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Analysis (wavelets and operators): Some useful results and formulae (2)

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Please let me know of any errors, omissions or obscurities!

Unbounded operators: examples on $L^2(0,\infty)$ include $(Tf)(t) = \int_0^t f(t)\,dt$, the Volterra integral operator; and $Tf = f'$, the differentiation operator; each being defined on a dense subspace. Standard convention: X, Y will be Banach spaces, $D(T) \subseteq X$, the *domain* of T, and $T : D(T) \to Y$.

Graph of an operator: $G(T) = \{(x, Tx) : x \in D(T)\} \subset X \times Y$. Usually we give $X \times Y$ the product norm $\|(x,y)\|_2 = \left(\|x\|^2 + \|y\|^2\right)^{1/2}$, which makes it a Hilbert space if X and Y are (or at least a Banach space if X and Y are).

Closed operators: T is closed if $G(T)$ is a closed subspace of $X \times Y$. By the closed graph theorem, a closed operator with $D(T) = X$ is necessarily bounded. But many unbounded operators, often with $D(T)$ a dense subspace of X, have closed graphs too.

The graph norm: $D(T)$ becomes a Banach/Hilbert space, isometric to $G(T)$, under the graph norm $\|x\|_g = \left(\|x\|^2 + \|Tx\|^2\right)^{1/2}$, and $T : (D(T), \|\,.\,\|_g) \to Y$ is bounded.

Closable operators: T is closable if $\overline{G(T)}$ is the graph of an operator, in which case define $\overline{T}$ to be the extension of T with $G(\overline{T}) = \overline{G(T)}$. In fact T is closable iff whenever $(x_n) \subset D(T)$ with $x_n \to 0$ and $Tx_n \to y$ for some $y \in Y$ then y is necessarily 0. So bounded $\Longrightarrow$ closed $\Longrightarrow$ closable.

Adjoints: now work on Hilbert spaces H, K, and $T : D(T) \to K$ with $D(T) \subseteq H$. Then $T' : D(T') \to H$ with $D(T') \subseteq K$ is an adjoint to T if $\langle Th, k\rangle = \langle h, T'k\rangle$ for all $h \in D(T)$ and $k \in D(T')$. If $D(T)$ is dense, then there is a maximal adjoint T^* such that all other adjoints are restriction of T^* to subspaces of $D(T^*)$. Indeed $D(T^*) = \{k \in K : \exists h_0 \in H \text{ s.t. } \langle Th, k\rangle = \langle h, h_0\rangle \ \forall h \in D(T)\}$, and then $T^*k = h_0$.

Duality of graphs: if T has dense domain, then $G(T)^\perp = \{(h,k) : k \in D(T^*), h = -T^*k\}$, the *reversed graph*, $G'(-T^*)$ of $-T^*$. Thus if T is closed and densely defined, then so is T^* and also $T^{**} = T$.

The gap metric: for V, W, closed subspaces of H (Hilbert), the gap between them is $\delta(V,W) = \|P_V - P_W\|$, where P_V, P_W denote the orthogonal projections onto V, W. This is a metric on closed subspaces.

Chordal distance. Take a, $b \in \mathbb{C}$; then $\delta(V_a, V_b) = |a-b|/(\sqrt{1+|a|^2}\sqrt{1+|b|^2})$, where $V_a = \{(x, ax) : x \in \mathbb{C}\} \subset \mathbb{C}^2$, and similarly for V_b. This is the chordal distance on the Riemann sphere between a and b (identified with $\mathbb{C} \cup \{\infty\}$ by stereographic projection).

Formulae for the gap: $\delta(V,W) = \max\{\vec{\delta}(V,W), \vec{\delta}(W,V)\}$, where $\vec{\delta}(V,W) = \|(I - P_W)P_V\|$ is the *directed gap* (not a metric). Define $\vec{\delta}(W,V)$ similarly! Then $\vec{\delta}(V,W) = \sup_{v \in V, \|v\|=1} \operatorname{dist}(v, W)$. In Banach spaces, this isn't always a metric, but $\delta_1(V,W) = \max\{\vec{\delta}_1(V,W), \vec{\delta}_1(W,V)\}$ is one, where $\vec{\delta}_1(V,W) = \sup_{v \in V, \|v\|=1} \operatorname{dist}(v, S_W)$, and $S_W = \{w \in W : \|w\| = 1\}$.

Operator gaps: let $\delta(A,B) = \delta(G(A), G(B))$, where $A : D(A) \to K$ and $B : D(B) \to K$ are closed operators with $D(A)$, $D(B) \subseteq H$, and H and K are Hilbert spaces. This is a metric on closed operators, and restricted to the bounded operators it gives the usual norm topology. In fact $P_{G(T)} = \begin{pmatrix} I \\ T \end{pmatrix} (I + T^*T)^{-1} \begin{pmatrix} I & T^* \end{pmatrix}$, for T bounded, using column vectors to denote elements of $H \times K$.

Semigroups: let $(T(t))$ be defined for $t \geq 0$ as bounded operators on a Banach space X, such that (i) $T(0) = I$; (ii) $T(s+t) = T(s)T(t)$ for s, $t \geq 0$; (iii) $t \mapsto T(t)x$ is continuous for all $x \in X$. These are C_0 *semigroups*, also called *strongly continuous semigroups*. Simplest example: $T(t) = e^{At}$, for A bounded, defined as the limit of the power series. Or on $L^2(0,\infty)$, let $(T(t)f)(s) = \begin{cases} 0 & \text{if } s < t, \\ f(s-t) & \text{if } s \geq t, \end{cases}$ (the right shift semigroup). *Growth bound:* for any C_0 semigroup $(T(t))$ there are M, $\alpha > 0$ such that $\|T(t)\| \leq Me^{\alpha t}$ for all $t \geq 0$.

Infinitesimal generator: this is a linear operator $A : D(A) \to X$ such that $Ax = \lim_{h\to 0+} \frac{T(h)x-x}{h}$, and $D(A)$ is the (linear) space of all x for which this limit exists. Now $T(t)$ maps $D(A)$ into $D(A)$ and $AT(t)x = T(t)Ax$ for $x \in D(A)$. In fact A is a closed operator and $D(A)$ is dense. If A is bounded, then $T(t) = e^{At}$.

Translations: if $(T(t))$ is a C_0 semigroup with generator A, and $\lambda \in \mathbb{C}$, then $(T(t)e^{-\lambda t})$ is a C_0 semigroup with generator $A - \lambda I$ (same domain of definition).

Resolvents: if $\|T(t)\| \leq Me^{\alpha t}$ for all $t \geq 0$, then $(A - \lambda I)$ is invertible whenever $\operatorname{Re} A > \alpha$, i.e., there is a bounded B s.t. $(A-\lambda I)Bx = x$ for $x \in X$ and $B(A-\lambda I)x = x$ for $x \in D(A)$. So for *contraction semigroups*, i.e., $\|T(t)\| \leq 1$ for all t, we can invert $A - \lambda I$ whenever $\operatorname{Re} \lambda > 0$. *Adjoints:* in the Hilbert space situation, if $(T(t))$ is a C_0 semigroup, with inf. gen. A, then $(T(t)^*)$ is a C_0 semigroup with inf. gen. A^*.

Hille–Yosida theorem: let A be closed, dense domain $D(A) \subseteq X$ (Banach). Then A is the inf. gen. of a contraction semigroup if and only if $(A - \lambda I)^{-1}$ exists for all $\lambda > 0$ and satisfies $\|(A - \lambda I)^{-1}\| \leq 1/\lambda$.

Feller–Miyadera–Phillips theorem: let A be closed, dense domain $D(A) \subseteq X$ (Banach). Then A is the inf. gen. of a C_0 semigroup with $\|T(t)\| \leq Me^{\alpha t}$ if and only if $(A-\lambda I)^{-1}$ exists for all $\lambda > \alpha$ and $\|(A - \lambda I)^{-n}\| \leq M/(\lambda - \alpha)^n$ for all $n \in \mathbb{N}$.

Mild solutions: the differential equation $\dot{x} = Ax$ with $x(0) = x_0 \in D(A)$ is said to have the mild solution $x(t) = T(t)x_0$ if A generates a C_0 semigroup $(T(t))$.

Shift-invariant operators: let S be the right shift on $\ell^2(\mathbb{Z}_+)$; look at T (poss. unbounded) s.t. $ST = TS$, that is, $G(T)$ is a shift-inv. subspace, i.e., $SG(T) \subseteq G(T) \subseteq \ell^2(\mathbb{Z}_+)^2$. *Automatic continuity:* every shift-inv. T with $D(T) = \ell^2(\mathbb{Z}_+)$ is bounded.

Natural equivalences: $\ell^2(\mathbb{Z}) \sim L^2(\mathbb{T})$, then $\ell^2(\mathbb{Z}_+) \sim H^2(\mathbb{T}) \sim H^2(\mathbb{D})$, by Fourier series/Taylor series. The operator S is unitarily equivalent to multiplication by $z = e^{it}$.

Wiener's theorem: a closed subspace $K \subseteq L^2(\mathbb{T})$ satisfies $SK = K$ if and only if $K = \chi_E L^2(\mathbb{T})$ for some measurable $E \subseteq \mathbb{T}$. In the vector-valued case, look at $L^2(\mathbb{T}, \mathbb{C}^m)$, and then $K = PL^2(\mathbb{T}, \mathbb{C}^m)$ for some measurable projection-valued function P, i.e., $P : \mathbb{T} \to \mathcal{L}(\mathbb{C}^m)$, and $P(e^{it})$ is the orthogonal projection onto a subspace $J(e^{it})$ of $\mathbb{C}^m$ for almost all t. The entries of $P(e^{it})$ (as a matrix), are measurable functions of t.

Beurling–Helson theorem: a closed subspace $K \subseteq L^2(\mathbb{T})$ satisfies $SK \subset K$, $SK \neq K$ if and only if $K = \phi H^2$, where $|\phi(e^{it})| = 1$ a.e., and ϕ measurable. There is also Beurling's theorem for $H^2(\mathbb{T}, \mathbb{C}^m)$, in which case K satisfies $SK \subset K$ if and only if $K = \Phi H^2(\mathbb{T}, \mathbb{C}^r)$ where $0 \leq r \leq m$ and $\Phi \in H^\infty(\mathbb{T}, \mathcal{L}(\mathbb{C}^r, \mathbb{C}^m))$ with $\Phi(e^{it})$ an isometry for almost all t, i.e., Φ is *inner*.

Georgiou–Smith theorem: let $T : D(T) \to H^2(\mathbb{T}, \mathbb{C}^p)$ be a closed shift-invariant operator with $D(T) \subseteq H^2(\mathbb{C}^m)$. Then $\exists r \leq m$, a nonsingular function $M \in H^\infty(\mathcal{L}(\mathbb{C}^r, \mathbb{C}^m))$ and $N \in H^\infty(\mathcal{L}(\mathbb{C}^r, \mathbb{C}^p))$ such that $G(T) = \begin{pmatrix} M \\ N \end{pmatrix} H^2(\mathbb{C}^r) = \Theta H^2(\mathbb{C}^r)$, say, where $\|\Theta u\| = \|u\|$ for all $u \in H^2(\mathbb{C}^r)$.

Scalar case: if $T : D(T) \to H^2$ with $D(T) \subseteq H^2$ is closed shift-inv. then $D(T) = \{0\}$ or $G(T) = \begin{pmatrix} m \\ n \end{pmatrix} H^2$ with m, $n \in H^\infty$ and $|m(e^{it})|^2 + |n(e^{it})|^2 = 1$ a.e. So T is mult. by n/m. Finally, T is bounded with $D(T) = H^2$ iff $n/m \in H^\infty$; then $\|T\| = \|n/m\|_\infty$.