University of Leeds, School of Mathematics

Postgraduate course in Analysis 2008–2009

Problems 1

1. Show that in the Bergman space A^2 of functions analytic in the disc, with inner product

$$\langle f, g\rangle = \frac{1}{\pi}\int_{r=0}^{1}\int_{\theta=0}^{2\pi} f(re^{i\theta})\overline{g(re^{i\theta})}\, r\, dr\, d\theta,$$

the functions z^n and z^m are orthogonal for distinct values of $n, m \geq 0$. Show also that $\|z^n\|_{A^2} = 1/\sqrt{n+1}$, and hence calculate the norm of a function given by a power series $\sum_{n=0}^{\infty} a_n z^n$, in the case that it is finite.

2. Show that for $f \in H^1$, and $|w| < 1$, we have $f(w) = \frac{1}{2\pi i}\int_{\mathbb{T}} f(z)\, dz/(z-w)$, integrating round the unit circle; that is, Cauchy's integral formula still holds, even though f need not be well-behaved on $\mathbb{T}$. (Hint: consider $f_r(z) = f(rz)$ for $0 < r < 1$.)
What do you get if you do the same integral for the non-analytic function $f(z) = 1/z^n$, with $n > 0$?

3. What are the harmonic extensions to $\mathbb{D}$ of the following functions defined on $\mathbb{T}$? (Hint: express them in terms of z and $\overline{z}$.)
(i) $\exp(1/z)$; (ii) $1/(z^2 - \frac{5}{2}z + 1)$; (iii) $\cos\theta$, where $z = e^{i\theta}$;
(iv) $\cos(1/\overline{z})$.
Which if any are in H^∞?

4. For which values of p with $1 \leq p \leq \infty$ is $(1-z)^{-1/2}$ in H^p? For which real values of r is $\sum_{n=0}^{\infty} n^r z^n$ in H^2?

5. Which of the following sets in the disc is the zero set of an H^2 function? For any that is, write down an inner function with precisely those zeroes.
(i) $\{\frac{1}{2}\}$; (ii) $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\}$; (iii) $\{1 - 2^{-n}\exp(i2^n) : n \geq 1\}$.

6. What is the norm of the operator M_ϕ on H^2, where $\phi(z) = z + 4$? Is there a function $f \neq 0$ in H^2 with $\|M_\phi f\|_2 = \|M_\phi\|\,\|f\|_2$?

7. Calculate Pf, where $f(z) = 1/(z^2 + 2z)$ and $P : L^2 \to H^2$ is the orthogonal projection.

8. What are the symbol and norm of the Toeplitz operator defined by the following matrix?

$$\begin{pmatrix} 2 & 1 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & \ldots \\ 0 & 1 & 2 & 1 & \ldots \\ 0 & 0 & 1 & 2 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

9. Let Γ denote the following Hankel matrix:

$$\begin{pmatrix} -8 & 3 & 0 & 0 & \dots \\ 3 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Show that Γ defines an operator of rank 2 and find its norm and a function $f \neq 0$ such that $\|\Gamma f\|_2 = \|\Gamma\| \, \|f\|_2$. Hence calculate the best H^∞ approximant of the $L^\infty(\mathbb{T})$ function $z^{10} - \frac{8}{z} + \frac{3}{z^2}$. Considering the same function as an element of $L^2(\mathbb{T})$, what is its best H^2 approximant?

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Problems 2

10. Given that the best H^∞ approximant to the function $1/(z-\frac{1}{2})$ is $h(z)=\frac{2}{3}$, find the minimum-norm H^∞ function f such that $f(0)=0$ and $f(\frac{1}{2})=1$. Now solve the same problem using the H^2 norm.

11. Use the alternative Möbius map $M_\lambda(z)=\lambda(1-z)/(1+z)$ for arbitrary $\lambda>0$ to construct a norm-preserving bijection between $L^2(\mathbb{T})$ and $L^2(i\mathbb{R})$, and thus write down an orthonormal basis for $L^2(i\mathbb{R})$ whose poles are at $\pm\lambda$.

12. The spectral radius of an element f in a Banach algebra is defined by $r(f)=\max\{|\lambda| : \lambda\in\sigma(f)\}$. Show that, for every $f\in A(\mathbb{D})$, we have $r(f)=\|f\|$.

Show that if $f\in W_+$, then its spectrum is the same whether we consider f as an element of W_+ or of $A(\mathbb{D})$.

13. Show that if f is a polynomial of degree 1, then the norm of f is the same in both algebras $A(\mathbb{D})$ and W_+. Show that the polynomial $f(z)=z^2+z-1$ satisfies $\|f\|_{A(\mathbb{D})}<3$ (in fact it is $\sqrt{5}$), but $\|f\|_{W_+}=3$.

14. For which real values of a does $1/(a+\cos z)$ lie in W? Answer the same question for W_+.

15. Show that the functions $f(z)=z^2$ and $g(z)=1+z$ are coprime in $A(\mathbb{D})$. Find a Bézout identity $fh+gk=1$ which they satisfy in $A(\mathbb{D})$.

16. Which of the following sequences (γ_n) determine a reproducing kernel Hilbert space of functions on the disc with $\|z^n\|=\gamma_n$ for all n?

(i) $\gamma_n=2^n$; (ii) $\gamma_n=1/2^n$; (iii) $\gamma_n=(1+n)^{1/4}$.

17. A function in PW(1) is given by $f(0)=1$, $f(\pi)=-1$ and $f(n\pi)=0$ for $n\neq 0,1$. What is f?

Write down a different function in PW(2) that takes the same values at the points $n\pi$ as f does.

18. Show that $(z_k)=(1-2^{-k})$ is an interpolating sequence. By taking the union of (z_k) and another sequence obtained by perturbing each element slightly, show that the union of two interpolating sequences need not be an interpolating sequence.

However, show that the union of two sequences satisfying the Newman condition (N) still satisfies (N).