Interpolation in Hardy spaces, with applications

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Plan of course

1. Hardy spaces, classical interpolation results.
2. Reproducing kernels.
3. Linear systems, Hardy spaces,
   extremal problems.
4. Semigroups, admissibility, controllability.

Part 1: Hardy spaces, classical interpolation results.

For $1 \le p < \infty$ the Hardy space H^p on the unit disc $\mathbb{D}$ consists of analytic functions with L^p boundary values on the circle $\mathbb{T}$.

More precisely, H^p is the Banach space of functions such that the norm

$$\|f\|_p = \left(\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{1/p} < \infty.$$

Likewise, H^∞ is the bounded analytic functions in $\mathbb{D}$, and

$$\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|.$$

H^p functions can be given boundary values on the circle $\mathbb{T}$ (almost everywhere) by setting

$$\widetilde{f}(e^{it}) = \lim_{r \to 1-} f(re^{it}),$$

and

$$\|\widetilde{f}\|_{L^p(\mathbb{T})} = \|f\|_{H^p}.$$

We'll usually write f instead of $\widetilde{f}$.

For us, $p = 2$ and $p = \infty$ are the most important.

Note that, if

$$f(z) = \sum_{n=0}^\infty a_n z^n, \qquad \text{then}$$

$$\|f\|_2 = \left(\sum_{n=0}^\infty |a_n|^2 \right)^{1/2}.$$

Indeed H^2 is a **Hilbert space** and $(e_n)_{n=0}^\infty$ form an orthonormal basis, where $e_n(z) = z^n$.

Where do H^p functions vanish?

If the zeroes of $f \not\equiv 0$ are (z_n) in $\mathbb{D}$ (a finite or countable set), then they satisfy the **Blaschke condition**

$$\sum_n (1 - |z_n|) < \infty,$$

so, if infinitely many, they go to the boundary quickly.

Given a Blaschke sequence (i.e., satisfying the Blaschke condition), we can find a function in H^∞ (and hence all H^p) that has precisely these zeroes.

N.B. We allow repeated zeroes.

Let

$$B(z) = z^p \prod_{n:z_n \neq 0} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z_n} z}.$$

This is called a **Blaschke product**, and converges pointwise (locally uniformly) to an analytic function.

Note that $|B(e^{it})| = 1$ a.e., i.e., B is an **inner function**.

There are other inner functions without zeroes (singular inner functions), for example

$$\Theta(z) = \exp\left(\frac{z-1}{z+1}\right).$$

Zeroes on the boundary

This is more delicate as the functions are now only defined a.e.

In fact, if f is in an H^p space, then

$$\int_0^{2\pi} \log|f(e^{it})|\, dt > -\infty,$$

so $f \neq 0$ a.e. on $\mathbb{T}$. More on this later.

Another way to get inner functions

Let $S : H^2 \to H^2$ be the shift operator,

$(Sf)(z) = zf(z)$.

What are its closed invariant subspaces, i.e., subspaces $\mathcal{M}$ such that $S\mathcal{M} \subseteq \mathcal{M}$?

Answer given by **Beurling's theorem**. Apart from $\{0\}$ and H^2, the nontrivial subspaces are

$$\mathcal{M} = \Theta H^2 = \{\Theta . f : f \in H^2\},$$

where Θ is inner.

Example: the set of functions vanishing at (z_n) is BH^2 where B is the Blaschke product with those points as zeroes.

The **cyclic** functions, i.e., that lie in no proper S-invariant subspace, are called **outer functions**.

For example, $z-1$ or $z-2$.

Every H^p function is a product

[inner function] × [outer function].

More on finite Blaschke products

We all know how $B(z) = z^n$ maps $\mathbb{T}$ to itself. Finite Blaschke products do something similar.

If

$$B(z) = \prod_{k=1}^{n} \frac{z - z_k}{1 - \overline{z_k} z}.$$

is a Blaschke product with n zeroes including multiplicity, then it maps $\mathbb{T} \to \mathbb{T}$ as n-to-1.

If $B(e^{i\theta}) = e^{i\phi}$, then $\dfrac{d\phi}{d\theta} > 0$.

(This is easy algebra.)

In fact, the zeroes of B' lie in the convex hull of 0 and the zeroes of B (Walsh, for finite products; Cassier–Chalendar for infinite products).

Nevanlinna–Pick interpolation

Suppose $z_1, \ldots, z_n$ are distinct points in $\mathbb{D}$ and $w_1, \ldots, w_n$ are also points in $\mathbb{D}$.

Q1. Can we find an interpolating function $\phi \in H^\infty$ of norm at most 1 with $\phi(z_k) = w_k$ for each k?

Q2. What is the minimal H^∞ norm of an interpolating function ϕ?

Pick's theorem

Consider the $n \times n$ matrix Q such that

$$Q_{j,k} = \frac{1 - w_j \overline{w_k}}{1 - z_j \overline{z}_k} \qquad (j,k = 1, \ldots, n).$$

Then we can find such an interpolating function ϕ if and only if $Q \geq 0$ (i.e., positive semi-definite).

When $Q \geq 0$, there is an interpolating ϕ, which is a Blaschke product of degree at most n.

One way to prove this is as follows.

We introduce the **Cauchy–Szegö (reproducing) kernels**, k_a for $a \in \mathbb{D}$,

where $\quad k_a(z) = 1/(1 - \overline{a} z)$,

satisfying

$$f(a) = \langle f, k_a \rangle, \qquad (f \in H^2).$$

The functions $k_{z_1}, \ldots, k_{z_n}$ form a basis for a finite-dimensional space K_B, the orthogonal complement of BH^2, where B is the Blaschke product with zeroes $z_1, \ldots, z_n$.

Now look at operators $M_\phi : H^2 \to H^2$,

$$M_\phi f = \phi . f.$$

Note $\langle f, M_\phi^* k_{z_j} \rangle = \langle M_\phi f, k_{z_j} \rangle = \phi(z_j) f(z_j)$,

so $\quad M_\phi^* k_{z_j} = \overline{\phi(z_j)} k_{z_j}$ for each j.

It can be shown that ϕ interpolates, and M_ϕ is a contraction, if and only if $I - TT^* \geq 0$, where $T = M^*_{\phi | K_B}$, which satisfies

$$T k_{z_j} = \overline{w_j} z_j \quad \text{for each } j.$$

But then $\langle (I - TT^*) k_{z_j}, k_{z_k} \rangle$ is just the $Q_{j,k}$ in the Pick matrix.

The minimal-norm interpolant

Let p be **any** function in H^∞ such that $p(z_j) = w_j$ for all j, e.g. a polynomial.

Then the set of all possible ϕ is $p + BH^\infty$, where B is the Blaschke product with zeroes $z_1, \ldots, z_n$.

Now $\quad \|p + Bg\|_\infty = \|B^{-1} p + g\|_{L^\infty(\mathbb{T})}$.

We think of all functions as being defined on $\mathbb{T}$.

It's a **best approximation problem:** given $f \in L^\infty(\mathbb{T})$, minimize $\|f - h\|_\infty$ over all $h \in H^\infty$.

Answer given by **Nehari's theorem** (it involves Hankel operators). Here $f = B^{-1} p$ is rational and $g \in H^\infty$ turns out to be rational.

Boundary interpolation

We can do more than just interpolate on finite sets.

The **Rudin–Carleson theorem** says that if

$K \subset \mathbb{T}$ closed, measure 0, then any $f_0 \in C(K)$ can be extended to a function $f_1 \in A(\mathbb{D}) = C(\overline{\mathbb{D}}) \cap H^\infty$, with $\|f_1\|_{H^\infty} = \|f_0\|_{C(K)}$.

$A(\mathbb{D})$ is the **disc algebra**, the closure of the polynomials in H^∞.

Indeed, the zero sets (sets of uniqueness) for $A(\mathbb{D})$ are the closed subsets $Z \subset \overline{\mathbb{D}}$ such that

(i) $Z \cap \mathbb{T}$ has measure 0; (ii) $Z \cap \mathbb{D}$ countable, satisfying the Blaschke condition $\sum(1 - |z_n|) < \infty$.

Carleman formulae

For $f \in H^p$, we know that f is uniquely determined by its values on any set S of positive measure; that is, we have a restriction mapping

$$R : H^p \to L^p(S),$$

which is injective, but not surjective.

It has dense range (for $1 \leq p < \infty$).

The problem: how do we invert R?

If we know f on all of $\mathbb{T}$ we can find it in $\mathbb{D}$ by harmonic extension (Dirichlet problem, Poisson kernel) or by complex analysis (Cauchy's formula):

$$f(w) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(z)\, dz}{z - w}.$$

We rewrite this as

$$f(w) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{it})\, dt}{1 - we^{-it}}.$$

Goluzin–Krylov technique

We explain this for a closed subset K of positive measure – an arc is the most important example.

We begin by constructing a function $H \in H^\infty$ (invertible) such that

$$|H(e^{it})| = \exp \chi_K(e^{it})$$

a.e. on $\mathbb{T}$. Here χ denotes characteristic function (1 on K, 0 off K).

Indeed, we may take

$$H(z) = \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \chi_K(e^{it})\, dt \right\}.$$

Now $|H| > 1$ in the open disc $\mathbb{D}$, and thus, for fixed $w \in \mathbb{D}$, $\left(\frac{H(e^{it})}{H(w)}\right)^n$ gets very small on $\mathbb{T} \setminus K$.

H is called a **quenching function**.

Given the values on K of the unknown f define

$$f_n(w) = \frac{1}{2\pi}\int_K f(e^{it})\left(\frac{H(e^{it})}{H(w)}\right)^n \frac{dt}{1-we^{-it}}.$$

If we could integrate on the whole circle we'd get $f(w)$ exactly (but we don't know f except on K).

The quenching property can be used to prove easily that $f_n(w) \to f(w)$ pointwise as $n \to \infty$ for any $f \in H^1$ (so if f in any H^p space for $p \geq 1$).

Theorem (Patil, 1972) For $f \in H^p$, and

$1 < p < \infty$, we have $\|f_n - f\|_p \to 0$.

The proof involves Toeplitz operators. Indeed

$$f_n = \frac{e^{2n}}{e^{2n}-1}(f - T_n f),$$

where T_n are operators (inverses of Toeplitz operators) such that $\|T_n f\| \to 0$ for all $f \in H^p$.

Later we'll look at what can be done if we only know f approximately on K (real applications).

For recovery from values in $\mathbb{D}$, one can recover $f \in H^p$ from $(f(z_n))_n$, provided that $\sum(1-|z_n|) = \infty$, i.e., a non-Blaschke sequence. For example, Totik provided a formula in 1984.

Carleson interpolation

Suppose that $(z_j)_j$ is a sequence of distinct points in $\mathbb{D}$. Then we get a contractive mapping

$$R : H^\infty \to \ell^\infty, \qquad f \mapsto (f(z_j))_j.$$

When is this surjective? If so we call $(z_j)_j$ a

Carleson sequence.

By the open mapping theorem, if it is surjective then, for some $M > 0$ it holds that all $(a_j) \in \ell^\infty$ there is an $f \in H^\infty$ with

$$f(z_j) = a_j \text{ for all } j$$

and

$$\|f\|_\infty \leq M\|(a_j)\|_\infty.$$

Easy observation: an **Carleson sequence** (z_j) must satisfy the Blaschke condition $\sum(1-|z_j|) < \infty$.

For there will be an $f \in H^\infty$ such that

$f(z_1) = 1$ and $f(z_j) = 0$ for all $j > 1$.

So there is a non-zero H^∞ function vanishing at $(z_j)_{j \geq 2}$, and hence the Blaschke condition holds.

We can take this further. For each $k = 1, 2, \ldots$ we can find $f_k \in H^\infty$ s.t. $\|f_k\|_\infty \leq M$ with

$f_k(z_k) = 1$ and $f_k(z_j) = 0$ for all $j \neq k$.

Then write $f_k = B_k g_k$, where B_k is a Blaschke product with zeroes $(z_j)_{j \neq k}$.

Now $1 = B_k(z_k) g_k(z_k)$ and $\|g_k\| \leq M$.

Hence

$$\prod_{j \neq k} \left| \frac{z_j - z_k}{1 - \overline{z_k} z_j} \right| = |B_k(z_k)| \geq \frac{1}{M} > 0.$$

Conclusion: the condition (C) that

$$\delta := \inf_{k \geq 1} \prod_{j \neq k} \left| \frac{z_j - z_k}{1 - \overline{z_k} z_j} \right| > 0$$

is necessary for an Carleson sequence.

This is much stronger than just being a Blaschke sequence.

It implies that the points are well-separated from each other (e.g. there is a minimum hyperbolic distance between any pair).

Note that finite unions of Carleson sequences don't need to be Carleson, whereas finite unions of Blaschke sequences are still Blaschke.

Theorem (Carleson, 1958) Condition (C) is necessary and sufficient for (z_j) to be an Carleson sequence.

Thus in this case $R : H^\infty \to \ell^\infty$ is surjective.

There are several proofs. Some (e.g. that of P. Jones) actually construct an interpolating function.

The H^2 version is also important, but now we have to weight the sequence.

Theorem (Shapiro–Shields, 1961) Condition (C) is necessary and sufficient for the following problem: given $(v_j)_j \in \ell^2$, find $f \in H^2$ such that

$f(z_j)(1 - |z_j|^2)^{1/2} = v_j$ for all j.

This is more subtle, since for an arbitrary sequence $(z_j)_j$ there might not even be a mapping from H^2 to ℓ^2 given by $Tf = (f(z_j)(1 - |z_j|^2)^{1/2})_j$.

To understand this better, we look at Carleson measures.

A **Carleson measure** on $\mathbb{D}$ is a Borel measure μ such that the natural mapping $H^2 \to L^2(\mathbb{D}, \mu)$ is well-defined and bounded.

That is, for some $C > 0$,

$$\int_{\mathbb{D}} |f(z)|^2 \, d\mu(z) \le C\|f\|_2^2$$

for all $f \in H^2$.

Thus the sequence (z_j) has the property that there's a bounded operator

$T : H^2 \to \ell^2, \qquad Tf = (f(z_j)(1 - |z_j|^2)^{1/2})_j,$

if and only if the measure

$$\mu = \textstyle\sum_j (1 - |z_j|^2)\delta_{z_j}$$

is a Carleson measure. Here δ_j is a Dirac point mass at z_j.

How do we recognise a Carleson measure?

Answer: μ is a Carleson measure if and only if there is a constant $M > 0$ such that

$\mu(Q_{h,\alpha}) \le Mh \quad$ for all $0 < h < 1$ and $\alpha \in [0, 2\pi]$,

where

$Q_{h,\theta} = \{z = re^{i\theta} : 1 - h < r < 1, |\theta - \alpha| < h\}$,

a curvilinear rectangle. We'll see another way of testing these later.

In fact: if $(z_j)_j$ is a Carleson sequence then

$$\mu = \textstyle\sum_j (1 - |z_j|^2)\delta_{z_j}$$

is a Carleson measure. Conversely, if μ is a Carleson measure, then $(z_j)_j$ is a finite union of Carleson sequences.

Interpolation by inner functions

Recall an inner function $\Theta \in H^\infty$, satisfies

$|\Theta(e^{it})| = 1$ a.e.,

i.e., Θ maps $\mathbb{D}$ to $\mathbb{D}$ and $\mathbb{T}$ to $\mathbb{T}$.

Let's look now at values on the circle.

Recall that for a finite Blaschke product, B is an n-to-1 mapping from $\mathbb{T}$ to $\mathbb{T}$.

Suppose now that Θ is inner, and extends analytically except at $z = 1$.

Suppose that we know the sequence (t_n) on $\mathbb{T}$ (accumulating at 1 on both sides) where $\Theta(t_n) = 1$. What can we say about Θ?

Transform to the upper half-plane by

$\psi(z) = i(1 + z)/(1 - z)$. Note $\psi(1) = \infty$.

We are now interested in $F := \psi \circ \Theta \circ \psi^{-1}$.

Then F is meromorphic on $\mathbb{C}$ with real poles (b_n) accumulating at $\pm\infty$.

It maps $\quad \mathbb{C}^+ \to \mathbb{C}^+ \quad$ and $\quad \mathbb{C}^- \to \mathbb{C}^-$.

Such functions are called **strongly real**. To simplify explanations suppose $F(0) \neq 0, \infty$.

The Hermite–Biehler (Krein) theorem says that the zeroes (a_n) and poles (b_n) are interlaced, i.e., $b_n < a_n < b_{n+1}$ for all n and

$$F(z) = c \prod_{n \in \mathbb{Z}} \frac{1 - z/a_n}{1 - z/b_n},$$

where $c > 0$ unless some $a_n b_n < 0$, when $c < 0$.

And we can choose the (a_n) how we like, if interlaced with the (b_n).

As given in the literature, there are other cases not always considered, e.g. accumulation at $-\infty$ but not $+\infty$, but they can be handled.

Conclusion: (if there is one limit point on $\mathbb{T}$)

The set $\Theta^{-1}(1)$ does not determine Θ.

The sets $\Theta^{-1}(1)$ and $\Theta^{-1}(-1)$ together do tell us Θ (to within composition by a Möbius map fixing 1 and -1).

Recent extensions due to Chalendar, Gorkin, P.

We understand the case of finitely-many singularities (so F has some finite essential singularities).

Certain unexpected non-uniqueness cases appear.

For example, if $a_n \searrow 1$ at $-\infty$ and

$a_n \nearrow +\infty$ at $+\infty$,

then the function can be

$$F(z) = c \prod_{n \in \mathbb{Z}} \frac{1 - z/a_n}{1 - z/b_n},$$

but it can also be

$$F(z) = c(z-1) \prod_{n \in \mathbb{Z}} \frac{1 - z/a_n}{1 - z/b_n},$$

with $c > 0$ (and that's all).

Part 2: Reproducing kernels.

We've seen the **Cauchy–Szegö (reproducing) kernels**, k_a for $a \in \mathbb{D}$,

where $\qquad k_a(z) = 1/(1 - \overline{a} z)$, satisfying

$$f(a) = \langle f, k_a \rangle, \qquad (f \in H^2).$$

Clearly, they play a key role in interpolation.

Write K for the closed linear span of a sequence $(k_{z_n})_n$.

Then $K^\perp$ is the space of all functions f such that

$$f(z_n) = \langle f, k_{z_n} \rangle = 0 \qquad \text{for all } n.$$

Suppose we take a Blaschke sequence $(z_n)_n$.

Then $K = (BH^2)^\perp$, where B is the Blaschke product with zeroes $(z_n)_n$.

If $(z_n)_n$ is non-Blaschke, then $K = H^2$, since

$$K^\perp = \{0\}.$$

Now reproducing kernels give another way of characterising Carleson sequences.

A **Riesz basic sequence** in a Hilbert space is a sequence $(g_n)_n$ such that for some $A, B > 0$,

$$A \sum |a_n|^2 \le \left\| \sum a_n g_n \right\|^2 \le B \sum |a_n|^2$$

for all scalar sequences (a_n).

If the closed linear span is H then it's a **Riesz basis**.

Equivalently, $(Tg_n)_n$ is an orthonormal basis (i.e.,

$A = B = 1$) for some linear isomorphism

$$T : H \to H.$$

Theorem. The sequence (z_n) is a Carleson sequence if and only if the normalized kernels

$$e_n := (1 - |z_n|^2)^{1/2} k_{z_n}$$

form a Riesz basis for K.

This gives yet another approach to constructing interpolating functions.

The Reproducing Kernel Thesis (RKT)

Refers to a body of powerful results.

Consider the complete collection of normalized reproducing kernels

$$h_a(z) = \frac{(1 - |a|^2)^{1/2}}{1 - \overline{a} z}, \qquad (z \in \mathbb{D}),$$

for $a \in \mathbb{D}$.

Clearly, if $T : H^2 \to X$ is any operator, we have

$$\sup_{a \in \mathbb{D}} \|Th_a\| \le \|T\|.$$

We have no right to expect a converse.

After all, the closed absolute convex hull of the reproducing kernels does not contain

$$\{f \in H^2 : \|f\| \le \epsilon\}$$

for any $\epsilon > 0$.

For example, if $\psi_n(f) = \langle f, e_n \rangle$, where $e_n(z) = z^n$, then $\|\psi_n\| = 1$, but

$$\sup_{a \in \mathbb{D}} |\psi_n(h_a)| = O(n^{-1/2}).$$

That is, estimating the norm by reproducing kernels is very inefficient.

However for certain classes of operator, we do have a converse.

RKT for Toeplitz operators on H^2.

Take $\phi \in L^\infty(\mathbb{T})$ and define

$$T_\phi f = P_{H^2}(\phi.f) \qquad (f \in H^2),$$

where $P_{H^2} : L^2(\mathbb{T}) \to H^2(\mathbb{D})$ is the orthogonal projection.

Fact: $\|T_\phi\| = \|\phi\|_\infty$ (clearly it is "$\le$").

Now

$\langle Th_a, h_a \rangle = \dfrac{\langle T_\phi k_a, k_a \rangle}{\|k_a\|^2} = \dfrac{\langle \phi, |k_a|^2 \rangle}{\|k_a\|^2} = \langle \phi, P_a \rangle$

where P_a is the Poisson kernel. By the general theory of harmonic extensions, we get

$$\sup_{a \in \mathbb{D}} \|T_\phi h_a\| = \|\phi\|_\infty = \|T_\phi\|.$$

Carleson–Vinogradov embedding theorem

Recall that μ is a Carleson measure when

$J : H^2 \to L^2(\mathbb{D}, \mu)$ is bounded, i.e.,

$$\int_{\mathbb{D}} |f(z)|^2 \, d\mu(z) \le C \|f\|_2^2$$

for all $f \in H^2$. (Some constant $C > 0$.)

Theorem. The embedding J is bounded if and only if $\sup_{a \in \mathbb{D}} \|Jh_a\|_{L^2(\mu)} < \infty$.

That is,

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{1-|a|^2}{|1-\overline{a}z|^2}\, d\mu(z) < \infty.$$

Indeed (Petermichl, Treil and Wick, 2008)

$$\|J\| \leq \sqrt{2e}\,\sup_{a\in\mathbb{D}} \|Jh_a\|.$$

Bonsall's theorem for Hankel operators

Write $L^2(\mathbb{T}) = H^2 \oplus \overline{H_0^2}$,

where H^2 spanned by $\{e^{int} : n \geq 0\}$, and

its complement $\overline{H_0^2}$ spanned by $\{e^{-int} : n \geq 1\}$.

For $\phi \in L^\infty$, define $\Gamma_\phi : H^2 \to \overline{H_0^2}$,

$$\Gamma_\phi f = P_{\overline{H_0^2}}(\phi . f) \qquad (f \in H^2).$$

Then $\|\Gamma_\phi\| = \operatorname{dist}(\phi, H^\infty) = \inf_{g\in H^\infty} \|\phi - g\|_\infty$.

(Nehari's theorem.)

Bonsall (1984): the RKT holds for Hankel operators; i.e., for some $C > 0$,

$$\|\Gamma_\phi\| \leq C \sup_{a\in\mathbb{D}} \|\Gamma_\phi h_a\|.$$

We may take $C = 4\sqrt{2e}$ (Jacob, P., Pott, 2010).

Many instances of the RKT can be derived from the Carleson–Vinogradov result.

Example: **weighted composition operators.**

Let $\phi : \mathbb{D} \to \mathbb{D}$ analytic. Define $C_\phi : H^2 \to H^2$

by $\qquad C_\phi f = f \circ \phi$.

Littlewood's subordination theorem: C_ϕ is always bounded on H^2.

Now let $g \in H^2$ and define $W_{g,\phi}$ by

$$W_{g,\phi} f = g.(f \circ \phi) \quad \text{for } f \in H^2.$$

NSC for $W_{g,\phi}$ to map boundedly into H^2

(Z. Harper, 2004) is $\quad \sup_{a\in\mathbb{D}} \|W_{g,\phi} h_a\| < \infty$,

$$\text{i.e.,} \quad \sup_{a\in\mathbb{D}} \left\| \frac{(1-|a|^2)^{1/2} g}{1-\overline{a}\phi} \right\|_{H^2} < \infty.$$

Part 3: Linear systems, Hardy spaces, and extremal problems.

Discrete-time linear systems

Informally. Systems have inputs $u(0)$, $u(1)$, $u(2), \ldots$, often vector-valued, and outputs $y(0)$, $y(1)$, $y(2), \ldots$, also vector-valued.

More formally. Look at operators T defined on an input space $\mathcal{U}$, such as $\ell^2(\mathbb{Z}_+, H)$, mapping into an output space $\mathcal{Y}$, such as $\ell^2(\mathbb{Z}_+, K)$.

Here H and K are Hilbert spaces, usually finite-dimensional in practice, say $H = \mathbb{C}^m$ and $K = \mathbb{C}^p$.

Sometimes work with SISO (single-input, single-output) systems, $m = p = 1$.

Physically we would expect inputs and outputs to be real, i.e., expect $\ell^2(\mathbb{Z}_+, \mathbb{R}^m)$ to map into $\ell^2(\mathbb{Z}_+, \mathbb{R}^p)$.

Our operators may also be unbounded, and defined on a domain $\mathcal{D}(T)$, a proper subspace of $\ell^2(\mathbb{Z}_+, \mathbb{C}^m)$.

Example.

Let $y(t) = \sum_{k=0}^{t} u(k)$, a discrete integrator or "summer".

Clearly even $(1, 0, 0, \ldots) \not\in \mathcal{D}(T)$.

Causality. If $u \in \mathcal{D}(T)$ and $u(t) = 0$ for $t \le n$, then $y(t) = 0$ for $t \le n$. The past cannot depend on the future.

Algebraically, $P_n T P_n u = P_n T u$, where

$$P_n u = (u(0), \ldots, u(n), 0, 0, \ldots).$$

The "summer" example above is causal, and has dense domain.

Causality corresponds to a lower triangular (block) matrix representation using the standard orthonormal basis of ℓ^2.

Shift invariance.

Let S be the right shift on $\mathcal{U} = \ell^2(\mathbb{Z}_+, \mathbb{C}^m)$, so

$$S(u_0, u_1, u_2, \ldots) = (0, u_0, u_1, \ldots).$$

We also use S for the analogous operator on

$\mathcal{Y} = \ell^2(\mathbb{Z}_+, \mathbb{C}^p)$.

Shift-invariance: if $y = Tu$, then $Su \in \mathcal{D}(T)$, and $\qquad Sy = T(Su)$.

Automatic continuity: if T is shift-invariant and $\mathcal{D}(T) = \mathcal{U}$ then T is a bounded operator, at least for $\mathcal{U} = \ell^2(\mathbb{Z}_+, \mathbb{C}^m)$.

Causality: Shift-invariant operators with $\mathcal{D}(T) = \mathcal{U}$ will also be causal (easy).

Transfer functions

Shift-invariant operators have a representation as multiplication operators (Hartman–Winter, Fourés–Segal, 1954/1955) using the theory of Hardy spaces.

We'll work with $H^2(\mathbb{D}, \mathbb{C}^m)$, analytic vector-valued functions

$$U(z) = \sum_{k=0}^{\infty} u(k) z^k,$$

with

$$\|U\|_2^2 = \sum_{k=0}^{\infty} \|u(k)\|^2 < \infty.$$

Can be regarded as power series in the disc $\mathbb{D}$, extending to give L^2 vector-valued functions on the circle $\mathbb{T}$.

Likewise, $H^\infty(\mathbb{D}, \mathcal{L}(\mathbb{C}^m, \mathbb{C}^p))$ consists of

bounded analytic matrix-valued functions in $\mathbb{D}$, extending also to L^∞ functions on $\mathbb{T}$. Here

$$\|G\|_\infty = \sup_{|z|<1} \|G(z)\|.$$

Using the obvious unitary equivalence between $\ell^2(\mathbb{Z}_+)$ and H^2, the shift-invariant operators become multiplications

$$Y(z) = G(z)U(z) \qquad \text{and}$$

$$\|T\| = \|G\|_\infty.$$

On $\ell^2(\mathbb{Z}_+)$ they look like convolutions

$$(Tu)(t) = \sum_{k=0}^{t} h(k) u(t-k),$$

where $h(0), h(1), \ldots$ are the Fourier coefficients of an H^∞ **transfer function**.

Finite-dimensional systems

These correspond to rational (matrix-valued) functions.

Convenient from a computational point of view.

These can be realized using finite state matrices,

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

where $x(t) \in \mathbb{C}^n$ denotes the **state** of the system.

If $x(0) = 0$, then the associated transfer function is

$$D + Cz(I - zA)^{-1}B.$$

Many infinite-dimensional systems can be realized using operators A, B, C, D, rather than matrices (see later).

Continuous-time systems

We work with operators

$$T : L^2(0,\infty;\mathbb{C}^m) \to L^2(0,\infty;\mathbb{C}^p).$$

Again notions such as causality and shift-invariance make sense.

For shift-invariance (i.e., time-invariance) we suppose that T commutes with all right shifts S_τ.

To translate this into function theory, use the Laplace transform

$$\mathcal{L} : L^2(0,\infty;\mathbb{C}^m) \to H^2(\mathbb{C}_+;\mathbb{C}^m),$$

$$(\mathcal{L}u)(s) = \int_0^\infty e^{-st}u(t)\,dt,$$

giving an isometry (up to a constant) between

$L^2(0,\infty;\mathbb{C}^m)$ and a Hardy space of analytic vector-valued functions on the right half-plane $\mathbb{C}_+$

(Paley–Wiener).

$$\|f\|^2 = \sup_{x>0} \int_{-\infty}^{\infty} \|f(x+iy)\|^2\,dy < \infty.$$

This is a closed subspace of $L^2(i\mathbb{R};\mathbb{C}^m)$.

Again the causal, bounded, everywhere-defined, shift-invariant operators correspond to **transfer functions**, i.e., multiplication by functions in

$H^\infty(\mathbb{C}_+, \mathcal{L}(\mathbb{C}^m, \mathbb{C}^p))$.

These are bounded analytic matrix-valued functions in $\mathbb{C}_+$, extending also to L^∞ functions on $i\mathbb{R}$.

Note that a shift by $T > 0$ in $L^2(0,\infty)$

(the **time domain**)

corresponds to a multiplication by e^{-sT} on $H^2(\mathbb{C}_+)$

(the **frequency domain**).

We may define a continuous-time linear system in state form by the equations

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned}$$

In the finite-dimensional case, these are matrices; more generally, they are operators (more details later).

The associated transfer function is

$$C(sI - A)^{-1}B + D,$$

supposed to be matrix-valued and analytic in some right half-plane.

Examples (all with zero initial conditions)

$$\frac{dy(t)}{dt} + ay(t) = u(t), \qquad G(s) = 1/(s+a)$$

this is $H^\infty(\mathbb{C}_+)$ stable only if $a > 0$.

$$\frac{dy(t)}{dt} + ay(t-1) = u(t), \qquad G(s) = 1/(s+ae^{-s})$$

this is a delay system, and is $H^\infty(\mathbb{C}_+)$ stable only if $0 < a < \pi/2$.

Graphs and invariant subspaces.

We deal now with operators

$$T : \mathcal{D}(T) \to H^2(\mathbb{C}^p)$$

that have closed shift-invariant graphs.

Why closed? Since in fact systems stabilizable by feedback (i.e., useful ones) will be closable.

We try not to specify whether we are in discrete or continuous time (i.e., $\mathbb{D}$ or $\mathbb{C}_+$).

Note that the graph $\mathcal{G}(T)$ is defined to be

$$\left\{ \begin{pmatrix} u \\ Tu \end{pmatrix} : u \in \mathcal{D}(T) \right\} \subset H^2(\mathbb{C}^m) \times H^2(\mathbb{C}^p) = H^2(\mathbb{C}^{m+p}).$$

We can use the Beurling–Lax theorems on shift-invariant subspaces of $H^2(\mathbb{C}^N)$ to classify the closed shift-invariant operators, by means of their graphs.

Theorem (Georgiou–Smith, 1993). Let $T : \mathcal{D}(T) \to H^2(\mathbb{C}^p)$ be closed, shift-invariant, with $\mathcal{D}(T) \subseteq H^2(\mathbb{C}^m)$. Then there exist $r \leq m$, a nonsingular $M \in H^\infty(\mathcal{L}(\mathbb{C}^r, \mathbb{C}^m))$, and $N \in H^\infty(\mathcal{L}(\mathbb{C}^r, \mathbb{C}^p))$ such that

$$\mathcal{G}(T) = \begin{pmatrix} M \\ N \end{pmatrix} H^2(\mathbb{C}^r) = \Theta H^2(\mathbb{C}^r),$$

where Θ is inner in the sense that

$\|\Theta u\| = \|u\|$ for all $u \in H^2(\mathbb{C}^r)$.

(If M is allowed to be singular, it's not a graph!)

What this means for SISO systems

Take $m = p = 1$. Then

$$\mathcal{G}(T) = \begin{pmatrix} M \\ N \end{pmatrix} H^2,$$

with M, $N \in H^\infty$ and $|M(z)|^2 + |N(z)|^2 = 1$ a.e. on $\mathbb{T}$ or $i\mathbb{R}$ (as appropriate).

This means that T acts as multiplication by N/M.

The domain $\mathcal{D}(T)$ is MH^2, dense provided that M is outer.

Causality can be characterized in terms of inner divisors of M and N.

Example

An unstable delay system

$$\frac{dy(t)}{dt} - y(t) = u(t-1),$$

$$G(s) = \frac{e^{-s}}{s-1}.$$

Take

$$N(s) = \frac{e^{-s}}{s+\sqrt{2}}, \qquad M(s) = \frac{s-1}{s+\sqrt{2}}.$$

This is a **normalized coprime factorization**.

$$\mathcal{G}(T) = \begin{pmatrix} M \\ N \end{pmatrix} H^2.$$

Recovery of functions from boundary values

Take $I \subset \mathbb{T}$ of positive measure, e.g. an arc.

Now, if $f \in H^2(\mathbb{D})$ and $f = 0$ on I, then $f = 0$ everywhere (F. and M. Riesz).

That is, $R : H^2(\mathbb{D}) \to L^2(I)$, the restriction mapping, is injective.

It also has dense range.

Problem: Given data $g \in L^2(I)$ can we find f in $H^2(\mathbb{D})$ such that $f_{|I} \approx g$?

Considered by Krein and Nudel'man in 1975.

FACT: For all $g \in L^2(I)$, $\exists (f_n)$ in H^2 with $f_{n|I} \to g$, but unless $g \in H^2{}_{|I}$, necessarily $\|f_n\| \to \infty$.

In applications (signal processing, control theory and inverse problems for PDEs), this is no use!

Motivates **Bounded Extremal Problem (BEP)**.

In its simplest form:

Given $g \in L^2(I)$ and $M > 0$, find $f \in H^2(\mathbb{D})$ such that $\|f\|_2 \le M$ and $\|f_{|I} - g\|_{L^2(I)}$ minimized.

Solution involves Toeplitz operators (later).

Alpay–Baratchart–Leblond (1993).

The H^∞ problem

Useful in control theory.

One difference: the closure of $H^\infty{}_{|I}$ is no longer $L^\infty(I)$. Open question: WHAT IS IT?

Bounded Extremal Problem (BEP)

Given $g \in L^\infty(I)$ and $M > 0$, find $f \in H^\infty(\mathbb{D})$ such that $\|f\|_\infty \le M$ and $\|f_{|I} - g\|_{L^\infty(I)}$ minimized.

The solution to the L^∞ version involves Hankel operators (Baratchart–Leblond–P., 1996).

Back to the L^2 problem

The solution is unique if $M > \|g\|$, and it saturates the constraint, i.e., $\|f\| = M$.

Since it is an approximation procedure it can be used to recover functions in the presence of (small) noise and errors in measurements.

Applications: Originally Krein and Nudel'man: signal processing, band-limited signals.

Alpay–Baratchart–Leblond, 1990s: control theory, system identification.

BLP, 2000s: inverse problems for PDEs (e.g. heat flux).

Example: if ϕ real harmonic on $\mathbb{D}$ and we know ϕ and $\dfrac{\partial\phi}{\partial r}$ on I, then $\phi = \operatorname{Re} f$, with f analytic in $\mathbb{D}$, and by Cauchy–Riemann, $f \approx \phi + i \int \dfrac{\partial\phi}{\partial r}$.

General constrained approximation problem (Chalendar–P., 2003)

(Banach space version, Chalendar–P.–Smith, 2004)

Let H, K, L be Hilbert spaces and $A : H \to K$ and $B : H \to L$ linear operators.

Given $k \in K$ and $\ell \in L$, find $h \in H$ to minimize $\|Ah - k\|$ subject to the condition $\|Bh - \ell\| \leq M$.

Solution (subject to conditions on A and B to make it well-posed):
There is a constant $\gamma > 0$ such that

$$(A^*A + \gamma B^*B)h = A^*k + \gamma B^*\ell$$

and $\quad \|Bh - \ell\| = M$.

In our application,

$H = H^2(\mathbb{D}), \quad K = L^2(I), \quad L = H^2(\mathbb{D})$,

$Af = f_{|I}, \quad Bf = f$,

$A^* : L^2(I) \to H^2, \quad A^*g = P_{H^2}(g \sqcup 0)$, so

A^*A is Toeplitz operator on H^2 with symbol χ_I.

Links to earlier (Carleman) interpolation.

The annulus

Take the domain $\mathbb{A} = \mathbb{D} \backslash \overline{r_0\mathbb{D}}$, $\quad$ with circular boundaries $r_0\mathbb{T}$ and $\mathbb{T}$, where $0 < r_0 < 1$.

One application is to fault detection in pipelines – Jaoua, Leblond, Mahjoub, P., 2006.

Functions in $H^2(\mathbb{A})$ have a Laurent series

$$f(z) = \textstyle\sum_{n=-\infty}^{\infty} a_n z^n,$$

with

$$\textstyle\sum_{n=-\infty}^{\infty} |a_n|^2 (1 + r_0^{2n}) < \infty.$$

Unfortunately harmonic functions in $\mathbb{A}$ are not always the real part of analytic functions: consider $\log |z|$.

But this is the only problem, and in physical situations can be dealt with.

Problem. Given (noisy) data $g \in L^2(K)$ of an unknown $H^2(\mathbb{A})$ function f on $K \subset \partial\mathbb{A}$, find the function.

Similarly for a harmonic function and its normal derivative.

BEP: $\quad \min \|f_{|K} - g\|_{L^2(K)}$ for $f \in H^2(\mathbb{A})$

with $\|f\| \leq M$.

Again the solution involves Toeplitz operators, and in the most important case $K = \mathbb{T}$ these are diagonal w.r.t. the orthonormal basis $\left(\dfrac{z^n}{(1 + r_0^{2n})^{1/2}} \right)_{n \in \mathbb{Z}}$.

Part 4: Semigroups, admissibility, controllability.

H a complex Hilbert space, $(T_t)_{t\geq 0}$ a strongly continuous semigroup of bounded operators; i.e.,

$T_{t+u} = T_t T_u$ and $t \mapsto T_t x$ is continuous ($x \in H$).

A the infinitesimal generator, defined on domain $\mathcal{D}(A)$.

$$Ax = \lim_{t\to 0} \frac{1}{t}(T_t - I)x.$$

Note that

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0$$

has "mild" solution

$$x(t) = T_t x_0,$$

for $x_0 \in \mathcal{D}(A)$.

A continuous-time linear system in state form:

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

with $x(0) = x_0$, say.

Often we take $D = 0$. Sometimes B and C (the control and observation operators) are bounded.

If unbounded, we may think of B as mapping into a larger Hilbert space (and define the semigroup there instead).

A PDE example (equation of an undamped beam)

$$\frac{\partial^2 y}{\partial t^2} = -\frac{\partial^4 y}{\partial x^4}, \qquad 0 \leq x \leq 1, \quad t \geq 0,$$

with initial conditions on the position and velocity,

$$y(x,0) = y_1(x) \qquad \text{and} \quad y_t(x,0) = y_2(x),$$

given, and boundary conditions

$$y(0,t) = y(1,t) = y_{xx}(0,t) = y_{xx}(1,t) = 0,$$

i.e., the beam is fixed at the endpoints.

Let

$$E = -\frac{d^2}{dx^2}$$

with domain

$$\mathcal{D}(E) = \{z \in L^2(0,1) : z, \frac{dz}{dx} \text{ abs. cont.}, \\ \frac{d^2z}{dx^2} \in L^2(0,1), z(0) = z(1) = 0\}.$$

We can rewrite the equation as

$$\frac{dz}{dt} = Az,$$

with

$$z = \begin{pmatrix} y \\ \frac{dy}{dt} \end{pmatrix}, \qquad \text{and} \qquad A = \begin{pmatrix} 0 & I \\ -E^2 & 0 \end{pmatrix},$$

where z lies in $\mathcal{D}(A)$, a subspace of the Hilbert space

$$H = \mathcal{D}(E) \oplus L^2(0,1),$$

equipped with the norm

$$\|(z_1, z_2)\|^2 = \|Ez_1\|^2 + \|z_2\|^2.$$

(Infinite-time) admissibility

There is a **duality** here between control and observation. Observation is simpler to explain.

Admissibility of control operators.

Consider

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t), \qquad x(0) = x_0, \quad t \geq 0,$$

where $u(t) \in \mathcal{U}$ is the input at time t,

$\mathcal{U}$ is a separable Hilbert space, and

$B : \mathcal{D}(B) \to H$ may be unbounded, $\mathcal{D}(B) \subseteq \mathcal{U}$.

How can we insist that the state $x(t)$ lies in H?

Sufficient that $B \in \mathcal{L}(\mathcal{U}, \mathcal{D}(A^*)')$ and $\exists m_0 > 0$ such that

$$\left\| \int_0^\infty T_t B u(t)\, dt \right\|_H \leq m_0 \|u\|_{L^2(0,\infty;\mathcal{U})}$$

(the admissibility condition for B).

Admissibility of observation operators.

Consider

$$\begin{aligned}\frac{dx(t)}{dt} &= Ax(t),\\ y(t) &= Cx(t),\end{aligned}$$

with $x(0) = x_0$, say.

Let $C : \mathcal{D}(A) \to \mathcal{Y}$, Hilbert, be an A-bounded 'observation operator', i.e., for some m_1, $m_2 > 0$,

$$\|Cz\| \le m_1\|z\| + m_2\|Az\|.$$

C is **admissible**, if $\exists m_0 > 0$ such that $y \in L^2(0,\infty;\mathcal{Y})$ and

$$\|y\|_2 \le m_0\|x_0\|.$$

Note $y(t) = CT_t x_0$.

The duality

B is an admissible control operator for $(T_t)_{t\ge 0}$ if and only if B^* is an admissible observation operator for the dual semigroup $(T_t^*)_{t\ge 0}$.

The Weiss conjecture

Suppose C admissible, take Laplace transforms:

$$y(t) = CT_t x_0, \qquad \text{so}$$

$$\begin{aligned}\hat{y}(s) &= \int_0^\infty e^{-st} y(t)\, dt,\\ &= C(sI - A)^{-1} x_0.\end{aligned}$$

Now if $y \in L^2(0,\infty;\mathcal{Y})$, then $\hat{y} \in H^2(\mathbb{C}_+,\mathcal{Y})$, Hardy space on RHP, and

$$\|\hat{y}(s)\| = \left\| \int_0^\infty e^{-st} y(t)\, dt \right\| \le \frac{\|y\|_2}{\sqrt{2\operatorname{Re} s}},$$

by Cauchy–Schwarz.

Thus admissibility, i.e.,

$$\|CT_t x_0\|_{L^2(0,\infty;\mathcal{Y})} \le m_0\|x_0\|,$$

implies the **resolvent condition**: $\exists m_1 > 0$ such that

$$\|C(sI - A)^{-1}\| \le \frac{m_1}{\sqrt{\operatorname{Re} s}}, \qquad \forall s \in \mathbb{C}_+.$$

George Weiss (1991) conjectured that the two conditions are equivalent.

As we'll see this would imply several big theorems in function theory in an elementary way.

In fact it is true for contraction semigroups if

$\dim \mathcal{Y} < \infty$ (Jacob, P., 2000), but not in general.

Example 1

$$H = L^2(\mathbb{C}_+, \mu).$$

$$(T_t(x))(\lambda) = e^{-\lambda t} x(\lambda).$$

$$(Ax)(\lambda) = -\lambda x(\lambda).$$

For which Borel measures μ on $\mathbb{C}_+$ does C defined by

$$Cf = \int_{\mathbb{C}_+} f(\lambda)\, d\mu(\lambda)$$

satisfy the resolvent condition?

Answer: if and only if

$$\int_{\mathbb{C}_+} \frac{d\mu(\lambda)}{|s+\lambda|^2} \le \frac{M}{\operatorname{Re} s} \qquad \forall s \in \mathbb{C}_+.$$

This actually means that μ is a Carleson measure for $\mathbb{C}_+$: μ-measure of square $[0, 2h] \times [a-h, a+h]$ is always at most const. $\times h$.

So when is C admissible?

Precisely when there is a continuous embedding

$$H^2(\mathbb{C}_+) \to L^2(\mathbb{C}_+, \mu).$$

This equivalence is the **Carleson–Vinogradov embedding theorem** for the half-plane: if

$$\left\| \frac{1}{s+\lambda} \right\|_{L^2(\mathbb{C}_+,\mu)} \le M \left\| \frac{1}{s+\lambda} \right\|_{H^2},$$

for each $\lambda \in \mathbb{C}_+$, then a similar inequality holds for all H^2 functions.

Thus the Weiss conjecture for the above semigroup is equivalent to the above embedding theorem (Weiss), an example of the RKT.

Example 2

Take the right shift semigroup on $H = L^2(0, \infty)$:

$$(T_t x)(\tau) = x(\tau - t), \qquad \tau \ge t.$$

Equivalently,

$$H = H^2(\mathbb{C}_+).$$

$$(T_t(x))(\lambda) = e^{-\lambda t}x(\lambda).$$

$$(Ax)(\lambda) = -\lambda x(\lambda).$$

Now $C : \mathcal{D}(A) \to \mathbb{C}$ is A-bounded iff it has the form

$$Cx = \int_{-\infty}^{\infty} \overline{c(i\omega)}x(i\omega)\,d\omega,$$

where $c(z)/(1+z) \in H^2(\mathbb{C}_+)$ (easy).

Then C is admissible, i.e.,

$$\int_0^\infty |CT_t x_0|^2\,dt \leq m_0\|x_0\|^2,$$

if and only if the following Hankel operator is bounded:

$$\Gamma_c : H^2(\mathbb{C}_-) \to H^2(\mathbb{C}_+), \qquad \Gamma_c u = \Pi_+(c.u),$$

where Π_+ is the orthogonal projection from

$$L^2(i\mathbb{R}) = H^2(\mathbb{C}_+) \oplus H^2(\mathbb{C}_-)$$

onto $H^2(\mathbb{C}_+)$.

Resolvent Condition equivalent to

$$\left\|\Gamma_c\left(\frac{1}{s-a}\right)\right\| \leq m'\left\|\frac{1}{s-a}\right\| \qquad \forall a \in \mathbb{C}_+.$$

Hence the Weiss conjecture for the shift semigroup is equivalent to **Bonsall's theorem** for the half-plane: a Hankel operator Γ_c bounded if and only if bounded on normalized reproducing kernels.

Proof of Weiss conjecture for contraction semigroups uses results for unitary semigroups, a stronger form of the result for the shift, and some auxiliary results.

Combining the 'smaller' theorems gives the big theorem, which implies the smaller theorems.

Controllability

Assume an exponentially stable semigroup $(T_t)_{t\geq 0}$, i.e.,

$$\|T_t\| \leq Me^{-\lambda t}, \qquad (t \geq 0),$$

for some $M > 0$ and $\lambda > 0$.

Look at the equation

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t),$$

with solution

$$x(t) = T_t x_0 + \int_0^t T_{t-s} Bu(s)\, ds,$$

suitably interpreted.

Let's assume B admissible (an easier case to describe).

Then we have a bounded operator

$\mathcal{B}_\infty : L^2(0,\infty;\mathcal{U}) \to H$, defined by

$$\mathcal{B}_\infty u = \int_0^\infty T_t Bu(t)\, dt.$$

The system is **exactly controllable**, if

$\operatorname{Im} \mathcal{B}_\infty = H$, i.e., we can steer the system where we like, using the input u.

Alternatively, it is **approximately controllable**, if $\operatorname{Im} \mathcal{B}_\infty$ is dense.

There are dual notions of exact and approximate observability (omitted).

Controllability involves more links with the theory of interpolation, as follows.

Diagonal semigroups

An important special case where most things are known (includes some heat equations, vibrating structures, etc.)

Suppose that

$$A\phi_n = \lambda_n \phi_n,$$

with (ϕ_n) normalized eigenvectors forming a Riesz basis.

Let (ψ_n) be the dual basis.

So every $x \in H$ can be written

$$x = \sum_{n=1}^\infty \langle x, \psi_n \rangle \phi_n$$

with

$$K_1 \sum |\langle x, \psi_n \rangle|^2 \le \|x\|^2 \le K_2 \sum |\langle x, \psi_n \rangle|^2.$$

Note that

$$T_t \sum_{n=1}^\infty c_n \phi_n = \sum_{n=1}^\infty c_n e^{\lambda_n t} \phi_n.$$

By exponential stability $\sup_n \operatorname{Re} \lambda_n < 0$.

We look at the case $\dim \mathcal{U} = 1$: interesting (and simpler).

Finite-dimensional $\mathcal{U}$ can be handled similarly.

If $\dim \mathcal{U} = 1$ then $B : \mathbb{C} \to H$ so is a vector,

$$b = \sum_{n=1}^{\infty} b_n \phi_n.$$

Let's calculate $\mathcal{B}_\infty$:

$$\mathcal{B}_\infty u = \int_0^\infty T_t B u(t)\, dt, \qquad \text{so}$$

$$\mathcal{B}_\infty u = \sum_{n=1}^{\infty} b_n \int_0^\infty e^{\lambda_n t} u(t)\, dt\ \phi_n.$$

This is just

$$\sum_{n=1}^{\infty} b_n \hat{u}(-\lambda_n) \phi_n.$$

(Laplace Transform!)

Since $\mathcal{L} : L^2(0,\infty) \to H^2(\mathbb{C}_+)$ is an isomorphism,

exact controllability is equivalent to:

for every $(c_n) \in \ell^2$ there is a function $g \in H^2(\mathbb{C}_+)$ such that

$$b_n g(-\lambda_n) = c_n.$$

This brings us back directly to Carleson interpolation problems in the Shapiro–Shields version (exact result needed due to McPhail (1990)).

Once again we have to rewrite things for the half-plane.

Scalar case Jacob-P. (2006), vector case Jacob–P.–Pott (2007) (with new interpolation results).

The necessary and sufficient condition for exact controllability is that

$$\nu = \sum_{n=1}^{\infty} \frac{|\mathrm{Re}\,\lambda_n|^2}{|b_n|^2} \prod_{k \neq n} \frac{|\overline{\lambda_n} + \lambda_k|^2}{|\lambda_k - \lambda_n|^2} \delta_{-\lambda_n}$$

is a Carleson measure on $\mathbb{C}_+$,

i.e., ν-measure of square $[0, 2h] \times [a-h, a+h]$ is $O(h)$.

Here δ_λ denotes a Dirac (point mass) at λ.

We don't need $(-\lambda_n)$ to be a Carleson sequence, but it does need to be a Blaschke sequence.

However $\lambda_n = -n^\beta$ with $0 < \beta \leq 1$ is never exactly controllable (if $\dim \mathcal{U} < \infty$).

Approximate controllability (dense range) is much easier and just requires $b_n \neq 0$ for all n and distinct eigenvalues (λ_n).

For finite-dimensional systems (matrices, not just operators) exact/approximate controllability coincide.

An intermediate concept is **null controllability**. We just require $\operatorname{Im} \mathcal{B}_\infty$ to contain $T_{t_1} H$ for some $t_1 \geq 0$.

The NSC now becomes that

$$\nu = \sum_{n=1}^{\infty} \frac{|\operatorname{Re} \lambda_n|^2}{|b_n|^2} e^{2t_1 \operatorname{Re} \lambda_n} \prod_{k \neq n} \frac{|\overline{\lambda_n} + \lambda_k|^2}{|\lambda_k - \lambda_n|^2} \delta_{-\lambda_n}$$

is a Carleson measure on $\mathbb{C}_+$.

Summary of ideas

* Interpolation in Hardy spaces enables construction of functions from given values.

* Carleson sequences and Carleson measures play a key role.

* Reproducing kernels link to interpolation.

* Boundedness of certain operators

(Carleson, Toeplitz, Hankel)

can be tested on reproducing kernels.

* Applications of these ideas appear in:

(i) identification from partial data;

(ii) admissibility and controllability of systems.