Robust numerical algorithms for the solution of Cauchy-type inverse problems in annular domains.

Mohamed Jaoua∗ Juliette Leblond† Moncef Mahjoub‡ and Jonathan R. Partington§

Short title: Cauchy-type inverse problems in annular domains


Abstract

We consider the Cauchy problem of recovering both Neumann and Dirichlet data on the inner part of the boundary of an annular domain, from measurements on some part of the outer boundary. The ultimate goal is to compute the Robin coefficient, which is the quotient of these extended data, on the inner boundary. Using tools from complex analysis and Hardy class approximations, we present constructive and robust identification schemes validated by a thorough numerical study.

1 Introduction

The problem we are dealing with in this contribution is the recovery of both Dirichlet and Neumann data on some part of the inner boundary of an annulus, from measurements on some part of the outer boundary. The so-extended data may be relevant by themselves in some applications, or used to compute the electrical impedance (Robin coefficient), which is needed in other applications.

Such a problem arises for example in corrosion detection in tubular domains. Corrosion may occur in many different forms, and several models are encountered in the literature [9, 16, 17, 21]. Evaluating the electric impedance, which is actually the Robin coefficient, on the internal wall of a hollow pipe from measurements performed on the external wall turns out actually to be an appropriate way to locate the corroded parts of the internal wall. Santosa and al [21] have given a simple linear model proving how corrosion affects

∗Laboratoire J.-A. Dieudonné, Université de Nice Sophia Antipolis, Parc Valrose, F-06108 Nice Cedex 02, jaoua@math.unice.fr, phone: +33 4 9207 6292.
†INRIA, BP 93, 06902 Sophia–Antipolis Cedex, FRANCE, leblond@sophia.inria.fr, phone: +33 4 92 38 78 76, fax: +33 4 92 38 78 58.
‡LAMSIN-ENIT, BP 37, 1002 Tunis Belvedere, TUNISIA, moncef.mahjoub@lamsin.rnu.tn, phone/fax: +216 71 871 022.
§School of Mathematics, University of Leeds, Leeds LS2 9JT, U.K., J.R.Partington@leeds.ac.uk, phone: +44 113 34 35123, fax: +44 113 34 35090.
the electric impedance. Since the Robin coefficient may be recovered from the completed Cauchy data, this problem reduces to a Cauchy problem for the Laplace operator.

These kinds of data completion problems have been widely studied in the case of simply-connected domains, which can be conformally mapped on the unit disk [9]. The method we wish to generalize here to annular domains is to construct analytic approximations by solving a bounded extremal problem there. Such a construction uses an explicit asymptotic expansion of the analytic approximant, and it needs to determine by some appropriate procedure the actual bound of that approximant in order to stabilize the whole algorithm.

The first issue to tackle is thus to get explicit asymptotic expansions in annular domains. Provided full data are available on the whole of the outer boundary, such formulae have already been obtained in [22], and stability estimates for the inverse problem (with suitable norms) have been established as consequences of boundedness properties for functions of weighted Hardy classes in [18]. In most practical cases, however, full data cannot be expected. In the present work, explicit formulae of the analytic approximant have therefore been sought and obtained for the incomplete data case. Continuity results of the approximants so-computed, with respect to the data, have also been proved; this makes it possible to use these formulae as a basic tool in the algorithmic part.

In order to produce an accurate approximant, it has already been noticed that the numerical algorithms need sharp information on the actual bound of the data sought. Both the issues of computing these data and the bound on them thus need to be dealt with together. This has been achieved by characterizing the actual bound as the unique zero of an appropriate function. Robustness properties of the procedure designed are improved by applying it to the n-th order derivatives of the data, instead of the data themselves, working in certain Sobolev classes of smoother functions, provided of course that this additional regularity is available. A whole family of algorithms, more robust as their order increases, is designed this way.

In Section 2 of this paper we introduce the inverse Robin problem and present the identifiability and stability results as obtained in [8, 18]. Section 3 is devoted to deriving the formulae we use to compute the solution in the incomplete external data case, and to proving continuity of these solutions with respect to the prescribed data. The identification algorithms are presented and studied in Section 4, and their numerical implementation and results are finally discussed in Section 5.

2 The inverse problem

Let $\mathcal{G} \subset \mathbb{R}^2$ be a doubly-connected domain with smooth boundary $\partial \mathcal{G} = \Gamma_i \cup \Gamma_e$ made of two Jordan closed curves $\Gamma_i, \Gamma_e$ such that $\Gamma_i \cap \Gamma_e = \emptyset$. Up to conformal mapping, we can consider $\mathcal{G}$ as an annular domain, see Figure 1.

More particularly, let $\mathbb{D}$ be the unit disk and $G$ be the annulus $G = \mathbb{D} \setminus s\mathbb{D}$ for some fixed $s$ with $0 < s < 1$ and denote $\partial G = T \cup sT$.

Let $I$ be a non–null measurable subset of $T$, and let $J = \partial G \setminus I$. We consider the following inverse problem: given functions $u_d$ and $\phi$, or a number of pointwise measurements, with
\[ \phi \neq 0, \text{ find a function } q, \text{ such that a solution } u \text{ to } \]
\[
\begin{align*}
\Delta u &= 0 \quad \text{in } G \quad (i) \\
u &= u_d \quad \text{on } I \quad (ii) \\
\partial_n u &= \phi \quad \text{on } I \quad (iii)
\end{align*}
\]
also satisfies
\[ \partial_n u + q u = 0 \quad \text{on } J, \quad (2) \]
where \( \partial_n \) stands for the partial derivative w.r.t. the outer normal unit vector to \( T \). In the electrical framework, \( u_d \) and \( \phi \) correspond to the measured electrical potential and to the imposed current flux on the outer boundary of some plane section of a tube, while \( q \) is the electrical impedance to be recovered on the associated inner boundary.

Let \( c, \tau > 0 \) and introduce the following class of “admissible” electrical impedance
\[ Q^n = \{ q \in C^n(J); |q^{(k)}(x)| \leq \tau, \quad 0 \leq k \leq n, \text{ and } q(x) \geq c \quad \forall x \in J \}. \]

For \( k \geq 1 \), let \( W^{k,2}(I) \) denote the usual Sobolev space of functions \( f \in L^2(I) \) the derivatives of which are also, up to the \( k \)-th denoted by \( f^{(k)} \) in \( L^2(I) \). For consistency, we shall also denote by \( W^{0,2}(I) \) the space \( L^2(I) \). The Sobolev spaces \( W^{k,2}(G) \) and \( W^{k,2}(\partial G) \) are defined analogously.

**Theorem 1 ([9])** Let \( n \geq 0, \phi \in W^{n,2}(I), \phi \geq 0, \phi \neq 0 \) and assume that \( q \in Q^n \) for some constants \( c, \tau > 0 \). Then there exists a unique function \( u \in W^{n+3/2,2}(G) \), whence \( u_{|_{\partial G}} \in W^{n+1,2}(\partial G) \), a solution to \( (1)i, (1)iii \) and \( (2) \). Further, there exist constants \( m > 0 \) and \( \kappa \) (depending on the class \( Q^n \)) such that for all \( q \in Q^n \) and \( \phi \in W^{n,2}(I) \),
\[ u \geq m > 0 \quad \text{on } J \] and \( \| u \|_{W^{n+1,2}(\partial G)} \leq \kappa. \]

The proofs of the above results rely on shift and Sobolev embedding theorems, together with the Hopf maximum principle.

The next identifiability property ensures the uniqueness of solutions \( q \) to the inverse problem, which is a necessary prerequisite for stability issues to make sense.

**Theorem 2 ([8])** Let \( \phi \in L^2(I), \phi \geq 0, \phi \neq 0 \) and \( q_1, q_2 \in Q^0 \). Let \( u_1 \) and \( u_2 \) be the associated solutions of \( (1)i, (1)iii \) and \( (2) \), with \( q_1, q_2 \) as impedances. If \( u_{1,i} = u_{2,i} \), then \( q_1 = q_2 \).
Stability results for the inverse Robin problem (with suitable norms) were established as consequences of boundedness properties for functions of weighted Hardy classes [18].

**Theorem 3 ([18])** Let $n \geq 1$, assume $\phi_1, \phi_2 \in W^{n,2}(I)$, $\phi_1, \phi_2 \geq 0, \neq 0$ on $I$ and $q_1, q_2 \in \mathbb{Q}^n$. Let $u_1, u_2$ solve (1), (2), with $q_1, q_2$ as impedances, and assume that:

$$\| u_1 - u_2 \|_{L^2(I)} \leq \varepsilon, \| \phi_1 - \phi_2 \|_{L^2(I)} \leq \varepsilon,$$

for some $\varepsilon > 0$. Then there exists a constant $K = K(s, \mathbb{Q}^n) > 0$ such that the following estimate holds:

$$\| q_1 - q_2 \|_{L^2(J)} \leq \frac{K}{\log \varepsilon^n}.$$

In the uniform norm, we have the following result:

**Theorem 4 ([18])** Let $n \geq 2$, assume $\phi_1, \phi_2 \in W^{n,2}(I)$, $\phi_1, \phi_2 \geq 0, \neq 0$ on $I$ and $q_1, q_2 \in \mathbb{Q}^n$. Let $u_1, u_2$ be the associated solutions to (1), (2) with $q_1, q_2$ as impedances, and assume that:

$$\| u_1 - u_2 \|_{L^\infty(I)} \leq \varepsilon, \| \phi_1 - \phi_2 \|_{L^\infty(I)} \leq \varepsilon,$$

for some $\varepsilon > 0$. Then there exists a constant $K = K(s, \mathbb{Q}^n) > 0$ such that the following estimate holds:

$$\| q_1 - q_2 \|_{L^\infty(J)} \leq \frac{K}{\log \varepsilon^{n-1}}.$$

These results can be viewed as extensions of those established in [2, 7], which hold on part of connected Lipschitz or Hölder smooth boundaries.

### 3 Bounded Extremal Problems

#### 3.1 Harmonic conjugate

Let $\phi \in L^2(I)$ and assume that $q \in \mathbb{Q}^0$. From Theorem 1, $u_{|\partial G} \in W^{1,2}(\partial G)$. Then there exists a function $v$ harmonic in $G$ such that $\partial_\theta v = \partial_n u$ on $\partial G$, where $\partial_\theta$ stands for the tangential partial derivative on $\partial G$, from the Cauchy–Riemann equations. Hence, $v$ is given on $I$ up to a constant by

$$v_I(e^{i\theta}) = \int_{\theta_0}^{\theta} \phi(e^{i\tau}) \, d\tau.$$

Further, from the M. Riesz theorem [12, Thm 4.1], the harmonic conjugate operator is bounded in $L^2(\partial G)$, whence $v_{|\partial G} \in W^{1,2}(\partial G)$. Thus, $f = u + i v$ is analytic in $G$ and $f_{|\partial G} \in W^{1,2}(\partial G)$; it is given on $I$ by

$$f(e^{i\theta}) = u_d(e^{i\theta}) + i \int_{\theta_0}^{\theta} \phi(e^{i\tau}) \, d\tau.$$

Also, on $J$,

$$q = -\frac{\partial_\theta u}{u} = -\frac{\partial_\theta \text{Im } f}{\text{Re } f},$$

which gives the link to be used between $q$ and $f$, in order to recover $q$ from approximations to $f$ on $I$ of the boundary $\partial G$. 

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3.2 Hardy classes of multiply connected circular domains

The Hardy spaces $H^p(G)$ on certain multiply-connected domains $G$ were defined by Rudin [20] in terms of analytic functions $f$ such that $|f(z)|^p$ has a harmonic majorant on $G$, that is, a real harmonic function $u(z)$ such that $|f(z)|^p \leq u(z)$ on $G$. It is also possible to define the Hardy spaces $H^p(\partial G)$ for $1 \leq p < \infty$, as the closure in $L^p(\partial G)$ of the set $R_G$ of rational functions whose poles lie in the complement of $G$. This approach, similar to the one in [4], was taken in [11]. The spaces $H^p(G)$ and $H^p(\partial G)$ are then isomorphic in a natural way, and so we identify the two spaces.

Below, we stick to the most completely analysed example of the annulus $G = \mathbb{D} \setminus s\mathbb{D}$ for some fixed $s$ with $0 < s < 1$ and to the Hilbert case $p = 2$. Here, the Lebesgue measure on $\partial G$ is normalized so that the circles $T$ and $sT$ each have unit measure.

The space $H^2(\partial G)$ has a canonical orthonormal basis consisting of the functions

$$e_n(z) := (z^n/\sqrt{1+s^{2n}})_{n\in\mathbb{Z}},$$

and it can be written as an orthogonal direct sum

$$H^2(\partial G) = H^2(\mathbb{D}) \oplus H^2_0(\mathbb{C} \setminus s\mathbb{D})$$

of elementary Hardy spaces, by taking the closed linear spans of $(e_n)_{n \geq 0}$ and $(e_n)_{n < 0}$ respectively. Here $H^2_0(\mathbb{C} \setminus s\mathbb{D})$ is the Hardy space of functions analytic on the complement of $s\mathbb{D}$, with $L^2$ boundary values, and vanishing at infinity. It should be noted that a similar decomposition applies to general spaces $H^p(\partial G)$, but the direct sum is no longer orthogonal in the case $p \neq 2$.

So, a function $f \in H^2(G)$ has the following expansion:

$$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n \text{ for } z \in G,$$

where $\|f\|_{H^2(G)}^2 = \sum_{n \in \mathbb{Z}} (1+s^{2n})|a_n|^2$.

We write $P_{L^2(I)}g = \chi_I g$ for the function in $L^2(\partial G)$ that coincides with $g$ on $I$ and vanishes on $J$. The definition of $P_{L^2(J)}$ is analogous.

3.3 Approximation in Hardy classes

We assume that $I = [-\theta_0, \theta_0] \subset \mathbb{T}$, $0 < \theta_0 < \pi$. We write $L^2(\partial G) = L^2(I) \oplus L^2(J)$. Whenever $\kappa_1$ is defined on $I$ and $\kappa_2$ on $J$, we write $\kappa_1 \vee \kappa_2$ for the function equal to $\kappa_1$ on $I$ and $\kappa_2$ on $J$.

Suppose that we are given $f \in L^2(I)$ and we wish to approximate $f$ as well as possible by the restriction to $I$ of an $H^2(\partial G)$ function, i.e., $P_{L^2(I)}g$ for $g \in H^2(\partial G)$. In view of the results established in [11], the space $P_{L^2(I)}H^2(\partial G)$ is dense in $L^2(I)$. Then there will exist a sequence $(g_n)$ of $H^2(\partial G)$ functions such that $\|P_{L^2(I)}g_n - f\|_{L^2(I)} \to 0$. However, if $f \neq P_{L^2(I)}g$ for any $g \in H^2(\partial G)$ then it will follow that $\|P_{L^2(J)}g_n\|_{L^2(J)} \to \infty$, i.e., the approximation problem is ill-posed.
In our work we are interested in the determination of an extension on $J$. To prevent instability one requires a bound for the approximation on $J$. This motivates the following bounded extremal problem (BEP), which is a problem of analytic approximation of incomplete data in Hardy classes. To fix ideas, given $f \in L^2(I) \setminus P_{L^2(I)} H^2(\partial G)$ and $M > 0$, find a function $g \in H^2(\partial G)$ which, under the norm constraint on $J$, solves the minimization problem:

\[
(BEP) \quad \begin{cases} 
\text{Given } f \in L^2(I) \setminus P_{L^2(I)} H^2(\partial G), \ f_1 \in L^2(J) \text{ and } M > 0, \\
\text{find a function } g \in H^2(\partial G) \text{ such that } \|g - f_1\|_{L^2(J)} \leq M \text{ and } \\
\|f - g\|_{L^2(I)} = \inf \{\|f - \psi\|_{L^2(I)} : \ \psi \in H^2(\partial G), \ \|\psi - f_1\|_{L^2(J)} \leq M\}. 
\end{cases}
\]

In practice $f$ corresponds to the data, $I$ is the part where these data can be measured, $f_1$ is a reference behaviour of the data on the part of the boundary where they are unknown. Such a problem is convex and admits a unique solution which can be obtained by solving a spectral equation for the Toeplitz operator $T$ with symbol $\chi_J$, the characteristic function of the component $J$:

\[
T : H^2(\partial G) \to H^2(\partial G) \\
g \mapsto P_{H^2(\partial G)} \chi_J g = P_{H^2(\partial G)} P_{L^2(J)} g,
\]

where $P_{H^2(\partial G)} : L^2(\partial G) \to H^2(\partial G)$ is the orthogonal projection. More precisely, the unique solution $g$ to the (BEP) problem solves the following:

**Proposition 5** ([11]) The unique solution $g$ of the (BEP) problem is given by the formula

\[
g = (Id + \lambda T)^{-1} P_{H^2(\partial G)} [f \vee (1 + \lambda) f_1],
\]

(4)

for the unique $\lambda > -1$ such that

\[
\|g - f_1\|_{L^2(J)} = M.
\]

(5)

**Remark 6** Let us note that $\lambda$ plays the role of a Lagrange multiplier which makes implicit the dependence of the solution on $M$, and which can be adjusted by dichotomy. A consequence of Proposition 5 is that the error $e(\lambda) := \|f - g(\lambda)\|_{L^2(I)}$ smoothly decreases to 0 as $\lambda \to -1$ and we refer to [23], that $\lambda \to M(\lambda)$ is $C^1$, bijective and decreasing on $]-1, +\infty[\to]0, +\infty[.$

When $f$ is a the trace on $I$ of some $H^2(\partial G)$–function, the (BEP) problem becomes one of interpolation. In this case, for simplicity, we will continue to denote by $f$ the $H^2(\partial G)$ function defined on the whole of $\partial G$. The error $e(\lambda) := \|f - g(\lambda)\|_{L^2(I)}$ decreases strictly to zero as $M$ increases to $\|f - f_1\|_{L^2(J)}$ and vanishes identically for $M \geq \|f - f_1\|_{L^2(J)}$.

Now, using a Fourier series development on the $(e_n)_n$ basis, we are able to propose a quasi-explicit method to solve equation (4).

Let $a_n$ and $b_n$ be respectively the Fourier coefficients of $\phi = f \vee (1 + \lambda) f_1 \in L^2(\partial G)$ on $T$ and on $s T$, defined by:

\[
a_n = \frac{1}{2\pi} \left( \int_{-\theta_0}^{\theta_0} f(\text{e}^{i\theta}) \text{e}^{-i n \theta} \text{d}\theta + (1 + \lambda) \int_{\theta_0}^{2\pi - \theta_0} f_1(\text{e}^{i\theta}) \text{e}^{-i n \theta} \text{d}\theta \right),
\]

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and
\[ b_n s^n = \frac{1 + \lambda}{2\pi} \int_0^{2\pi} f_1(s e^{i\theta}) e^{-in\theta} \, d\theta. \]

Moreover, let, \( B \) denote the sequence \((B_n)_{n \in \mathbb{Z}}\), where
\[ B_n = \frac{a_n + b_n s^{2n}}{\sqrt{1 + s^{2n}}}. \]

The following theorem then holds.

**Theorem 7** The solution \( g \) of the (BEP) problem, viewed as the (infinite) vector as defined by its Fourier coefficients \((g_n)_{n \in \mathbb{Z}}\), solves the following equation:
\[ (\text{Id} + \lambda T) g = B, \tag{6} \]

where \( T \) is the Toeplitz operator represented in the \( \{e_n\} \) basis by the infinite Toeplitz matrix defined by:
\[ T_{n,m} = \begin{cases} \frac{1}{1 + s^{2n}} \left( 1 + s^{2n} - \frac{\theta_0}{\pi} \right) & \text{when } n = m, \\ -\frac{1}{\sqrt{(1 + s^{2n})(1 + s^{2m})}} \frac{\sin (m-n) \theta_0}{\pi (m-n)} & \text{when } n \neq m. \end{cases} \tag{7} \]

**Remark 8** 1. This result is similar to the one obtained in the unit disk [15]. Both lead to an infinite linear system, here indexed by \( \mathbb{Z} \) whereas it was indexed by \( \mathbb{N} \) for the problem in the unit disk. Let us denote by \( g_N \) the approximate solution obtained by solving the truncated system in the basis \((e_n)_{-N \leq n \leq N}\)
\[ ((\text{Id} + \lambda T) g_N)_N = B_N \tag{8} \]

where \( B_N \) is the truncated Fourier series of \( B \).

The linear system so-obtained has a symmetric positive-definite matrix, which can be factorized using the Cholesky method. Iterating then on \( \lambda \) until (5) holds leads to the solution of the (BEP) problem for a given bound \( M \). Further details are given in Section 4.

2. A particular case is that of full external data \((J = s\mathbb{T})\). In that case, it has been established in [1] that the Toeplitz operator is diagonalizable, and an explicit expression of the (BEP) solution has been obtained in [22]:
\[ g(z) = \sum_{n \in \mathbb{Z}} a_n + \alpha b_n s^{2n} e_n(z), \]

where \( \alpha > 0 \) is the unique constant such that
\[ \sum_{n \in \mathbb{Z}} \frac{|(a_n - b_n)|^2 s^{2n}}{(1 + \alpha s^{2n})^2} = M^2. \]
The proof of Theorem 7 is now a straightforward consequence of the following two lemmas, whose proofs are provided in the appendix (Section 7).

**Lemma 9**

\[ P_{H^2(\partial G)}(z) = \sum_{n \in \mathbb{Z}} \frac{a_n + b_n s^{2n}}{\sqrt{1 + s^{2n}}} e_n(z), \]

**Lemma 10** Let \( g \in H^2(\partial G) \) such that \( g(z) = \sum_{n \in \mathbb{Z}} g_n e_n(z) \) for \( z \in G \) and \( T \) the Toeplitz operator. Then

\[ T g(z) = \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{1 + s^{2n}}} \left( \frac{g_n}{\sqrt{1 + s^{2n}}} \left( 1 + s^{2n} - \frac{\theta_0}{\pi} \right) - \sum_{m \neq n} \frac{g_m}{\sqrt{1 + s^{2m}}} \frac{\sin (m - n) \theta_0}{\pi (m - n)} \right) e_n(z). \]

Since we have no information on how the data behave on the part \( J \) of the boundary, we shall choose from now on \( f_1 = 0 \), whence \( b_n = 0, \forall n \in \mathbb{Z} \).

### 3.4 Continuity of the solutions with respect to the data

In this section, we shall investigate continuity properties of the solutions of (BEP) problem with respect to the data \( f \) and \( M \). Let \( g \) be the mapping defined by:

\[ g : L^2(I) \times \mathbb{R}_+^* \rightarrow H^2(\partial G), \]

\[ (f, M) \mapsto g(f, M), \]

where \( g(f, M) \) solve the (BEP) problem associated to the data \( f \) and \( M \). Let \( \mathcal{D} = \{ (h, M) \in H^2(\partial G)_I \times \mathbb{R}_+^* \mid \|h\|_{L^2(I)} < M \} \).

**Theorem 11** The mapping \( g \) is continuous on \( (L^2(I) \times \mathbb{R}_+^*) \setminus \mathcal{D} \), but not on the whole of \( L^2(I) \times \mathbb{R}_+^* \). However, if \( (f_n, M_n) \to (f, M) \) in \( L^2(I) \times \mathbb{R}_+^* \), then \( g(f_n, M_n) \to g(f, M) \) weakly in \( H^2(\partial G) \), whereas \( g(f_n, M_n) \to g(f, M) \) in \( L^2(I) \).

**Proof** First, consider the mapping \( e_f \) defined by

\[ e_f : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^* \]

\[ M \mapsto \|g(f, M) - f\|_{L^2(I)}, \]

The mapping \( e_f \) is convex and decreasing, thus continuous on \( \mathbb{R}_+^* \).

Next, let \( (f_n) \) be a sequence in \( L^2(I) \) such that \( \|f_n - f\|_{L^2(I)} \to 0 \) and suppose that \( (M_n) \) is a sequence in \( \mathbb{R}_+^* \) such that \( M_n \to M \). We claim that:

\[ \lim_{n \to \infty} e_{f_n}(M_n) = e_f(M). \quad (9) \]

Indeed, let \( \delta > 0 \) and assume that either \( e_{f_n}(M_n) > e_f(M) + \delta \) or \( e_{f_n}(M_n) < e_f(M) - \delta \) infinitely often. In the first case, since

\[ \|g(f, M_n) - f_n\|_{L^2(I)} \leq \|g(f, M_n) - f\|_{L^2(I)} + \|f_n - f\|_{L^2(I)}, \]
and because \( e_f \) is continuous, we have infinitely often
\[
\| g(f, M_n) - f_n \|_{L^2(I)} < e_f(M) + \delta < e_f(M_n),
\]
which contradicts the fact that \( g(f_n, M_n) \) is optimal. In the second case,
\[
\| g(f_n, M_n) - f \|_{L^2(I)} \leq e_f(M_n) + \| f_n - f \|_{L^2(I)},
\]
which implies that we have infinitely often
\[
e_f(M_n) < e_f(M) - \frac{\delta}{2},
\]
and contradicts the continuity of \( e_f \) established above.

Next, the sequence \( (g(f_n, M_n)) \) is bounded. We show that each of its subsequences admits a (sub-)subsequence which converges to \( g(f, M) \). We pass to a subsequence that converges weakly to, say, \( \tilde{g} \in H^2(\partial G) \). By relabelling, we still call it \( (g(f_n, M_n)) \). It follows directly from the assumptions and from (9) that
\[
\| \tilde{g} - f \|_{L^2(I)} \leq e_f(M), \quad \| \tilde{g} \|_{L^2(J)} \leq M.
\]
Now, because the solution to (BEP) is unique (by the strict convexity of the norm), we necessarily have that \( \tilde{g} = g(f, M) \). This shows the weak convergence in \( H^2(\partial G) \).

On the other hand, it holds from (9) that \( \| g(f_n, M_n) - f_n \|_{L^2(I)} \to \| g(f, M) - f \|_{L^2(I)} \) which implies that strong convergence always holds in \( L^2(I) \).

Finally, whenever \( (f, M) \notin D \), then
\[
\limsup_{n \to \infty} \| g(f_n, M_n) \|_{L^2(J)} \leq \limsup_{n \to \infty} M_n = M = \| g(f, M) \|_{L^2(J)},
\]
and since we have also \( g(f_n, M_n) \) converging weakly to \( g(f, M) \) in \( H^2(\partial G) \), then we obtain a strong convergence on \( J \).

In order to achieve convergence of the reconstruction scheme, continuity ensured by Theorem 11 is hardly sufficient. Aiming to ensure strong convergence of the extended data, one needs to deal with higher order methods. These methods consist in solving the (BEP) problem for the data derivatives, instead of the data themselves, provided some additional regularity is available in order to allow that. Let us define to that end the appropriate Hardy–Sobolev spaces:

For \( n \geq 1 \), define
\[
H^{n,2}(\partial G) := H^2(\partial G) \cap W^{n,2}(\partial G)
\]
\[
= \left\{ f \in H^2(\partial G); f^{(k)} \in H^2(\partial G), 1 \leq k \leq n \right\}.
\]
For consistency, we shall also denote by \( H^{0,2}(\partial G) \) the space \( H^2(G) \).

Let now \( g_n \) be the mapping:
\[
g_n : W^{n,2}(I) \times \mathbb{R}^*_+ \to H^{n,2}(\partial G)
\]
\[
\text{(11)}
\]
defined by:
\[
[g_n(f, M)]^{(n)} = g(f^{(n)}, M), \quad [g_n(f, M)]^{(k)}(z_0) = f^{(k)}(z_0), \quad 0 \leq k \leq n - 1
\]
for some fixed \(z_0 \in I\). Note that \(g_0 = g\). An order-\(n\) version of the (BEP) problem consists in solving (BEP) with bound \(M\) for the \(n\)-th derivative \(f^{(n)}\) of \(f\), and then integrating \(n\) times using the initial conditions provided above, in order to get \(g_n(f, M)\) as a function of \(H^{n,2}(\partial G)\), see also [5].

Finally, let us define as above:
\[
D_n = \{ (h, M) \in H^{n,2}(\partial G)|_I \times \mathbb{R}^*_+ \mid (h^{(n)}, M) \in \mathcal{D} \}.
\]
Similarly to the previous theorem, the convergence result that holds in \(D_n\) is weaker than the one holding outside \(D_n\).

**Theorem 12** The mapping \(g_n\) is continuous on \((W^{n,2}(I) \times \mathbb{R}^*_+) \setminus D_n\), but not on the whole of \(W^{n,2}(I) \times \mathbb{R}^*_+\). However, if \((f_k, M_k) \to (f, M)\) in \(W^{n,2}(I) \times \mathbb{R}^*_+\), then \(g(f_k, M_k) \to g(f, M)\) weakly in \(H^{n,2}(\partial G)\), whereas \(g(f_k, M_k) \to g(f, M)\) in \(W^{n,2}(I)\). Thus, \(g_n\) is continuous on \(H^{n-1,2}(\partial G)\).

**Proof** The first two statements are direct consequences of Theorem 11 applied to the \(n\) first derivatives of the function \(g_n\). Regarding the third one, this follows since if \((f_k)\) is a sequence in \(W^{1,2}(J)\) such that \(f_k(z_0) = f(z_0), \ z_0 \in J\), with derivative \(f'_k\) converging weakly to \(f' \in L^2(J)\), then \(f_k \to f\) pointwise in \(J\) and hence, by the Lebesgue dominated convergence theorem, \(\|f_k - f\|_{L^2(J)} \to 0\).

## 4 Identification algorithms

We present in this section an original family of numerical algorithms permitting one to compute the Robin inverse problem solution. Still in the electrical framework, once the current flux and the electrical potential have been computed on the inaccessible boundary \(J\), we can evaluate the impedance (or Robin coefficient) \(q\) from equation (3):
\[
q = -\frac{\partial \theta}{\text{Im} g(f, M)} \quad \text{on} \ J,
\]
where \(f\) is the prescribed data, and \(g(f, M)\) the extended data computed by solving the (BEP) problem using \(f\) and the bound \(M\).

Actually, the data we are dealing with are usually noisy ones
\[
f_\varepsilon := f + \varepsilon,
\]
where \(f \in H^2(\partial G)|_I\) and \(\varepsilon \in L^2(I)\). In that case, what can be derived from the above section, namely from Theorem 11, is that in order to provide with extended data “close” to the actual ones, the (BEP) problem needs to use a bound \(M\) close enough to the actual one \(M_0 := \|f\|_{L^2(J)}\). Moreover, since the prescribed data do not belong to the Hardy class...
\( H^2(\partial G) \), the computed extension will saturate the prescribed bound whatever its value is, i.e.,
\[
\|g(f_\varepsilon, M)\|_{L^2(J)} = M.
\]
The point is that the actual bound \( M_0 \) is unknown, since it depends on the unknown part of the data. Any constructive algorithm will thus need to tackle together the tasks of computing the extended data and the bound on them. To make the paper easier to read, we shall however describe separately in the sequel how to go through each of these tasks.

Let \( f_\varepsilon = f + \varepsilon \) be the noisy data (\( f \in H^2(\partial G) \), \( \varepsilon \in L^2(I) \) but \( \varepsilon \notin H^2(\partial G) \)), and \( M_0 := \|f\|_{L^2(J)} \) be the actual (unknown) bound.

### 4.1 Determination of the actual bound

In [9], the authors have proposed, in order to determine the bound, a cross-validation procedure that needs to devote some part of the prescribed data to that task. Though efficient, this method turns out to be penalizing in the present case, since a smaller amount of data is available, due to the multiply connected geometry. It is thus preferable to devote the whole of these data to the reconstruction task, which requires one to build up an alternative “non-data-consuming” method in order to compute the bound. We shall be presenting that alternative method in the sequel.

Given a some positive real number \( M \), \( g(f, M) \) denotes as usual the solution of the (BEP) problem with data \( f \) and bound \( M \), whereas \( g_\varepsilon := g(f_\varepsilon, M) \) solves the same problem with \( f_\varepsilon \) as a data set and the same bound \( M \). The previous section convergence results indicate that \( g(f_\varepsilon, M) \) is close to \( f \), which is equal to \( g(f, M_0) \), provided \( M \) is close to \( M_0 \), and \( f_\varepsilon \) close to \( f \). Since we do not know \( M_0 \), let us try to evaluate the difference \( f - g_\varepsilon \). An approximation of this function on \( I \) may be given by \( f_\varepsilon - g_\varepsilon \), whereas a rough estimate of the bound may be given by
\[
e_{f_\varepsilon}(M) := \|f_\varepsilon - g_\varepsilon\|_{L^2(I)}.
\]
Let then \( w_\varepsilon \) solve the (BEP) problem with these data
\[
w_\varepsilon := g(f_\varepsilon - g_\varepsilon, e_{f_\varepsilon}(M)).
\]
Therefore, \( g_\varepsilon + w_\varepsilon \) is likely to provide a better approximation to \( f \) than \( g_\varepsilon \). As a matter of fact, let us define:
\[
\tau_\varepsilon: \mathbb{R}^+ \rightarrow \mathbb{R}^+_+ \quad M \mapsto |M - \|g_\varepsilon + w_\varepsilon(M)\|_{L^2(J)}|.
\]
The closer \( M \) becomes to the actual bound, the better the approximation becomes, and the closer to zero \( \tau_\varepsilon(M) \) becomes. Minimizing \( \tau_\varepsilon(M) \) seems thus a reasonable way to find out the actual bound \( M_0 \). This is what we are going to prove in Theorem 13 for analytic data. First, let us notice that:
\[
\tau_\varepsilon(M) \leq e_{f_\varepsilon}(M), \quad \forall M \in \mathbb{R}^+_+.
\]
Indeed,
\[
\tau_\varepsilon(M) = \| \| g_\varepsilon \|_{L^2(I)} - \| g_\varepsilon + w_\varepsilon(M) \|_{L^2(I)} \|
\leq \| w_\varepsilon(M) \|_{L^2(I)}
= e_f(M).
\]

**Theorem 13** (Bound determination for analytic data)
In case the data are analytic (i.e., \( \varepsilon = 0 \)), then \( M_0 \) is the smallest positive real number that minimizes the mapping \( \tau_0 \) and moreover \( \tau_0(M_0) = 0 \).

**Proof** Since \( f \in H^2(\partial G)|_I \), then for each \( M \geq M_0 \) one has \( g(f, M) = f \) on \( I \), therefore \( e_f(M) = 0 \). From (13), we get \( \tau_0(M) = 0, \forall M \geq M_0 \).

On the other hand, suppose \( M < M_0 \). Since \( g(f, M) \) solves the (BEP) problem with respect to \((f, M)\), we have
\[
e_f(M) = \| f - g(f, M) \|_{L^2(I)} = \inf \{ \| f - g \|_{L^2(I)} : g \in H^2(\partial G), \| g \|_{L^2(J)} \leq M \} > 0
\]
and, since \( w_0(M) \) solves the (BEP) problem with respect to \((f - g(f, M), e_f(M))\), we have
\[
\| f - g(f, M) - w_0(M) \|_{L^2(I)} = \inf \{ \| f - g(f, M) - w \|_{L^2(I)} : w \in H^2(\partial G), \| w \|_{L^2(J)} \leq e_f(M) \}.
\]
Since the null function \( w = 0 \) is in \( H^2(\partial G) \) (and \( \| w \|_{L^2(J)} = 0 < e_f(M) \)) then
\[
\| f - g(f, M) - w_0(M) \|_{L^2(I)} \leq \| f - g(f, M) \|_{L^2(I)}.
\tag{14}
\]
If there exists a real \( M < M_0 \) such that \( \tau_0(M) = 0 \), then \( \| g(f, M) + w_0(M) \|_{L^2(J)} = M \). Therefore, from (14) and uniqueness of the solution of the (BEP) problem, we have \( g(f, M) + w_0(M) = g(f, M) \) on \( G \) and then \( w_0(M) = 0 \) on \( G \). This implies that \( e_f(M) = \| w_0(M) \|_{L^2(J)} = 0 \), which contradicts the fact that \( e_f(M) \neq 0 \).

![Figure 2: Plots of \( \tau_0(M) \) (left) and \( \tau_\varepsilon(M) \), \( \varepsilon \neq 0 \) (right)](image)

The non-analytic data case is however the one we are interested in. Figure 2 illustrates the behaviour of the functions \( \tau_0(M) \) and \( \tau_\varepsilon(M) \) for the quite smooth function \( f(z) = c + \frac{2(\varepsilon - 1)}{z - 0.1} \). For \( \varepsilon \neq 0 \), the function \( \tau_\varepsilon \) seems to have a minimum, the argument of which is close to the correct value \( M_0 \) of the bound. At this stage, the result we are able to prove is unfortunately somewhat weaker.
Theorem 14 Let \( \alpha \) and \( \beta \) be two positive numbers such that \( 0 < \alpha < \beta \), and \( M_0 \in [\alpha, \beta] \), and let \( \varepsilon \in L^2(I) \) be a positive function.

(i) The function \( \tau_{\varepsilon} \) has at least one minimum \( M_\varepsilon \) in \([\alpha, \beta]\). Moreover, defining \( \delta_\varepsilon := \inf_{M \in [\alpha, \beta]} \tau_\varepsilon(M) = \tau_\varepsilon(M_\varepsilon) \), we have \( \lim_{\|\varepsilon\|_{L^2(I)} \to 0} \delta_\varepsilon = 0 \).

(ii) Let \( \mathcal{I}_\varepsilon = \{ M_\varepsilon \in [\alpha, \beta] : \delta_\varepsilon = \tau_\varepsilon(M_\varepsilon) \} \). Then \( \mathcal{I}_\varepsilon \) has a minimum point \( M_\varepsilon \).

(iii) Any accumulation point \( M_0 \) of the family \( (M_\varepsilon)_\varepsilon \) is such that \( M_0 \geq M_0 \).

(iv) When \( \|\varepsilon\|_{L^2(I)} \to 0 \), then \( g(f_\varepsilon, M_\varepsilon) \to f \) weakly in \( H^2(\partial G) \), hence also in the weak topology of \( L^2(J) \), and \( g(f_\varepsilon, M_\varepsilon) \to f \) in \( L^2(I) \).

Proof

(i) Since the data \( f_\varepsilon \) are not analytic, we get from Theorem 11:

\[
\lim_{M_n \to M} \|g(f_\varepsilon, M_n) - g(f_\varepsilon, M)\|_{L^2(J)} = 0,
\]

and also:

\[
\lim_{M_n \to M} \|w_\varepsilon(M_n) - w_\varepsilon(M)\|_{L^2(J)} = 0,
\]

\( \tau_\varepsilon \) is thus continuous on the compact set \([\alpha, \beta]\), and there exists some real number \( M_\varepsilon \in [\alpha, \beta] \) such that \( \tau_\varepsilon(M_\varepsilon) = \delta_\varepsilon \).

Let \((\varepsilon_n)_n\) a sequence such that \( \lim_{n \to \infty} \varepsilon_n = 0 \). Since \( M_0 \in [\alpha, \beta] \), we have:

\[
0 \leq \delta_{\varepsilon_n} \leq \tau_{\varepsilon_n}(M_0) \leq e_{f_{\varepsilon_n}}(M_0).
\] (15)

From equation (9), we obtain

\[
0 \leq \lim_{n \to \infty} \delta_{\varepsilon_n} \leq e_f(M_0) = 0.
\] (16)

(ii) Let \( \mathcal{I}_\varepsilon = \{ M_\varepsilon \in [\alpha, \beta] : \delta_\varepsilon = \tau_\varepsilon(M_\varepsilon) \} \). Now \( \mathcal{I}_\varepsilon \) is a closed subset of \([\alpha, \beta]\) since \( \mathcal{I}_\varepsilon = \tau_\varepsilon^{-1}(\delta_\varepsilon) \), then it is a compact set and therefore \( M_\varepsilon \) exists.

(iii) Assume there exists a subsequence \((M_{\varepsilon_n})_n\) of \((M_\varepsilon)_\varepsilon\) such that \( \lim_{n \to \infty} M_{\varepsilon_n} = M_0 < M_0 \). Introduce the notation \( g_\varepsilon(M) = g(f_\varepsilon, M) \), \( g_0(M) = g(f, M) \), and similarly for the functions \( w_\varepsilon \) and \( w_0 \). By Theorem 11, we have

\[
\|g_{\varepsilon_n}(M_{\varepsilon_n})\|_{L^2(J)} \to \|g_0(M_0)\|_{L^2(J)},
\]

and by (9),

\[
\|w_{\varepsilon_n}(M_{\varepsilon_n})\|_{L^2(J)} \to \|w_0(M_0)\|_{L^2(J)}.
\]

Then

\[
0 = \lim_{n \to \infty} \delta_{\varepsilon_n} = \tau_0(M_0),
\]

13
and we deduce that
\[ M_0 = \|g_0(M_0) + w_0(M_0)\|_{L^2(J)} \cdot \]
Since
\[ \|f - g_0(M_0) - w_0(M_0)\|_{L^2(I)} \leq \|f - g_0(M_0)\|_{L^2(I)}, \]
therefore \( g_0(M_0) + w_0(M_0) = g_0(M_0) \), which implies that \( w_0(M_0) = 0 \), and in this case we deduce that \( e_f(M_0) = 0 \); then \( \|f - g_0(M_0)\|_{L^2(I)} = 0 \), i.e., \( f = g_0(M_0) \) and \( M_0 = M_0 \), which is a contradiction.

(iv) This is a straightforward consequence of Theorem 11 and the point (iii) above.

Note that weak convergence in \( H^2(\partial G) \) implies weak convergence when restricted to \( L^2(J) \), since the traces of functions in \( H^2(\partial G) \) are dense in \( L^2(J) \).

Remark 15 Theorem 14 does not provide us with the actual bound for non-analytic data as Theorem 13 does for analytic data. However, it provides us with a family of bounds permitting to compute extended data that converge – although weakly – to the required extension. The uncoupling of both tasks (bound determination and data extension) here reaches its limits. In the following sub-sections, we shall need to couple them again, in order to build up robust reconstruction algorithms.

4.2 The zero–order algorithm (A_0)

The above Theorem 14 does not ensure the convergence of the bound \( M_\varepsilon \) to \( M_0 \), since only weak convergence of the analytic extensions \( g(f_\varepsilon, M_\varepsilon) \) to \( f \) holds on \( J \). Still, despite its ineffectiveness at least from a theoretical point of view, it is interesting to describe the so-called “zero–order” algorithm that we shall use next as an intermediate tool.

<table>
<thead>
<tr>
<th>The (A_0) algorithm:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Given ( M &gt; 0 ), solve the (BEP) problem with respect to ( (f_\varepsilon, M) ) and get ( g_\varepsilon(M) := g(f_\varepsilon, M) ), ( e_f(M) := |f_\varepsilon - g_\varepsilon(M)|_{L^2(I)} ).</td>
</tr>
<tr>
<td>2. Solve the (BEP) problem w.r.t. ( (f_\varepsilon - g_\varepsilon(M)</td>
</tr>
<tr>
<td>3. Compute ( M_\varepsilon := \text{Argmin}<em>{M &gt; 0} \tau</em>\varepsilon(M) ) by some numerical method such as the golden section search ;</td>
</tr>
<tr>
<td>4. Compute ( q_\varepsilon = -\frac{\partial_\theta \text{Im} g_\varepsilon(M_\varepsilon)}{\text{Re} g_\varepsilon(M_\varepsilon)} ) on ( J ).</td>
</tr>
</tbody>
</table>

The numerical implementation of this algorithm has been done using Matlab 7.1. [19]. The discrete Fourier transform function \( \text{fft} \) and the inverse discrete Fourier transform one \( \text{ifft} \) have been used in order to compute the Fourier coefficients, whereas the Toeplitz matrix coefficients have been computed using the function \( \text{toeplitz} \). The finite differences function \( \text{diff} \) has been used to compute the function derivatives.
4.2.1 Solving the (BEP) problem for a prescribed multiplier $\lambda$

We are given data $(f, \varepsilon)$, a prescribed bound $M$, and the related multiplier $\lambda$; we shall describe how to derive $\lambda$ from the bound $M$, and compute it, in the next subsection. The solution of the (BEP) problem is obtained by solving the infinite linear equation (6), (7) given by Theorem 7. A discretization is needed, using a finite basis of Fourier functions \( \{e_n(z), -N \leq n \leq N\} \). The proper value $N$ to choose has been derived from an error study: given some data $f$, and the noisy data $f_\varepsilon$ derived from them, we have plotted the error $\|f - f_{\varepsilon,N}\|_{L^2(I)}$ between the actual data and the truncated noisy data (Figure 3).

![Figure 3: Error approximations w.r.t. the number of Fourier functions used](image)

It turns out actually that a value $N = 7$ (15 basis functions) is sufficient to bring this error below the noise level, if the level is around 15%. Furthermore, the figure shows that it is not worthwhile to choose more than 35 basis functions ($N = 17$), since the error is stabilized starting from that point. In order to fix our computations, and since these computations are quite cheap, we have however chosen $N = 25$ (51 basis functions).

Let us now describe the computations. Having prescribed data $f$ on the part $I$ of the boundary, discretizing equation (6) leads to the following:

\[
((Id + \lambda T_I) g_N)_N = P_{H^2}(\chi_J f_\varepsilon)_N, \tag{17}
\]

which can also be written as follows:

\[
[(T_I + (1 + \lambda) T_I) g_N]_N = P_{H^2}(\chi_J f_\varepsilon)_N, \tag{18}
\]

where $T_I := P_{H^2} \chi_I$ is the Toeplitz operator associated to the characteristic function related to the part $I$ of the boundary (in our implementation, $I := (e^{-i\theta_0}, e^{i\theta_0}), \theta_0 \in [0, \pi]$).

The Toeplitz matrices of operators $T_I$ (and $T_I = Id - T_I$) with respect to the basis \( \{e_n(z), n = -N, \cdots, N\} \) are obtained by truncating the infinite matrix \( (T_{n,m})_{(n,m)\in\mathbb{Z}\times\mathbb{Z}} \).
by (7):

\[ T_{n,m} = \begin{cases} \frac{1}{1 + s^{2n}} \left( 1 + s^{2n} - \frac{\theta_0}{\pi} \right) & \text{when } n = m, \\ \frac{1}{\sqrt{(1 + s^{2n})(1 + s^{2m})}} \sin (m - n)\theta_0 \pi (m - n) & \text{when } n \neq m. \end{cases} \]

The linear system (18) is finally solved using the Matlab Cholesky inversion routine named \texttt{pinv}.

We have described so far how to go through items (1)–(2) of the above (A0) algorithm, provided the multiplier \( \lambda \) is known. Let us now describe how to derive it from the bound \( M \).

### 4.2.2 Determining the multiplier \( \lambda \) associated to the bound \( M \)

It has already been mentioned in Proposition 5 and Remark 6 that the mapping \( \lambda \to M(\lambda) \) is \( C^1 \), bijective and decreasing from \((-1, +\infty)\) to \((0, +\infty)\). Using the change of variables

\[ \lambda = \frac{r}{1 - r} - 1 \]

gives us a function \( M(\lambda(r)) \) decreasing on \([0, 1)\). Furthermore, we know that the right value of \( \lambda \) is that ensuring that the computed (BEP) solution w.r.t. \( \lambda \) saturates the bound \( M \), which actually means:

\[ V(\lambda) := \|g(f, \lambda)\|_{L^2(J)} = M. \]

Having prescribed some threshold, a bisection method has been used to find \( r \), increasing \( r \) if \( V(\lambda(r)) > M \) and decreasing it otherwise.

### 4.2.3 Computing the right bound \( M \)

Let us now describe how to compute the right bound \( M \), which is Step 3 of the (A0) algorithm. Given a bound \( M \), one needs to solve two (BEP) problems – as described by items (1) – (2) of the (A0) algorithm – in order to compute \( \tau_\varepsilon(M) \)

1. Solve the (BEP) problem w.r.t. to \((f_\varepsilon, M)\) and get \( g_\varepsilon(M) := g(f_\varepsilon, M) \)

2. Solve the (BEP) problem w.r.t. \( (f_\varepsilon - g_\varepsilon(M)|_I, \|f_\varepsilon - g_\varepsilon(M)\|_{L^2(I)}) \) and get \( w_\varepsilon(M) \)

Having done so, we have, as defined above in (12):

\[ \tau_\varepsilon(M) := |M - \|g_\varepsilon + w_\varepsilon(M)\|_{L^2(J)}|. \]

Minimizing \( \tau_\varepsilon \) w.r.t. \( M \) has been done using the golden section search method for \( M \in [A, B] \).

In case the bound provided by the algorithm is equal to \( A \) (resp. \( B \)), one needs to run it once again, after enlarging the interval \([A, B]\) on the left hand side (resp. on the right hand side).

Finally, we compute the (BEP) extension \( g_\varepsilon \) associated with the bound \( M_\varepsilon \) so-obtained, in order to compute the Robin coefficient, which requires us first to differentiate its imaginary part using finite differences.
4.3 The higher order algorithms \((A_n)\)

The fourth statement in Theorem 14 can be seen as a weak robustness result for the \((A_0)\) algorithm. This is not enough even as regards the data extension, and it is definitely less than our needs for the impedance computation. This is the reason why, in search of better robustness properties, we shall now investigate higher order algorithms based on the same tools.

The basic idea is actually to apply the above described zero-order algorithm to the \(n^{th}\) derivatives of the prescribed data, and then to integrate \(n\) times the so-extended derivatives.

Let \(f_\varepsilon = f + \varepsilon\), where \(\varepsilon\) is a non–analytic, but still smooth, perturbation \((\varepsilon \in W^{n,2}(I) \setminus H^{n,2}(\partial G)_I))\), and assume \(f \in H^{n,2}(\partial G)\). The \((A_n)\) algorithm is thus expressed as follows:

**The \((A_n)\) algorithm:**

1. Compute the \(n^{th}\) derivative \(f_\varepsilon^{(n)}\) of \(f_\varepsilon\) on \(I\);
2. Apply the zero order method to the data \(f_\varepsilon^{(n)}\), and get \(g_\varepsilon^{(n)}\);
3. Integrate \(n\) times \(g_\varepsilon^{(n)}\) and get \(g_{n,\varepsilon}\);
4. Compute
   \[
   q_{n,\varepsilon} = -\frac{\partial_\theta \text{Im} g_{n,\varepsilon}}{\text{Re} g_{n,\varepsilon}} \quad \text{on} \quad J.
   \]

Thanks to the continuity properties of Section 3.4, these algorithms possess much better robustness properties than the zero-order one. This is the content of the next theorem.

**Theorem 16** *(Robustness of the \(n^{th}\) order method)*

Suppose \(\phi \in W^{n,2}(I), q \in \mathcal{Q}^n, n \geq 1\). Let then \(f_\varepsilon = u_d + i \int \phi \, d\theta + \varepsilon \in W^{n,2}(I)\) and \(g_{n,\varepsilon}\) as above. As \(\|\varepsilon\|_{W^{n,2}(I)} \to 0\) it holds that:

\[
\text{Re} \, g_{n,\varepsilon} \to u \quad \text{in} \quad W^{n,2}(\partial G), \quad \partial_\theta \text{Im} \, g_{n,\varepsilon} \to \partial_n u \quad \text{in} \quad W^{n-1,2}(\partial G).
\]

Also

\[
q_{n,\varepsilon} \to q \quad \text{in} \quad W^{n-1,2}(J).
\]

**Proof** This is an immediate consequence of Theorem 12. ■

**Remark 17** This “robustness” result is obtained for smooth noise \((\varepsilon \in W^{n,2}(I))\), a feature that noise is actually not expected to have. Suppose now \(\varepsilon \in L^\infty(I)\) with \(|\varepsilon(x)| \leq \varepsilon\) for \(x\) a.e. in \(I\). Let us denote by \(\hat{f}_\varepsilon\) the smoothed function obtained by using cubic B-splines with a path length \(h\). It has been proved in [13] that we then have the following estimates:

\[
\|\hat{f}_\varepsilon - f\|_{L^\infty(I)} \leq c(\varepsilon + h^2), \quad \|\frac{\partial}{\partial x}(\hat{f}_\varepsilon - f)'\|_{L^\infty(I)} \leq c \left(\frac{\varepsilon}{h} + h\right).
\]
Choosing now $h = O(\sqrt{\varepsilon})$, we get a $\sqrt{\varepsilon}$ error on $f'$, which means $(\hat{f}^\varepsilon)'$ can be seen as noisy data w.r.t. $f'$, with a noise level $\sqrt{\varepsilon}$. By “bootstrapping” with the B-spline approximation, we can thus get an estimate of order $\varepsilon^{1/2}$ on the $p$-th derivative of $f$.

This means that the smoothing of noisy data by using proper B-splines provides with “smoothed noisy data” that meet the above theorem assumptions. Actually, this is the way numerical results are usually run: data are smoothed prior to being processed.

In Section 5, we are going to confirm these robustness properties by a thorough numerical study which shows the efficiency of the higher order methods in both the tasks of extending the data and recovering the electrical impedance coefficients.

5 Numerical validation

In the numerical results we are presenting in this section, we have considered both cases of full prescribed data (i.e., data prescribed on the whole of the outer boundary), and of incomplete data (i.e., data prescribed on some part of the outer boundary). The latter case is actually the most realistic one, particularly concerning non-destructive control applications. The impact on the outcome of several parameters has been studied:

- smoothness of the data to be reconstructed,
- amount of prescribed data,
- noise level.

The smooth data we have considered are those resulting from $f(z) = \exp(z)$, whereas non-smooth data have been generated by

$$f(z) = c + 2 \frac{z - 1}{z - a},$$

where $c$ is some constant and $a$ some point inside the internal disk. The closer $a$ becomes to the circle $s\Gamma$, the more “singular” the data to reconstruct become.

5.1 Case of full external data

Extension formulae at order zero are provided by [10, 11, 22], and formulae for the $(\mathbf{A}_n)$ algorithms ($n = 0, 1, 2$) have been straightforwardly derived from them.

First, the smooth data to be reconstructed are those resulting from the function $f(z) = \exp(z)$. Figure 4 shows that the three methods (zero, first and second order) provide very accurate results, as regards the analytic extension as well as the electrical impedance computed from it.

Things change however when it comes to noisy data, as can be noticed in Figure 5. Although the extended data using the zero and first order methods remain acceptable up to a 10% level noise, the accuracy of the reconstructed electrical impedance drops dramatically when the noise level increases. Actually, the zero order method turns out to be definitely unsuitable for the electrical impedance recovery task, whereas the first and second order ones behave quite well in that respect.
These conclusions were predictable from the theoretical results on robustness proved in Section 4. The zero order method possesses only weak robustness properties, regarding the extended data but not the electrical impedance. From Theorem 16, we derive that the first order method is the lowest possible ensuring $L^2$ convergence for the electrical impedance.

The sensitivity of the reconstruction method with respect to $a$, which parametrizes the smoothness of the data generated by the function (19) is summarized in Figure 6. As expected, the accuracy on the electrical impedance computed drops when $a$ gets too close to the internal boundary. The first and second order methods do not show qualitative differences, though the second order one is more accurate, even for data lacking smoothness. The next part of the study will thus focus on the second order algorithm.

## 5.2 Case of incomplete external data

In this section, we are concerned with the behaviour of the algorithm when data are lacking on some part of the outer boundary. This situation is likely to happen quite often.
in practice, and this is the reason why we have run quite an extensive numerical study, investigating the impact of the following parameters on the result:

- amount of prescribed data, as measured by the ratio $\rho = \frac{|I|}{2\pi}$ where $|I|$ is the Lebesgue measure of the prescription area $I$ on the outer boundary, whose length is $2\pi$;
- non-smoothness of the data, as parametrized by $\delta := \frac{1}{s}d(a, sT)$, where $a$ is the complex number defined in the previous section (location of a singularity inside the inner disk);
- noise level.

In the case of non-noisy smooth data, Figure 7 shows that the error on the Robin coefficient remains acceptable for as small a quantity of data as that prescribed on half the outer boundary, and the error decreases quite fast with respect to the amount of prescribed data.

Let us now study the sensitivity of the reconstruction method to the data smoothness. By making $a$ closer to the circle $sT$, the behaviour of the function (19) gets harsher, though remaining “smooth” as stated in Theorem 16. Unsurprisingly, the left-hand plot
in Figure 8 shows that the harsher the data, the lower the accuracy on the computed electrical impedance. However, the plots in the right-hand side of Figure 8 also indicate how to make up for the lack of smoothness by increasing the amount of prescribed data. Highly singular functions need an almost complete set of external data in order to compute the electrical impedance with an acceptable accuracy.

Figure 8: Plots of errors w.r.t. amount of data (left) and of \( \rho \) versus \( \delta \) for a 1\%, 5\% and 10\% error level (right)

The noise effects are somewhat similar. The right-hand plot of Figure 9 displays curves relating the noise level to the amount of prescribed data for different targeted error levels. Once again, we observe that to some extent, one can make up for the noise effects by increasing the amount of available data.

Figure 9: Plots of errors w.r.t. \( \rho \) in the case of noisy data (left), and of \( \rho \) versus noise at a 5\%, 7\% and 10\% error level

6 Conclusion

The methods we have been presenting in this work constitute a family of fast data completion algorithms solving the Cauchy problem for the Laplace equation in an annular domain, and up to a conformal mapping, in similar domains in the plane. The main goal was to compute accurately from these data the electrical impedance on the inaccessible inner part of the boundary from the extended data.

To that end, we have derived new explicit formulae in the case when the set of available data on the outer boundary is not complete. These formulae have been implemented
in order to build up algorithms using the bounded extremal problems, and needing the actual bound on the unknown data to be computed at the same time as the data are extended. These algorithms use a new stabilization technique that proves to be fast and efficient. Beside their efficiency, the so-designed algorithms have been proved to be robust with respect to noise, and a thorough numerical study has been run that widely confirms these theoretical predictions.

Despite their valuable qualities (accuracy and robustness at a low computational cost), these algorithms have two limitations. The first is related to the exclusive focus on the Laplace equation. Though not restrictive for corrosion detection, this limitation would need to be lifted, and extensions to other differential operators are already under study. Secondly, extending these methods to 3D problems is the crucial issue to investigate in order to deal with “real-life problems”. However, these extensions are not at all straightforward. Finally, besides tackling these two challenging developments, the next step is to study “real 2D problems”, in other domains than the annulus, in order to obtain a clearer idea of how the conformal mapping affects the numerical results.

7 Appendix

Proof of Lemma 9.
By [22], we can write

\[ P \mathcal{H}^2(\partial G) \phi(z) = \sum_{n \in \mathbb{Z}} \frac{a_n + b_n s^{2n}}{1 + s^{2n}} z^n \]

where

\[ a_n = \frac{1}{2\pi} \int_0^{2\pi} \phi(e^{i\theta}) e^{-i n \theta} \, d\theta \]
\[ = \frac{1}{2\pi} \int_{\theta_0}^{\theta_0 + 2\pi} \phi(e^{i\theta}) e^{-i n \theta} \, d\theta + \frac{1 + \lambda}{2\pi} \int_{\theta_0}^{2\pi-\theta_0} \phi(e^{i\theta}) e^{-i n \theta} \, d\theta \]
\[ = \frac{1}{2\pi} \int_{-\theta_0}^{\theta_0} f(e^{i\theta}) e^{-i n \theta} \, d\theta + \frac{1 + \lambda}{2\pi} \int_{\theta_0}^{2\pi-\theta_0} f_1(e^{i\theta}) e^{-i n \theta} \, d\theta, \]

and

\[ b_n s^n = \frac{1}{2\pi} \int_0^{2\pi} \phi(se^{i\theta}) e^{-i n \theta} \, d\theta \]
\[ = \frac{1 + \lambda}{2\pi} \int_0^{2\pi} f_1(se^{i\theta}) e^{-i n \theta} \, d\theta. \]

\[ \square \]

Proof of Lemma 10.
Let \( \phi = \chi_J g \). Then we have from Lemma 9

\[ P \mathcal{H}^2(\partial G) \phi(z) = \sum_{n \in \mathbb{Z}} \frac{c_n + d_n s^{2n}}{1 + s^{2n}} z^n, \]
where
\[ c_n = \frac{1}{2\pi} \int_{\theta_0}^{2\pi-\theta_0} g(e^{i\theta}) e^{-in\theta} \, d\theta, \]
and
\[ d_n s^n = \frac{1}{2\pi} \int_0^{2\pi} g(se^{i\theta}) e^{-in\theta} \, d\theta. \]

Let \( g(z) = \sum_{n \in \mathbb{Z}} g_n z^n \) for \( z \in G \), then
\[ c_n = \frac{1}{2\pi} \int_{\theta_0}^{2\pi-\theta_0} \sum_{m \in \mathbb{Z}} g_m e^{im\theta} e^{-in\theta} \, d\theta \]
\[ = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} g_m \int_{\theta_0}^{2\pi-\theta_0} e^{i(m-n)\theta} \, d\theta \]
\[ = g_n \left( 1 - \frac{\theta_0}{\pi} \right) - \sum_{m \neq n} g_m \frac{\sin (m-n)\theta_0}{\pi(m-n)}. \]

and
\[ d_n s^n = \frac{1}{2\pi} \int_0^{2\pi} \sum_{m \in \mathbb{Z}} g_m s^m e^{im\theta} e^{-in\theta} \, d\theta \]
\[ = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} g_m s^m \int_0^{2\pi} e^{i(m-n)\theta} \, d\theta \]
\[ = g_n s^n. \]

Then we deduce that
\[ P_{H^2(\partial G)} \chi_J g(z) = \sum_{n \in \mathbb{Z}} \frac{1}{1 + s^{2n}} \left( g_n \left( 1 - \frac{\theta_0}{\pi} \right) + g_n s^{2n} - \sum_{m \neq n} g_m \frac{\sin (m-n)\theta_0}{\pi(m-n)} \right) z^n, \]
therefore
\[ T g(z) = \sum_{n \in \mathbb{Z}} \frac{1}{1 + s^{2n}} \left( g_n \left( 1 + s^{2n} - \frac{\theta_0}{\pi} \right) - \sum_{m \neq n} g_m \frac{\sin (m-n)\theta_0}{\pi(m-n)} \right) z^n. \]

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