \in \mathbb{N}} \left| \text{Re} \lambda_n \right| e^{2 \text{Re} \lambda_n \tau} \frac{\| b_n \|}{\| b_k \|} < \infty.
\]  

(23)

Further, using Corollary 2.16, null-controllability in time \( \tau \) implies that the sequence \((- \lambda_n) \in \mathbb{N}\) has to be a Blaschke sequence.

Theorem 3.12 We have

1. If the system (14) is null-controllable in time \( \tau \), then there exists a constant \( m > 0 \) such that for all \( n \in \mathbb{N} \) and all intervals \( I \subset \mathbb{R} \),

\[
\frac{\delta(-\lambda_1, \ldots, -\lambda_n)^2}{\log^2 \delta(-\lambda_1, \ldots, -\lambda_n)} \sum_{-\lambda_k \in Q_I} (\text{Re} \lambda_k)^2 e^{2 \text{Re} \lambda_k \tau} \| b_k \|^2 \prod_{j=1 \atop j \neq k}^n \left| \frac{\lambda_j + \overline{\lambda_k}}{\lambda_j - \lambda_k} \right|^2 \leq m|I|.
\]

(24)

2. If there exists a constant \( m > 0 \) such that for all \( n \in \mathbb{N} \) and all intervals \( I \subset \mathbb{R} \),

\[
\sum_{-\lambda_k \in Q_I} (\text{Re} \lambda_k)^2 e^{2 \text{Re} \lambda_k \tau} \| b_k \|^2 \prod_{j=1 \atop j \neq k}^n \left| \frac{\lambda_j + \overline{\lambda_k}}{\lambda_j - \lambda_k} \right|^2 \leq m|I|,
\]

(25)

then the system (14) is null-controllable in time \( \tau \).

As a practical application of the above theory, we study the one-dimensional heat equation.

Corollary 3.13 Consider the one-dimensional heat equation with infinitesimal generator \( A \) such that \( A \phi_k = \lambda_k \phi_k \), where \( \phi_k(x) = \sqrt{2} \sin k \pi x \) and \( \lambda_k = -\pi^2 k^2 \) as in [8]. Write \( s_k = -\lambda_k \) for each \( k \). Suppose that the finite-dimensional control operator \( B \) is given by a sequence \((b_k)\) of vectors in \( \mathbb{C}^N \). For (14) to be null-controllable in time \( \tau \) it is sufficient that the measures

\[
\mu_n := \frac{n}{k^4 e^{-2 \pi^2 k^2}} \prod_{j \neq k} \frac{|s_j + \overline{s_k}|^2}{(s_j - s_k)^2} \delta_{s_k}
\]

(26)

be uniformly Carleson, and it is necessary that the measures

\[
\nu_n := \frac{\delta(s_1, \ldots, s_n)^2}{\log^2 \delta(s_1, \ldots, s_n)} \sum_{k=1}^n \frac{k^4 e^{-2 \pi^2 k^2}}{\| b_k \|^2} \prod_{j \neq k} \frac{|s_j + \overline{s_k}|^2}{(s_j - s_k)^2} \delta_{s_k}
\]

(27)

be uniformly Carleson.
We know from [8, Example 2.5] that in this case
\[ \delta(s_1, \ldots, s_n) \geq \exp(-2n(1 + \log n)). \]
Thus in many examples the above corollary can give fairly precise estimates
for controllability at time \( \tau \), and sometimes it seems to depend only on the
\[ \|b_n\| \] and not on the dimension \( N \). If we take a Carleson square \( Q_I \), then the
significant case is when \( |I| = \pi^2n^2 \), and it is then sufficient that
\[ \sum_{k=1}^{n} \frac{k^4e^{-2\pi^2k^2}e^{4k(1+\log k)}}{\|b_k\|^2} \leq Cn^2 \] (28)
and necessary that
\[ \sum_{k=1}^{n} \frac{k^4e^{-2\pi^2k^2}e^{4k(1+\log k)}}{\|b_k\|^2} \leq Cn^2e^{4n(1+\log n)}(2n(1 + \log n))^2. \] (29)
If \( \|b_k\| = k \exp(-k^2) \), as in [8], then the left hand side becomes
\[ \sum_{k=1}^{n} k^2e^{-2\pi^2k^2+2k^2+4k(1+\log k)} \]
and we see that the sufficient condition is satisfied for \( \tau > 1/\pi^2 \) whereas the
necessary condition is not satisfied for \( \tau < 1/\pi^2 \). This is the same behaviour as
seen in the scalar case.

3.3 Conditions for approximate controllability

Next we characterize approximately controllable systems in terms of their eigen-
values and the operator \( B \). By \( e_n \) we denote the \( n \)th unit vector of \( \mathbb{C}^N \).

**Theorem 3.14** Suppose that \( \{\lambda_n | n \in \mathbb{N}\} \) is totally disconnected, that is, no
two points \( \lambda, \mu \in \{\lambda_n | n \in \mathbb{N}\} \) can be joint by a segment lying entirely in
\( \{\lambda_n | n \in \mathbb{N}\} \). Then the following properties are equivalent:

1. The system (14) is approximately controllable.
2. \( \text{rank}(\langle Be_1, \phi_n \rangle, \cdots, \langle Be_N, \phi_n \rangle) = 1 \) for all \( n \in \mathbb{N} \).

**Proof** It is easy to see that statement 1 implies statement 2. Next we show
that statement 2 implies statement 1. We recall that the statements of this
theorem are known to be equivalent if, additionally, \( B \in \mathcal{L}(\mathbb{C}^N, \mathcal{H}) \), see (Curtain
and Zwart [4, page 164]).

To deduce the result in the general case \( B \in \mathcal{L}(\mathbb{C}^N, \mathcal{H}_{\alpha}) \), say, we fix an integer
\( m > \alpha \). Now we know that the system \( (A, \beta) \), where
\[ \beta := \sum_{j=1}^{\infty} \frac{\langle \cdot, B^* \phi_j \rangle}{(1 - \lambda_j)^m} \phi_j \in \mathcal{L}(\mathbb{C}^N, \mathcal{H}), \]
is approximately controllable, by the result in [4]. Using the fact that

\[ B_\infty f = \sum_{j=1}^{\infty} \int_0^{\infty} e^{\lambda_n t} (f(t), B^* \phi_j) \, dt \phi_j = \sum_{j=1}^{\infty} \langle \hat{f}(-\lambda_j), B^* \phi_j \rangle \phi_j, \]

where \( \hat{f} \) denotes the Laplace transform of \( f \), we get that the set

\[ S_\beta := \left\{ \sum_{j=1}^{\infty} \langle \hat{f}(-\lambda_j), B^* \phi_j \rangle \phi_j : \hat{f} \in H^2(\mathbb{C}_+, \mathbb{C}^N) \right\} \]

is dense in \( \mathcal{H} \). Similarly, let

\[ S_B := \left\{ \sum_{j=1}^{\infty} \langle \hat{f}(-\lambda_j), B^* \phi_j \rangle \phi_j : \hat{f} \in H^2(\mathbb{C}_+, \mathbb{C}^N) \right\}. \]

Now if \( \hat{f} \in H^2(\mathbb{C}_+, \mathbb{C}^N) \), then so is the function \( \hat{g} : s \mapsto \hat{f}(s)/(s+1)^m \), and then

\[ \sum_{j=1}^{\infty} \langle \hat{f}(-\lambda_j), B^* \phi_j \rangle \phi_j = \sum_{j=1}^{\infty} \langle \hat{g}(-\lambda_j), B^* \phi_j \rangle \phi_j. \]

Hence \( S_\beta \subseteq S_B \), which implies that \( S_B \) is dense in \( \mathcal{H} \), as required. \( \Box \)

**Remark 3.15** In [8] this theorem was proved for \( N = 1 \). However, in [8] the authors omitted to mention that the proof only works if the closure of the eigenvalues is totally disconnected. The final result shows that the statement of the theorem does not hold for every diagonal system.

**Theorem 3.16** There exists a system (14) with \( B \in \mathcal{L}(\mathbb{C}, \mathcal{H}_{-1}) \) such that the system (14) is not approximately controllable, but \( \langle B e_1, \phi_n \rangle \neq 0 \) for all \( n \in \mathbb{N} \).

**Proof** We note that by duality the system (14) is approximately observable if and only if the following holds: If \( B^* x \) is well-defined for some \( x \in \mathcal{H} \) and \( B^* x = 0 \), then necessarily \( x = 0 \). Let \( \mathcal{H} = \ell^2 \) and let \( \phi_n, n \in \mathbb{N} \), be the nth unit vector of \( \ell^2 \). We assume that \( (\phi_n)_{n \in \mathbb{N}} \) are the eigenvectors of pairwise distinct eigenvalues \( (\lambda_n)_{n \in \mathbb{N}} \) of the operator \( A \). The sequence \( (\lambda_n)_{n \in \mathbb{N}} \) will be chosen later on, but \( (\lambda_n)_{n \in \mathbb{N}} \) should satisfy \( \text{Re} \lambda_n < -1, n \in \mathbb{N}, \) \( \sup_{n \in \mathbb{N}} \text{Re} \lambda_n \leq -1 \) and \( \sum_{n \in \mathbb{N}} \frac{-\text{Re} \lambda_n - 1}{|\lambda_n|^2} < \infty \). These assumptions guarantees that \( A \) generates an exponentially stable \( C_0 \)-semigroup and that the operator \( Bu := \sum_{n \in \mathbb{N}} \sqrt{-1 - \text{Re} \lambda_n} \phi_n \cdot u, u \in \mathbb{C} \), satisfies \( B \in \mathcal{L}(\mathbb{C}, \mathcal{H}_{-1}) \). It remains to show that there is a sequence \( (x_n)_{n \in \mathbb{N}} \in \ell^2 \setminus \{0\} \) such that

\[ \sum_{n \in \mathbb{N}} x_n \sqrt{-1 - \text{Re} \lambda_n} e^{\lambda_n t} = 0, \quad t \geq 0, \]

where the sum converges in \( L^2(0, \infty) \).
In [5] (see also [11]) it is shown that there exist two sequences \((a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}\) in \(D = \{z \in \mathbb{C} : |z| < 1\}\) with \(a_n \neq a_m, b_n \neq b_m, n \neq m,\) and \(a_n \neq b_m, n, m \in \mathbb{N},\) such that
\[
1 = \sum_{n \in \mathbb{N}} \alpha_n \frac{(1 - |a_n|^2)^{1/2}}{1 - a_n z} = \sum_{n \in \mathbb{N}} \beta_n \frac{(1 - |b_n|^2)^{1/2}}{1 - b_n z} \tag{30}
\]
and \(\sum_{n \in \mathbb{N}} (1 - |a_n|^2)^{1/2}, \sum_{n \in \mathbb{N}} (1 - |b_n|^2)^{1/2} < \infty\) for some sequences \((\alpha_n)_{n \in \mathbb{N}}, (\beta_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})\) and every \(z \in D.\) Here the sums in (30) converge in \(H^2(D).\) Subtracting these two sums in (30) we have the following: there exists a sequence \((a_n)_{n \in \mathbb{N}}\) with \(a_n \neq a_m, n \neq m,\) such that
\[
\sum_{n \in \mathbb{N}} \alpha_n \frac{(1 - |a_n|^2)^{1/2}}{1 - a_n z} = 0
\]
and \(\sum_{n \in \mathbb{N}} (1 - |a_n|^2)^{1/2} < \infty\) for some sequences \((\alpha_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})\) and every \(z \in D.\) In order to obtain a result for the open right half plane we write \(M\) for the self-inverse mapping \(s \mapsto (1 - s)/(1 + s)\) providing a conformal bijection between \(\mathbb{C}^+\) and \(D.\) Then we recall [18, p. 24] that
\[
g \in H^2(D) \iff G : s \mapsto (1 + s)^{-1}g(Ms) \in H^2(\mathbb{C}^+),
\]
with an equivalence of norms. Thus it is easy to see that there exists a sequence \((\omega_n)_{n \in \mathbb{N}} \subset \mathbb{C}^+,\) with \(\omega_n \neq \omega_m, n \neq m,\) such that
\[
\sum_{n \in \mathbb{N}} \frac{\sqrt{\text{Re} \omega_n}}{s + \omega_n} = 0
\]
and
\[
\sum_{n \in \mathbb{N}} \frac{\text{Re} \omega_n}{|1 + \omega_n|^2} < \infty,
\]
for some sequence \((y_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})\) and every \(s \in \mathbb{C}^+.\) Using the inverse Laplace transform we obtain
\[
\sum_{n \in \mathbb{N}} z_n \sqrt{\text{Re} \omega_n} e^{-\omega_n t} = 0,
\]
for some sequence \((z_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})\) and every \(t \geq 0.\) On defining \(\lambda_n := -\omega_n - 1,\) \(n \in \mathbb{N},\) the statement of the theorem follows. \(\blacksquare\)

4 Conclusions

The main results of this paper indicate that, in many cases, the minimal norms of vector-valued interpolants can be estimated in terms of the Carleson constants of scalar measures. For the application presented, that of analysing controllability properties of linear systems, these estimates are in many cases
precise enough; however, in other cases, to understand the distribution of eigenvalues seems to require complicated techniques from number theory, which are beyond the scope of this paper, as the example of the two-dimensional heat equation on the rectangle $\Omega = [0, a] \times [0, b]$ shows. Here the corresponding eigenvalues are given by

$$\lambda_{n,m} := -\pi^2 \left( \frac{n^2}{a^2} + \frac{m^2}{b^2} \right),$$

which clearly have a rather complicated distribution, and it is not hard to see that they do not even form a finite union of Carleson sets. Nonetheless, when the distribution of eigenvalues is more regular, as in the one-dimensional heat equation of Section 3.2, then the interpolation results of this paper can be applied to obtain useful information.

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References


