University of Leeds, School of Mathematics
Graduate course in Analysis 2003-04

Jonathan R. Partington, 5.12.03

Bases. A sequence $(x_n)_1^\infty$ in a Banach space X is a *(Schauder) basis*, if every $x \in X$ can be written uniquely $x = \sum_{n=1}^\infty c_n x_n = \lim_{N\to\infty} \sum_{n=1}^N c_n x_n$, converging in norm. Then we can identify X with a space of scalar sequences $(c_n)_1^\infty$ in the obvious way. A *basic sequence* is a Schauder basis for its closed linear span.

Standard basis. In ℓ^p for $1 \le p < \infty$ or c_0 take $(e_n)_1^\infty$, where e_k is the sequence with a 1 in the kth position, otherwise 0.

Basis constant. Define $P_k\left(\sum_{n=1}^\infty c_n x_n\right) = \sum_{n=1}^k c_n x_n$, so each P_k is a linear bounded projection and $K := \sup \|P_k\|$ is finite, the *basis constant* of (x_n). A *monotone basis* is one where $K = 1$.

Testing for a basis. The sequence (x_n) of nonzero vectors is a basic sequence if and only if there is a K such that $\|\sum_{i=1}^n a_i x_i\| \le K \|\sum_{i=1}^m a_i x_i\|$ whenever $n \le m$.

Haar functions. In $L^p(0,1)$, for $1 \le p < \infty$ we have a monotone basis that consists of $h_1 = 1$, $h_2 = \chi_{(0,1/2)} - \chi_{(1/2,1)}$, $h_3 = \chi_{(0,1/4)} - \chi_{(1/4,1/2)}$, $h_4 = \chi_{(1/2,3/4)} - \chi_{(3/4,1)}$, $h_5 = \chi_{(0,1/8)} - \chi_{(1/8,1/4)}$, and so on.

Schauder functions. In $C[0,1]$, the sequence (ϕ_n) forms a monotone basis, where $\phi_1 = 1$, and $\phi_n(t) = \int_0^t h_{n-1}(u)\,du$ for $n \ge 2$.

Biorthogonal functionals. $x_m^*\left(\sum_{n=1}^\infty c_n x_n\right) = c_m$; these are bounded linear functionals (elements of X^*) and $x_m^*(x_n) = \delta_{m,n}$. Note that (x_n) is *fundamental*, i.e., their linear span is dense, and (x_n^*) is *total*, i.e., if $x_n^*(x) = 0$ for all n then $x = 0$.

Minimal sequences. We say (x_n) is *minimal*, if there exist (x_n^*) such that $x_m^*(x_n) = \delta_{m,n}$.

Markusevich's theorem. Every separable Banach space contains a fundamental total biorthogonal sequence. By working harder one can for each $\epsilon > 0$ find a f.t.b.o.s. with $\sup \|x_n\|\,\|x_n^*\| < 1 + \epsilon$ (Pełczyński).

Unconditional convergence. For vectors (y_n) in a Banach space, TFAE: (i) $\sum \epsilon_n y_n$ converges for all sign choices $\epsilon_n = \pm 1$; (ii) $\sum y_{n_k}$ converges for all $n_1 < n_2 < \ldots$; (iii) for all $\epsilon > 0$ there's an N s.t. $\left\|\sum_{n\in S} y_n\right\| < \epsilon$ whenever S is finite and $\min S > N$; (iv) the series $\sum y_{\pi(n)}$ converges for all permutations π of $\mathbb{N}$. We call such sequences *unconditionally convergent*. In (iv), the sum does not depend on π.

Unconditional bases. A basis (x_n) is unconditional if for all x the sum $x = \sum_{n=1}^\infty c_n x_n$ is unconditionally convergent. For example, the basis (e_n) in c_0 or ℓ^p (with $1 \le p < \infty$), or the Haar basis in $L^p(0,1)$ for $1 < p < \infty$. The basis $u_n(t) = |t|^\alpha e^{int}$, for $n \in \mathbb{Z}$, is **conditional** in $L^2(0,1)$, if $0 < \alpha < 1/2$.

More projections. Suppose that (x_n) is an unconditional basis. Then $\sup_{S\subseteq\mathbb{N}} \|P_S\| < \infty$, where $P_s\left(\sum c_n x_n\right) = \sum_{n\in S} c_n x_n$.

Other operators. For (x_n) unconditional, we have $\sup_{\epsilon:\mathbb{N}\to\{\pm1\}} \|M_\epsilon\| < \infty$, where $M_\epsilon\left(\sum c_n x_n\right) = \sum \epsilon_n c_n x_n$. This is called the *unconditional basis constant* of (x_n). The mapping $\sum c_n x_n \mapsto \sum a_n c_n x_n$ is also bounded, whenever (a_n) is a bounded sequence of scalars.

Riesz bases. A sequence (x_n) in a Hilbert space H is a *Riesz basis*, if (Ux_n) is an orthonormal basis for some isomorphism $U : H \to H$ (i.e., U and U^{-1} are bounded). Equivalently, if there are A, $B > 0$ such that $A\sum_{k=1}^\infty |c_k|^2 \le \left\|\sum_{k=1}^\infty c_k x_k\right\|^2 \le B \sum_{k=1}^\infty |c_k|^2$ for all (c_k) in ℓ^2, and the closed linear span of the (x_k) is H. First inequality means it's a "Besselian" sequence; the second, that it is "Hilbertian".

Köthe–Toeplitz theorem. The unconditional bases of a Hilbert space are precisely the Riesz bases.

Shrinking bases. If (x_n) is a basis in a Banach space X, then the biorthogonal functionals (x_n^*) are a basic sequence in X^*. The basis (x_n) is *shrinking*, if for every $x^* \in X^*$ the norm of x^* restricted to $\text{lin}\{x_i : i > n\}$ tends to 0 as $n \to \infty$. Examples: c_0 and ℓ^p for $1 < p < \infty$. A theorem of James: (x_n^*) is a basis in X^* if and only if (x_n) is shrinking.
Boundedly complete bases. If (x_n) is shrinking, then X^{**} can be identified with the space of all sequences (c_k) such that condition (C): $\sup_n \|\sum_{k=1}^n c_k x_k\| < \infty$ holds. We identify (x^{**}) with the sequence of values $(x^{**}(x_k^*))$. Example, $c_0^{**} = \ell^\infty$. Then (x_n) is *boundedly complete*, if condition (C) implies that $\sum_{k=1}^\infty c_k x_k$ converges. For example ℓ^p with $1 < p < \infty$, but not c_0. The space X with a basis (x_n) is *reflexive* (i.e., $X = X^{**}$ under the natural embedding), if and only if the basis is both shrinking and boundedly complete.
Universal spaces. Every separable Banach space is both (isometrically) a subspace of $C[0, 1]$ and a quotient of ℓ^1, both of which are spaces with bases. Not every separable Banach space has a basis (Enflo).
Convexity and smoothness. A normed space X is *strictly convex (rotund)*, if $x, y \in S_X$ (unit sphere of X, the vectors of norm 1) with $x \neq y$ implies that $\|(x + y)/2\| < 1$ (i.e., unit sphere has no line segments in it). Then X is *smooth*, if for each $x \in S_X$ there is a **unique** $f \in S_{X^*}$ with $f(x) = 1$ (there's always at least one, by Hahn–Banach) (i.e., unit sphere has no corners). These properties are inherited by subspaces. Hilbert spaces have both, but ℓ^1 and ℓ^∞ have neither.
Equivalent forms. The space X is strictly convex if and only if $\|x+y\| < \|x\|+\|y\|$ whenever x and y are independent vectors in X. The space X is smooth if and only if the norm is *Gâteaux-differentiable*, i.e., $\lim_{t\to 0}(\|x + ty\| - \|x\|)/t$ exists for all $x \neq 0$ in X and $y \in X$.
Nearest points. Let K be nonempty, convex and compact in a normed linear space X. Each $x_0 \in X$ has a nearest point in K (i.e., minimizing $\|x_0 - y\|$ over $y \in K$). If X is strictly convex, then this nearest point is unique, say $p(x_0) \in K$. Moreover the mapping $x_0 \mapsto p(x_0)$ is continuous.
Duality. If X^* is strictly convex (resp. smooth), then X is smooth (resp. strictly convex). No implication in the other direction. All L^p spaces are both s.c. and smooth for $1 < p < \infty$.
Renorming. Any normed space with a total sequence of functionals has an equivalent s.c. norm (so, in particular, any separable normed space). Also every separable normed space has an equivalent smooth norm.
Uniform convexity. X is uniformly convex if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $x, y \in B_X$ (unit ball of X) and $\|x - y\| \geq \epsilon$, then $\|(x + y)/2\| \leq 1 - \delta$. A stronger notion than strict convexity, and equivalent for finite-dimensional spaces. The *modulus of convexity* is $\delta_X(\epsilon) = \inf\{1 - \|x + y\|/2 : x, y \in S_X, \|x - y\| = \epsilon\}$.
Clarkson inequalities. For $f, g \in L^p$ with $1 < p \leq 2$ and $q = p/(p-1)$, we have $\|f+g\|_p^q+\|f-g\|_p^q \leq 2\left(\|f\|_p^p + \|g\|_p^p\right)^{q-1}$. And for $2 \leq p < \infty$, we have $\|f + g\|_p^p + \|f - g\|_p^p \leq 2^{p-1}\left(\|f\|_p^p + \|g\|_p^p\right)$. These imply easily that L^p is uniformly convex whenever $1 < p < \infty$.
Uniform smoothness. For $\dim X \geq 2$ and $\tau > 0$, the modulus of smoothness is defined to be $\rho_X(\tau) = \sup\{(\|x + y\| + \|x - y\|)/2 - 1 : x, y \in X, \|x\| = 1, \|y\| = \tau\}$. Then X is uniformly smooth if $\lim_{\tau\to 0} \rho_X(\tau)/\tau = 0$. This implies smoothness. It is also equivalent to a stronger form of differentiability of the norm (Fréchet differentiability).
Duality. For $\tau > 0$, we have $\rho_{X^*}(\tau) = \sup\{\tau\epsilon/2 - \delta_X(\epsilon) : 0 \leq \epsilon \leq 2\}$, and so X is uniformly convex if and only if X^* is uniformly smooth. Thus L^p is uniformly smooth, for $1 < p < \infty$.
Radon–Riesz property. If X is uniformly convex, if (x_n) in X tends weakly to x (i.e., $f(x_n) \to f(x)$ for all $f \in X^*$), and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$. Example: in H^2, if $f_n \to f$ pointwise and also $\|f_n\| \to \|f\|$, then we have norm convergence.
Reflexivity. Every uniformly convex space, and every uniformly smooth space, is reflexive.
Goldstine's theorem. Give X^{**} the weak-* topology, so that $\phi_n \to \phi$ if and only if $\phi_n(f) \to \phi(f)$ for all $f \in X^*$. Then $B_{X^{**}}$ is compact in this topology, and it is the closure of B_X. This means that for any $\phi \in B_{X^{**}}$ there is a net (x_α) in B_X with $f(x_\alpha) \to \phi(f)$ for all $f \in X^*$.
Isometries on Hilbert spaces. T is an *isometry* if $\|Tx\| = \|x\|$ for all x, equivalently if $T^*T = I$.

Wold decomposition. If T is an isometry on H then $H = K \oplus L$, an orthogonal direct sum of closed subspaces, each invariant under T and T^*, such that $S = T_{|K}$ satisfies $\bigcap_{n=0}^{\infty} S^n K = \{0\}$ and $U = T_{|L}$ is unitary (i.e., a surjective isometry). Also $L = \bigcap_{n=0}^{\infty} T^n H$.
Unilateral shifts. The operator S above decomposes K as $K = M \oplus SM \oplus S^2 M \oplus \ldots$, an orthogonal direct sum, where $M = (TH)^{\perp}$. We call S a *unilateral shift*, and its multiplicity is $\dim M$, which may be infinite. Note that S is equivalent to M_z, multiplication by z on the Hardy space $H^2(\mathbb{D}, M)$ of functions with values in M.
Universal model. We say that T is a $C_{.0}$ *contraction* if $\|T\| \le 1$ and $\|T^n x\| \to 0$ for all $x \in H$. For such a $T \in L(H)$, let S be a unilateral shift on some Hilbert space with multiplicity $\dim H$. Then there is an invariant subspace F for S^* such that T is unitarily equivalent to $S^*_{|F}$.
Dilations. Let H be a Hilbert space and K a closed subspace. Then $A \in L(H)$ is a *dilation* of $B \in L(K)$ if $B^n = P_K A^N_{|K}$ for $n = 0, 1, 2, \ldots$. *Corollary of the universal model:* if R is a $C_{0.}$ *contraction* (i.e., R^* is a $C_{.0}$ contraction), then it has a dilation to a unilateral shift, since $(S^{*n}_{|F})^* = P_F S^n_{|F}$. Indeed, R dilates also to a unitary operator, a bilateral shift, since we can extend S to a bilateral shift, M_z on $L^2(\mathbb{T}, M)$.
Spectral theory for normal operators I. If $TT^* = T^*T$, then there is a measure space (Ω, μ) and a function $\phi \in L^\infty(\Omega, \mu)$ such that T is unitarily equivalent to M_ϕ acting on $L^2(\Omega, \mu)$. If H is separable then we can take $\mu(\Omega) < \infty$. Hence a normal operator T is self-adjoint/ unitary/ positive/ a projection, if and only if $\sigma(T)$ is a subset of $\mathbb{R}$/ $\mathbb{T}$/ $\mathbb{R}_+$/ $\{0, 1\}$. Also positive operators have (unique) positive square roots.
A spectral mapping theorem. If T is normal and p is a polynomial in 2 variables, then $p(T, T^*)$ is normal and $\sigma(p(T, T^*)) = \{p(z, \overline{z}) : z \in \sigma(T)\}$.
Spectral measures. E maps the Borel subsets of $\mathbb{C}$ to the set of orthogonal projections in $L(H)$, satisfies $E(\mathbb{C}) = I$, $E(\bigcup_{n=1}^{\infty} S_n)x = \sum_{n=1}^{\infty} E(S_n)x$ for all $x \in H$ whenever the (S_n) are pairwise disjoint (*strongly countably additive*), and has compact support ($E(\mathbb{C} \setminus K) = 0$ for some compact K). Note $E(S)E(T) = 0$ if $S \cap T = \emptyset$. Define $\mu_{x,y}(S) = \langle E(S)x, y \rangle$, an ordinary complex measure. Let f be a bounded Borel function on $\mathbb{C}$; then there is a unique $T \in L(H)$ s.t. $\langle Tx, y \rangle = \int f(\lambda)\, d\mu_{x,y}(\lambda)$ for all x, $y \in H$. Also $\|T\| \le \|f\|_\infty$. We write $T = \int f(\lambda)\, dE_\lambda$.
Spectral theory for normal operators II. If T is a normal operator then there is a unique spectral measure supported on $\sigma(T)$ such that $T = \int \lambda\, dE_\lambda$.
Functional calculus for normal operators. Define $\psi : L^\infty(\sigma(T)) \to L(H)$ by $\psi(f) = f(T) = \int f(\lambda)\, dE_\lambda$. Then ψ is an algebra homomorphism (linear, multiplicative) and $\psi(1) = I$, $\psi(I) = T$, where $I(\lambda) = \lambda$ for all λ, and $\psi(\overline{f}) = \psi(f)^*$ with $\|\psi(f)\| \le \|f\|_\infty$.
Polynomially bounded operators. For these there is a constant $C > 0$ such that $\|p(T)\| \le C\|p\|_{H^\infty}$ for any polynomial p. Any Hilbert space contraction satisfies this with $C = 1$ (the *Von Neumann inequality*, proved easily using isometric dilations). These have a disc algebra functional calculus, extending the idea of $f(T)$ from polynomials to obtain a contractive algebra homomorphism from $A(\mathbb{D})$ into $L(H)$. (This fails in general for operators on other Banach spaces, e.g. ℓ^1.)
Integrals along curves. We can define the integral of a continuous Banach space Y-valued function F along a piecewise C^1 curve $\gamma : [a, b] \to \mathbb{C}$ as the limit of Riemann sums

$$\int_\gamma F = \int_a^b F(\gamma(t))\gamma'(t)\, dt \approx \sum_{k=1}^{n} F(\gamma(t_k))(\gamma(t_k) - \gamma(t_{k-1})),$$

as the mesh of the partition $a = t_0 < t_1 < \ldots < t_n = b$ tends to 0, giving a value in Y.
Analytic functions. A Y-valued function F on an open set U is *analytic* if $\lim_{z \to z_0}(F(z) - F(z_0))/(z - z_0)$ exists for all $z_0 \in U$. Example: for an operator $T \in L(X)$, we have the *resolvent equation* $(z - T)^{-1} - (z_0 - T)^{-1} = (z_0 - z)(z - T)^{-1}(z_0 - T)^{-1}$, showing that $z \mapsto (z - T)^{-1}$ is analytic on $\mathbb{C} \setminus \sigma(T)$.
Cauchy's theorem. If γ is a closed curve, then its *interior* is the set of points about which it has nonzero winding number. Now if $F : U \to Y$ is analytic and γ and its interior lie in U, then $\int_\gamma F = 0$.

Riesz–Dunford functional calculus. Let $T \in L(X)$ and let F be a function analytic on a neighbourhood U of $\sigma(T)$. We define $f(T) = \frac{1}{2\pi i}\int_\gamma f(z)(z-T)^{-1}\,dz$, where γ is a finite union of closed (piecewise C^1) curves with γ and its interior in U, which winds exactly once about each point of $\sigma(T)$. The definition is independent of the choice of γ, by Cauchy's theorem. The mapping $f \mapsto f(T)$ is linear, and if $f_n \to f$ uniformly on a neighbourhood of $\sigma(T)$ then $\|f_n(T) - f(T)\| \to 0$. Indeed it is also multiplicative (harder), and maps 1 to I and $I(z) = z$ to T.

Spectral density. For $T \in L(H)$ and $x,\, y \in H$ we write $f = x \overset{T}{\cdot} y$ if $f \in L^1(\mathbb{T})$ with Fourier coefficients $\hat{f}(n) = \langle x, \overline{T^n y\rangle \text{ and } \hat{f}}(-n) = \langle T^n x, y\rangle$ for $n \ge 0$. Note that then $(x \overset{T^*}{\cdot} y)(e^{it}) = (x \overset{T}{\cdot} y)(e^{-it})$ and $(y \overset{T}{\cdot} x)(e^{it}) = \overline{(x \overset{T}{\cdot} y)(e^{it})}$. The spectral density exists if T has a unitary dilation with absolutely continuous spectral measure.

Sz.-Nagy–Foias functional calculus. Let T have a unitary dilation U with absolutely continuous spectral measure. So $\langle f(U)x, y\rangle = \int f(\lambda)\, d\langle E_\lambda x, y\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})(x \overset{U}{\cdot} y)(e^{it})\,dt$, using the Radon–Nikodym theorem, and $x \overset{U}{\cdot} y$ is the spectral density defined above. Now we define $\psi : L^\infty(\mathbb{T}) \to L(H)$, $\psi(f) = f(T) = P_H f(U)_{|H}$ using the functional calculus for normal operators to define $f(U)$. Note that $\langle f(T)x, y\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})(x \overset{T}{\cdot} y)(e^{it})\,dt$ and $\|f(T)\| \le \|f\|_\infty$. However ψ is not multiplicative on $L^\infty(\mathbb{T})$, unless T is unitary, although it is on $H^\infty(\mathbb{D})$.

The invariant subspace problem. Does an operator T on a separable infinite-dimensional complex Hilbert space H always have a nontrivial closed invariant subspace K, i.e., with $Tx \in K$ whenever $x \in K$? Does it (except when a multiple of the identity) have a *hyperinvariant subspace*, i.e., invariant under A for all A with $AT = TA$?

The i.s. problem in Banach spaces. The answer is negative on Banach spaces in general (Enflo, Read, c. 1984), even on ℓ^1. The question remains unsolved for all reflexive spaces.

Normal operators and the i.s.p. Normal operators all have reducing subspaces (K and $K^\perp$ invariant), from the spectral theorem. Unless T is a multiple of the identity, they are even hyperinvariant. Much harder (Scott Brown): every *subnormal operator*, i.e., $T = N_{|H}$, where H is an invariant subspace of the normal operator N, has invariant subspaces. They needn't be reducing, e.g. the unilateral shift has no reducing subspaces by Beurling's theorem.

Disconnected spectra. If $T \in L(X)$ and $\sigma(T)$ is disconnected then $X = Y \oplus Z$, where Y and Z are nontrivial closed invariant subspaces for T. The proof uses the Riesz–Dunford functional calculus to construct projections P and $I - P$ that commute with T.

Compact operators. Every nonzero compact operator $T \in L(X)$ has a nontrivial closed hyperinvariant subspace (Aronszajn–Smith–Lomonosov). For the proof, it is sufficient to assume that T is *quasinilpotent*, i.e., $\sigma(T) = \{0\}$, as well as compact. Indeed (harder), if T is not a multiple of the identity and T commutes with a nonzero compact operator, then T has a hyperinvariant subspace (Lomonosov).

Universal operators. Let X be a complex Banach space and $U \in L(X)$. Then U is *universal* for X if for every $T \in L(X)$ there is a $\lambda \ne 0$, an invariant subspace X_0 for U, and an isomorphism $J : X \to X_0$ such that $\lambda T = J^{-1}(U_{|X_0})J$. So invariant subspaces for T are $J^{-1}Z$, where $Z \subseteq X_0$ is invariant for U. *Caradus's theorem:* if $U \in L(H)$, $\dim \operatorname{Ker} U = \infty$ and $\operatorname{Im} U = H$, then U is universal for H (Hilbert space). Example: S^* when S is a unilateral shift of infinite multiplicity.

Composition operators. Let $\phi : \mathbb{D} \to \mathbb{D}$ be analytic. Then $C_\phi : f \mapsto f \circ \phi$ is bounded on H^2 *(Littlewood's subordination theorem)*. Example: $\phi(z) = z^2$; then C_ϕ is a shift of infinite multiplicity, hence C_ϕ^* is universal. Indeed, C_ϕ is an isometry if and only if ϕ is inner and $\phi(0) = 0$. Also $C_\phi - I$ is universal if $\phi(z) = (z + r)/(1 + rz)$ with $0 < r < 1$.

Cyclic and hypercyclic vectors. $T \in L(X)$ has a *cyclic vector* x if $\overline{\operatorname{lin}}\{x, Tx, T^2x, \ldots\} = X$, and a *hypercyclic vector* x if $\{x, Tx, T^2x, \ldots\}$ is already dense in X. There are no hypercyclic vectors if $0 < \dim X < \infty$; but (Rolewicz) αS^* has hypercyclic vectors on ℓ^2 whenever $|\alpha| > 1$. Similarly for $X = \ell^p$ if $1 \le p < \infty$ and $X = c_0$.