Spaces of analytic functions (Postgraduate course)

Jonathan R. Partington, University of Leeds, School of Mathematics

May 1, 2009

These notes are based on the postgraduate course given in Leeds in January–May 2009. Proofs are mostly omitted, and the numbering of definitions and results differs slightly from that used in the course.

Books:
K. Hoffman, Banach spaces of analytic functions.
P. Koosis, Introduction to H_p spaces.
W. Rudin, Real and complex analysis.
N. Nikolski, Operators, functions and systems, an easy reading, Vol. 1.

Introduction

We are going to work with Banach and Hilbert spaces whose elements are functions.

0.1 *Examples (treated informally for the moment)*

1. The *Hardy spaces* H^p $(1 \le p \le \infty)$ are Banach spaces consisting of analytic functions in the unit disc $\mathbb{D}$ whose boundary values are in $L^p(\mathbb{T})$, where $\mathbb{T}$ is the unit circle. Thus $f(z) = \sum_{n=0}^\infty a_n z^n$ for $|z| < 1$, and on $|z| = 1$, f corresponds to an L^p function. We use normalized Lebesgue measure on $\mathbb{T}$.
Recall that functions of period 2π have Fourier series, that is, $g(t) \sim \sum_{n=-\infty}^\infty c_n e^{int}$, where

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} g(t)e^{-int}\,dt.$$

If $g \in L^p(0, 2\pi)$ for some $1 \le p \le \infty$, then $g \in L^1(0, 2\pi)$, and

$$|c_n| \le \frac{1}{2\pi}\int_0^{2\pi} |g(t)||e^{int}|\,dt = \|g\|_{L^1},$$

so the sequence (c_n) is bounded. Now if $c_n = 0$ for all $n < 0$, we can define

$$f(z) = \sum_{n=0}^{\infty} c_n z^n,$$

and it converges for $|z| < 1$.
So we either think of H^p functions a defined on the disc, or as functions on the circle, putting $f(e^{it}) = g(t)$, or indeed as functions of period 2π.
In the special case $p = \infty$, H^∞ consists of all bounded analytic functions on $\mathbb{D}$.

2. The *harmonic Hardy spaces.* We may proceed similarly for the set of *all* L^p functions on the circle. Now the natural extension to the disc of $\sum_{-\infty}^{\infty} c_n e^{int}$ is going to be harmonic rather than analytic, i.e., we take $\sum_{n=0}^{\infty} c_n z^n + \sum_{n=1}^{\infty} c_{-n} \overline{z}^n$. (Recall that harmonic functions satisfy $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ and include analytic functions such as z^n and anti-analytic ones such as $\overline{z}^n$.) We define h_p to be the space of harmonic functions in the disc with L^p boundary values.

3. The *disc algebra* $A(\mathbb{D})$. These are the functions continuous on the closed unit disc $\overline{\mathbb{D}}$, and analytic on the open unit disc. We use the supremum norm.
This is a closed subspace of H^∞.

4. The *Bergman spaces.* These are again spaces of analytic functions in the disc, but now in L^p on the disc, i.e., $f \in A^p$ if and only if f is analytic in $\mathbb{D}$ and

$$\|f\|_p = \left(\frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^p \, dA(z) \right)^{1/p} < \infty,$$

where $dA(z)$ is two-dimensional Lebesgue measure, i.e.,

$$\|f\|_p^p = \frac{1}{\pi} \int_{r=0}^{1} \int_{\theta=0}^{2\pi} |f(re^{i\theta})|^p r \, dr \, d\theta.$$

Note that $A^\infty = H^\infty$.

5. The *Wiener algebra*, W. Take all functions of period 2π that can be written as $g(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$, with $\|g\|_W = \sum_{-\infty}^{\infty} |c_n| < \infty$. Here the series converges uniformly so g is actually continuous. Transferring to the circle we can write $f(z) = \sum_{-\infty}^{\infty} c_n z^n$, with $\sum_{-\infty}^{\infty} |c_n| < \infty$.
The *positive Wiener algebra* W_+ consists of all analytic functions $f(z) = \sum_{n=0}^{\infty} c_n z^n$ in the disc with $\|f\|_w = \sum_0^\infty |c_n| < \infty$. These functions all lie in $A(\mathbb{D})$.

Elementary inclusions:

$$W_+ \subset A(\mathbb{D}) \subset H^\infty \subset H^2 \subset H^1$$

(and many others).

6. The *Paley–Wiener space* $PW(K)$, where $K \subset \mathbb{R}$ is compact. Let f be a function on $\mathbb{R}$ which can be written as

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwt}\, dw,$$

where $\hat{f}(w) = 0$ for $w \notin K$, and $\hat{f}_{|K} \in L^2(K)$. Here $\hat{f}$ is the Fourier transform of f, and can be defined (for $f \in L^1(\mathbb{R})$) by

$$\hat{f}(w) = \int_{-\infty}^{\infty} f(t) e^{-iwt} dt.$$

It turns out that such f can be extended to be analytic on the whole of $\mathbb{C}$, and that

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(w)|^2\, dw,$$

makes $PW(K)$ into a Banach space, indeed even a Hilbert space.
An important special case is $K = [-a, a]$, where $a > 0$. Then $PW(K)$ is the space of band-limited signals (no high-frequency components).

7. The *Segal–Bargmann (or Fock) space* consisting of all functions f analytic on the whole of $\mathbb{C}$ such that

$$\|f\|^2 = \int_{\mathbb{C}} |f(z)|^2 e^{-\pi|z|^2}\, dA(z) < \infty.$$

This has applications in quantum mechanics.

1 Hardy spaces on the disc

We begin with H^2, which is a Hilbert space. Write $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta} : 0 \le \theta \le 2\pi\}$. Normalize it to have measure 1, so all integrals will be with respect to $\frac{d\theta}{2\pi}$.

1.1 *Proposition*

The set of functions $\{z^n : n \in \mathbb{Z}\} = \{e^{in\theta} : n \in \mathbb{Z}\}$ forms an orthonormal basis for $L^2(\mathbb{T})$, and its linear span is dense in $L^p(\mathbb{T})$ for all $1 \le p, \infty$.
Here we use the inner product on $L^2(\mathbb{T})$ given by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})}\, d\theta.$$

1.2 *Definition*

The space H^2 consists of all functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic in the unit disc $\mathbb{D}$ such that $\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty$. It can naturally be regarded as the closed subspace of $L^2(\mathbb{T})$ consisting of all functions $\sum_{n=0}^{\infty} a_n e^{in\theta}$ with $\sum_{n=0}^{\infty} |a_n|^2 < \infty$, i.e., the closed linear span of $\{e^{in\theta} : n \geq 0\}$.
Note that if $|z| < 1$ then

$$\sum |a_n z^n| \leq \left(\sum |a_n|^2\right)^{1/2} \left(\sum |z^{2n}|\right)^{1/2} < \infty,$$

by Cauchy–Schwarz, so the power series have radius of convergence at least 1, and define analytic functions in $\mathbb{D}$.

1.3 *Proposition*

H^2 is a Hilbert space with inner product

$$\left\langle \sum_0^{\infty} a_n z^n, \sum_0^{\infty} b_n z^n \right\rangle = \sum a_n \overline{b}_n,$$

or equivalently

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \, d\theta.$$

Analytic and harmonic extensions

Recall the *Dirichlet problem:* given f continuous on the unit circle, find g harmonic on the interior which extends it to a continuous function on the closed disc. These can be real or complex functions.
Recall that if g is analytic, then its real and imaginary parts are both harmonic, so g itself is. but if $g = u + iv$, then $\overline{g} = u - iv$ is also harmonic.
So if $f(e^{i\theta}) = e^{in\theta}$ for some $n \in \mathbb{Z}$, then a solution is

$$g(re^{i\theta}) = \begin{cases} r^n e^{in\theta} = z^n & \text{if} \quad n \geq 0, \\ r^m e^{-im\theta} = \overline{z}^m & \text{if} \quad n = -m < 0, \end{cases}$$

so that it $f(e^{in\theta}) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$, then $g(re^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta}$ is the 'natural' extension to $\mathbb{D}$.

1.4 *Definition*

The *Poisson kernel* $P(z, w)$, defined for $0 \leq |w| < |z| \leq 1$, is the function

$$P(z, w) = \frac{|z|^2 - |w|^2}{|z - w|^2}.$$

When $|z| = 1$, this equals $\frac{1-|w|^2}{|1-w\overline{z}|^2}$, since $|z-w| = |z\overline{z} - w\overline{z}| = |1 - w\overline{z}|$. This equals

$$\begin{aligned} \frac{1 - w\overline{w}}{(1-w\overline{z})(1-\overline{w}z)} &= \frac{1}{1-\overline{w}z} + \frac{1}{1-w\overline{z}} - 1 \\ &= \sum_{n=0}^{\infty} \overline{w}^n z^n + \sum_{n=1}^{\infty} w^n \overline{z}^n. \end{aligned}$$

Thus $P(e^{it}, re^{i\theta}) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(t-\theta)} = P_r(t-\theta)$, say.
Clearly the Poisson kernel is a positive continuous function of z and w for $0 \le |w| < |z| \le 1$.
Note that $\sum_{n=-\infty}^{\infty} r^{|n|} e^{in(t-\theta)} = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-t)}$, so $P_r(t-\theta) = P_r(\theta - t)$.

1.5 *Theorem*

If $f \in L^1(\mathbb{T})$, then the function F defined on $\mathbb{D}$ by

$$F(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t) f(e^{it})\, dt,$$

is a harmonic function, the harmonic extension of f.

1.6 *Proposition*

If $f(e^{int}) = e^{int}$ and $0 \le r < 1$, then $F(re^{i\theta}) = r^{|n|} e^{in\theta}$, i.e., $F(w) = w^n$ for $n \ge 0$ and $\overline{w}^{-n}$ for $n \le 0$.

1.7 *Corollary*

If $f \in L^1(\mathbb{T})$ and $c_n = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int}\, dt$, for $n \in \mathbb{Z}$, then the harmonic extension of f is given by

$$F(w) = \sum_{n=0}^{\infty} c_n w^n + \sum_{n=1}^{\infty} c_{-n} \overline{w}^n,$$

converging uniformly on $|w| \le r$ for $0 \le r < 1$.

1.8 *Definition*

The *Hardy space* H^p for $1 \le p \le \infty$ is the subspace of $L^p(\mathbb{T})$ consisting of all functions f such that $\frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int}\, dt = 0$ for all $n < 0$ (i.e., $\langle f(z), z^n \rangle = 0$). Such functions have an analytic extension to $\mathbb{D}$ and can be represented by

$$F(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad \text{with} \qquad c_n = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int}\, dt \qquad (n \ge 0).$$

Also

$$F(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t) f(e^{it})\, dt.$$

Note that H^p is a closed subspace, since it equals $\bigcap_{n=1}^{\infty} \phi_n^{-1}\{0\}$, where $\phi_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{int}\,dt$.
Since $|\phi_n(f)| \le \|f\|_p \|e^{int}\|_q = \|f\|_p$, where $\frac{1}{p}+\frac{1}{q}=1$ (Hölder's inequality), we see that ϕ_n is continuous and so $\phi_n^{-1}\{0\}$ is closed.
Normally we don't distinguish between f and F.
We normally think of H^p as the class of analytic functions in the disc with L^p boundary values (i.e., they are Poisson integrals of L^p functions).

1.9 *Corollary*

H^p is a Banach space.

1.10 *Theorem*

If $f \in L^\infty(\mathbb{T})$ with

$$\frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\,dt = 0 \qquad \text{for} \quad n < 0, \tag{1}$$

then $F(z)$ as given by Theorem 1.5 satisfies $\|F\|_\infty = \|f\|_\infty$. Conversely, any bounded analytic function F in the disc is the harmonic extension of a function f satisfying (1).

1.11 *Proposition*

If $f \in H^1$, then $\sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\,d\theta \le \|f\|_{H^1}$.

We later prove that in fact we have equality above.

1.12 *Definition*

A function of the form

$$B(z) = e^{i\phi}\prod_{j=1}^{n}\frac{z-z_j}{1-\overline{z}_j z},$$

where $\phi \in \mathbb{R}$ and $|z_j| < 1$ for $j = 1, \ldots, n$, is called a *finite Blaschke product.*

1.13 *Proposition*

Let B be a finite Blaschke product as above. Then
(i) B is analytic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$.
(ii) B has zeroes at $z_1, \ldots, z_n$ only, and poles at $1/\overline{z}_1, \ldots, 1/\overline{z}_n$ only.
(iii) $|B(e^{i\theta})| = 1$ for $\theta \in \mathbb{R}$.

1.14 *Definition*

An H^∞ function that has unit modulus almost everywhere on $\mathbb{T}$ is called an *inner function.*

Examples are finite Blaschke products and the 'singular inner function' $\exp\left(\frac{z-1}{z+1}\right)$. If we have an H^p function, we can use Blaschke products to factor out its zeroes, i.e., we can write $f(z) = B(z)g(z)$, where B is an infinite Blaschke product and $g \neq 0$. We now show this.

1.15 *Lemma*

If $f \in H^p$, $f \not\equiv 0$, then the zeroes (z_n) of f are at most countable in number and satisfy

$$\sum_n (1 - |z_n|) < \infty.$$

1.16 *Theorem*

Let $f \in H^p$. Then the infinite (or finite) Blaschke product

$$B(z) = z^m \prod_{z_n \neq 0} \frac{-\overline{z}_n}{|z_n|} \frac{z - z_n}{1 - \overline{z}_n z},$$

where (z_n) are the zeroes of f, with m of them at 0, converges uniformly on compact subsets of $\mathbb{D}$ to define an H^∞ function whose only zeroes are at the (z_n), with the correct multiplicities. Moreover, $|B(z)| \leq 1$ in $\mathbb{D}$ and $|B(e^{i\theta})| = 1$ a.e. for $e^{i\theta} \in \mathbb{T}$.

1.17 *Proposition*

Let $P(z, w)$ be the Poisson kernel. Then:
(i) If f is analytic in $\mathbb{D}$, then we have $f(w) = \frac{1}{2\pi} \int_0^{2\pi} P(re^{i\theta}, w) f(re^{i\theta})\, d\theta$ for $|w| < r < 1$;
(ii) $\frac{1}{2\pi} \int_0^{2\pi} P(re^{i\theta}, w)\, d\theta = 1$ for $|w| < r$;
(iii) $\frac{1}{2\pi} \int_0^{2\pi} P(z, se^{i\phi})\, d\phi = 1$ for $|z| > s$.

1.18 *Corollary*

If $0 < s < r < 1$ and f is analytic in $\mathbb{D}$, then

$$\frac{1}{2\pi} \int_0^{2\pi} |f(se^{i\phi})|\, d\phi \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|\, d\theta.$$

1.19 *Corollary*

For $f \in H^1$, the limit

$$\lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|\, d\theta$$

exists and equals $\|f\|_{H^1}$. Moreover $\|f_r - f\|_1 \to 0$, where $f_r(e^{i\theta}) = f(re^{i\theta})$.

For $f \in H^2$, we also have $\|f_r - f\|_2 \to 0$ (easy to prove), while for $f \in H^\infty$ we don't always have $\|f - f_r\|_\infty \to 0$, since the f_r are continuous on $\mathbb{T}$ and f needn't be. However $\|f_r\|_\infty \to \|f\|_\infty$.

1.20 *Theorem (Riesz)*

Let $f \in H^p$ with $f \not\equiv 0$, and let B be the Blaschke product formed using the zeroes (z_n) of f. Then $f = gB$, where $\|f\|_p = \|g\|_p$.

We prove this in the course just for $p = 1, 2$ and ∞.

1.21 *Theorem (Riesz factorization theorem)*

A function f lies in H^1 if and only if there exist $g, h \in H^2$ such that $f = gh$. We can choose g and h such that $\|f\|_1 = \|g\|_2 \|h\|_2$.

1.22 *Definition*

The *disc algebra* $A(\mathbb{D})$ is the space
$\{f \in C(\mathbb{T}) : \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})e^{-int}\, dt = 0 \text{ for all } n < 0\}$.
It is a closed subspace of $C(\mathbb{T})$, hence a Banach space. All functions in $A(\mathbb{D})$ lie in H^2 so have analytic extensions to $\mathbb{D}$ given by $f(e^{it}) = \sum_{n=0}^\infty c_n e^{int}$ (convergence in L^2), and $f(re^{it}) = \sum_{n=0}^\infty c_n r^n e^{int}$ (convergence locally uniformly).

1.23 *Theorem*

The closure of the space of polynomials in $L^p(\mathbb{T})$ is H^p for $1 \le p < \infty$ and $A(\mathbb{D})$ for $p = \infty$.

2 Operators on H^2 and $L^2(\mathbb{T})$

We look at three important classes of operators: multiplication (Laurent) operators, Toeplitz operators, and Hankel operators.

Notation: $P : L^2(\mathbb{T}) \to H^2$ denotes the orthogonal projection, $\sum_{n=-\infty}^\infty a_n e^{int} \mapsto \sum_{n=0}^\infty a_n e^{int}$. Clearly $\|Pf\|_2 \le \|f\|_2$.

2.1 *Proposition*

Let $\phi \in L^\infty(\mathbb{T})$. Then the transformation $M_\phi : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ defined by $M_\phi f(e^{it}) = \phi(e^{it}) f(e^{it})$ is a bounded operator of norm $\|M_\phi\| = \|\phi\|_\infty$.
Moreover, $\sup\{\|M_\phi f\|_2 : f \in H^2, \|f\|_2 = 1\} = \|\phi\|_\infty$.

2.2 *Corollary*

If $\phi \in H^\infty$ then $M_\phi : H^2 \to H^2$ satisfies $\|M_\phi\| = \|\phi\|_\infty$.

Matrix notation.

Write $(e_n)_{n=-\infty}^\infty$ for the orthonormal basis of $L^2(\mathbb{T})$ given by $e_n(z) = z^n$ for $z \in \mathbb{T}$.
Then $(e_n)_{n=0}^\infty$ is an orthonormal basis of H^2.
If $\phi(z) = \sum_{k=0}^\infty d_k z^k$, then calculating $M_\phi e_n$ we get the infinite matrix

$$\begin{pmatrix} d_0 & 0 & 0 & 0 & \dots \\ d_1 & d_0 & 0 & 0 & \dots \\ d_2 & d_1 & d_0 & 0 & \dots \\ d_3 & d_2 & d_1 & d_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which is constant on the main diagonals, i.e., a *Toeplitz matrix.* It is also lower triangular.

2.3 *Definition*

If $\phi \in L^\infty(\mathbb{T})$, then the *Toeplitz operator with symbol* ϕ, T_ϕ, is the operator $T_\phi : H^2 \to H^2$ defined by $T_\phi f = P(M_\phi f)$, for $f \in H^2$.

Clearly, $\|T_\phi\| \le \|\phi\|_\infty$, and we have equality if $\phi \in H^\infty$. We'll see that in fact we have equality always.

Matrix of T_ϕ.

Let $\phi(e^{it}) = \sum_{k=-\infty}^\infty d_k e^{ikt}$. We then have the (full) Toeplitz matrix.

$$\begin{pmatrix} d_0 & d_{-1} & d_{-2} & d_{-3} & \dots \\ d_1 & d_0 & d_{-1} & d_{-2} & \dots \\ d_2 & d_1 & d_0 & d_{-1} & \dots \\ d_3 & d_2 & d_1 & d_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

Special cases: if $\phi \equiv 1$, then T_ϕ is the identity. If $\phi(z) = z$, then T_ϕ is the shift, $z^n \mapsto z^{n+1}$.

2.4 *Theorem*

For $\phi \in L^\infty(\mathbb{T})$, $\|T_\phi\| = \|\phi\|_\infty$, that is, $\sup_{f\in H^2, \|f\|=1} \|T_\phi f\| = \|\phi\|_\infty$.

2.5 *Definition*

We write $(H^2)^\perp$ for the orthogonal complement of H^2 in $L^2(\mathbb{T})$, i.e., the closed subspace spanned by the negative basis vectors $e_n(z) = z^n$, for $z \in \mathbb{T}$, with $n < 0$. If $\phi \in L^\infty(\mathbb{T})$ then the *Hankel operator* $\Gamma_\phi : H^2 \to (H^2)^\perp$ is defined by $\Gamma_\phi f = (I - P)M_\phi f$, i.e., $M_\phi = T_\phi + \Gamma_\phi$. There are several equivalent definitions, but we shall use this one.

2.6 *Proposition*

We have $\|\Gamma_\phi\| \le \|\phi\|_\infty$, and if $\phi(e^{it}) = \sum_{k=-\infty}^{\infty} d_k e^{ikt}$, then the matrix of Γ_ϕ with respect to the bases $\{e_0, e_1, e_2, \ldots\}$ of H^2 and $\{e_{-1}, e_{-2}, \ldots\}$ of $(H^2)^\perp$ is

$$\begin{pmatrix} d_{-1} & d_{-2} & d_{-3} & \cdots \\ d_{-2} & d_{-3} & d_{-4} & \cdots \\ d_{-3} & d_{-4} & d_{-5} & \cdots \\ d_{-4} & d_{-5} & d_{-6} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

constant on 'minor' diagonals. Also $\Gamma_\phi = 0$ if and only if $\phi \in H^\infty$ (whereas $T_\phi = 0$ if and only if $\phi = 0$).

2.7 *Theorem (Nehari)*

If $\Gamma_\phi : H^2 \to (H^2)^\perp$ is a bounded Hankel operator, then there exists a symbol $\psi \in L^\infty(\mathbb{T})$ such that $\Gamma_\phi = \Gamma_\psi$ and $\|\Gamma_\psi\| = \|\psi\|_\infty$. Hence

$$\|\Gamma_\phi\| = \inf\{\|\phi + h\| : h \in H^\infty\} = \operatorname{dist}(\phi, H^\infty).$$

2.8 *Example*

Hilbert's Hankel matrix

$$\Gamma = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

has norm π as an operator on H^2 (take as symbol $\psi(e^{i\theta}) = i(\theta - \pi)$ for $0 \le \theta < 2\pi$). We thus have the inequality $|\langle \Gamma x, y\rangle| \le \pi \|x\| \, \|y\|$, or

$$\left| \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{x_n \overline{y}_m}{n+m+1} \right| \le \pi \left(\sum_{n=0}^{\infty} |x_n|^2 \right)^{1/2} \left(\sum_{m=0}^{\infty} |y_m|^2 \right)^{1/2}.$$

To get a more explicit solution we need a technical lemma on H^2 functions.

2.9 *Lemma*

If $f \in H^2$ and $f \not\equiv 0$, then

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| \, d\theta > -\infty,$$

and hence $f \neq 0$ a.e. on $\mathbb{T}$.

(Of course, arbitrary L^2 functions can vanish on any set of positive measure that we like.)

2.10 *Corollary*

If $f \in A(\mathbb{D})$, and $f \not\equiv 0$, then, regarding f as a function on $\overline{\mathbb{D}}$, its zero set $Z = \{z \in \overline{\mathbb{D}} : f(z) = 0\}$ satisfies:
(i) $Z \cap \mathbb{D}$ is countable and $\sum_{z_n \in Z \cap \mathbb{D}} (1 - |z_n|) < \infty$;
(ii) $Z \cap \mathbb{T}$ has measure 0;
(iii) Z is closed.

Solution of the Nehari problem

2.11 *Theorem (Sarason)*

If $\Gamma : H^2 \to (H^2)^\perp$ is a bounded Hankel operator and if $f \in H^2$ with $f \not\equiv 0$ satisfies $\|\Gamma f\| = \|\Gamma\| \, \|f\|$, then there is a unique symbol ψ for $\Gamma = \Gamma_\psi$ of minimal norm, i.e., $\|\psi\|_\infty = \|\Gamma_\psi\|$, and it is given by $\psi = \frac{\Gamma f}{f}$, i.e.,

$$\psi(e^{i\theta}) = \frac{(\Gamma f)(e^{i\theta})}{f(e^{i\theta})} \qquad \text{a.e.}$$

Moreover $|\psi(e^{i\theta})|$ is constant almost everywhere.

2.12 *Example*

Take a rank-1 Hankel operator with matrix

$$\begin{pmatrix} 1 & \alpha & \alpha^2 & \dots \\ \alpha & \alpha^2 & \alpha^3 & \dots \\ \alpha^2 & \alpha^3 & \alpha^4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $|\alpha| < 1$. One can check that $\|\Gamma\| = 1/(1 - |\alpha|^2)$. Let $u(z) = 1/(1 - \overline{\alpha} z)$, so that

$$(\Gamma u)(z) = \frac{1}{1 - |\alpha|^2} \left(\frac{1}{z} + \frac{\alpha}{z^2} + \frac{\alpha^2}{z^3} + \dots \right),$$

and thus

$$\psi(z) = \frac{(\Gamma u)(z)}{u(z)} = \frac{1}{1-|\alpha|^2}\frac{1-\overline{\alpha}z}{z-\alpha}.$$

Note that $(z-\alpha)/(1-\overline{\alpha}z)$ is a Blaschke product, so ψ has modulus $1/(1-|\alpha|^2)$ a.e.

Best approximation problems (Nehari problems)

2.13 *Corollary*

If $\phi \in L^\infty(\mathbb{T})$ is such that Γ_ϕ attains its norm (i.e., $\|\Gamma_\phi f\| = \|\Gamma_\phi\| \, \|f\|$ for some $f \in H^2$ with $f \not\equiv 0$), then ϕ has a unique best approximant h in H^∞ (i.e., a closest element, so that $\|\phi - h\|_\infty = \operatorname{dist}(\phi, H^\infty)$), given by

$$h = \phi - \frac{\Gamma f}{f}, \qquad \text{and} \qquad \|\phi - h\|_\infty = \|\Gamma_\phi\|.$$

Moreover $|(\phi - h)(e^{i\theta})| = \|\Gamma_\phi\|$ a.e.

Note. If $\phi \in C(\mathbb{T})$, then there is always a best approximant in H^∞, but it does not always lie in $A(\mathbb{D})$.

2.14 *Example*

Take $\phi(z) = \frac{3}{z} + \frac{2}{z^2}$, which lies in $C(\mathbb{T})$. Note that the closest element in H^2 (in the $L^2(\mathbb{T})$ norm) is just the orthogonal projection onto H^2, which is 0. To get the closest element in H^∞ (in the $L^\infty(\mathbb{T})$ norm), form the rank-two Hankel matrix

$$\begin{pmatrix} 3 & 2 & 0 & \dots \\ 2 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This is symmetric, eigenvalues 4 and -1, and orthogonal eigenvectors $(2,1,0,0,\dots)^T$ and $(1,-2,0,0,\dots)^T$. Hence a maximizing vector is $f(z) = 2 + z$, mapped into $(\Gamma f)(z) = \frac{8}{z} + \frac{4}{z^2}$.
Then $h = \phi - \frac{\Gamma f}{f}$, and $h(z) = \frac{3}{2+z}$, and $h \in H^\infty$.
Indeed $\|\phi\|_\infty = 5$, so 0 is not the best uniform approximant, whereas

$$(\phi - h)(z) = \frac{4}{z^2}\frac{1+2z}{z+2},$$

which has constant modulus 4.

Two classical problems

1. The *Carathéodory–Fejér extension problem.*

Given a polynomial $a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$, choose coefficients $a_{n+1}, a_{n+2}, \ldots$ to minimize $\left\| \sum_{k=0}^{\infty} a_k z^k \right\|_\infty$; i.e., minimize

$$\left\| a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + z^{n+1} h(z) \right\|_\infty$$

over all h analytic and bounded in $\mathbb{D}$.

2. The *Nevanlinna–Pick interpolation problem.*

Given $z_1, \ldots, z_n \in \mathbb{D}$ and $w_1, \ldots, w_n \in \mathbb{C}$, find a function g analytic in $\mathbb{D}$ such that $g(z_k) = w_k$ for $k = 1, \ldots, n$, and $\|g\|_\infty$ is minimized.

2.15 *Theorem*

The Carathéodory–Fejér problem reduces to the Nehari problem.

Example. Let $g(z) = 2 + 3z$. Then $\|g\|_\infty = 5$. The minimal-norm extension is $f(z) = 4\frac{1+2z}{z+2}$, which has $\|f\|_\infty = 4$.

2.16 *Theorem*

The Nevanlinna–Pick problem reduces to the Nehari problem.

Example. Find a minimal-norm interpolant g such that $g(0) = 1$, $g(\frac{1}{2}) = 0$. We start with an interpolant $f(z) = 1 - 2z$ of norm 3, and then the best interpolant is $h(z) = -2\frac{z-2}{1-\frac{1}{2}z}$, of norm 2.

Finite-rank Toeplitz and Hankel operators.

2.17 *Proposition*

The only finite-rank Toeplitz operator is $T_0 = 0$.

2.18 *Theorem (Kronecker)*

The Hankel operator

$$\Gamma = \begin{pmatrix} d_{-1} & d_{-2} & d_{-3} & \ldots \\ d_{-2} & d_{-3} & d_{-4} & \ldots \\ d_{-3} & d_{-4} & d_{-5} & \ldots \\ d_{-4} & d_{-5} & d_{-6} & \ldots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

has finite rank if and only if $f(z) := \sum_{k=-\infty}^{-1} d_k z^k$ is a rational function of z, and its rank is the number of poles of f (which must lie in $\mathbb{D}$).

2.19 *FACT*

(i) There are no compact Toeplitz operators except 0.

(ii) Γ_ϕ is compact if and only if $\phi \in C(\mathbb{T}) + H^i nfty$.

2.20 *Definition*

The *Hilbert transform* is the operator defined on $L^2(\mathbb{T})$ by

$$(Hf)(e^{i\phi}) = -\frac{1}{\pi}\mathrm{PV}\int_{-\infty}^{\infty} \frac{f(e^{i\theta})}{\theta - \phi}\,d\theta,$$

where PV denotes the Cauchy principal value, i.e., $\lim_{\delta\to 0}\int_{\phi+\delta}^{\infty} + \int_{-\infty}^{\phi-\delta}$.

Strictly speaking, define on the orthonormal basis (e_n), where $e_n(z) = z^n$, and extend by linearity and continuity.
Indeed

$$He_n = \begin{cases} -ie_n & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ ie_n & \text{if } n < 0. \end{cases}$$

Then H is an operator of norm 1 and $H^4(\sum a_n e_n) = \sum_{n\neq 0} a_n e_n$.
Note $\cos n\theta \mapsto \sin n\theta$ and $\sin n\theta \mapsto -\cos n\theta$.
Thus $f + iHf$ is analytic in $\mathbb{D}$, that is, if u is real harmonic in $\mathbb{D}$, then Hu is the harmonic conjugate v such that $u + iv$ is analytic in $\mathbb{D}$, and $v(0) = 0$; this is unique.
Indeed,

$$\sum_0^\infty a_n r^n \cos n\theta + \sum_1^\infty b_n r^n \sin n\theta \mapsto \sum_0^\infty a_n r^n \sin n\theta - \sum_1^\infty b_n r^n \cos n\theta.$$

3 Hardy spaces on the half-plane

3.1 *Proposition*

The Möbius map $M : z \mapsto \frac{1-z}{1+z}$ is a self-inverse bijection from the disc $\mathbb{D}$ to the right half-plane $\mathbb{C}_+ = \{x + iy : x > 0\}$. Let $1 \le p < \infty$. Then a function g defined on $\mathbb{T}$ is in $L^p(\mathbb{T})$ if and only if the function $G : i\mathbb{R} \to \mathbb{C}$ defined by

$$G(s) = \pi^{-1/p}(1+s)^{-2/p} g(Ms)$$

is in $L^p(i\mathbb{R})$, and moreover $\|g\|_p = \|G\|_p$.

We also have

$$g(z) = 2^{2/p}\pi^{1/p}(1+z)^{-2/p}G(Mz).$$

For $p = \infty$, we take $G(s) = g(Ms)$ and $g(z) = G(Mz)$. Then $\|g\|_\infty = \|G\|_\infty$.

3.2 *Corollary*

The functions E_n defined by

$$E_n(s) = \frac{1}{\sqrt{\pi}}\frac{(1-s)^n}{(1-s)^{n+1}}, \qquad (n \in \mathbb{Z}),$$

form an orthonormal basis for $L^2(i\mathbb{R})$.

3.3 *Definition*

The space $H^\infty(\mathbb{C}_+)$ consists of all functions analytic and bounded in $\mathbb{C}_+$, with norm $\|G\| = \sup_{s\in\mathbb{C}_+} |G(s)|$. Looking at boundary values, we regard $H^\infty(\mathbb{C}_+)$ as the closed subspace of $L^\infty(i\mathbb{R})$ consisting of all G with $G(s) = g\left(\frac{1-s}{1+s}\right)$, where $g \in H^\infty$.

Likewise $H^2(\mathbb{C}_+)$ consists of those functions analytic in $\mathbb{C}_+$ with L^2 boundary values, i.e., the closed subspace of $L^2(i\mathbb{R})$ consisting of all G with

$$G(s) = \frac{1}{\sqrt{\pi}}\frac{1}{1+s}g\left(\frac{1-s}{1+s}\right), \qquad \text{for} \quad g \in H^2.$$

3.4 *Corollary*

The functions E_n for $n \geq 0$ form an orthonormal basis of $H^2(\mathbb{C}_+)$. (Note that E_n is analytic in $\mathbb{C}_+$ if and only if $n \geq 0$.)

Laplace and Fourier transforms

3.5 *Definition*

For $f \in L^2(\mathbb{R}) \cap L^2(\mathbb{R})$ define the 2-sided Laplace transform of f by

$$(\mathcal{L}f)(s) = \int_{-\infty}^{\infty} e^{-st} f(t)\, dt,$$

and the Fourier transform of f by

$$(\mathcal{F}f)(w) = \int_{-\infty}^{\infty} e^{-iwt} f(t)\, dt. \tag{2}$$

Thus $(\mathcal{F}f)(w) = (\mathcal{L}f)(iw)$. We are most interested in $w \in \mathbb{R}$ and $s \in i\mathbb{R}$, when, since $f \in L^1(\mathbb{R})$, the integrals converge absolutely.

We regard $L^2(\mathbb{R})$ as $L^2(-\infty, 0) \oplus L^2(0, \infty)$, decomposing a function into its values on $t < 0$ and $t > 0$.

3.6 *Theorem (Paley–Wiener–Plancherel)*

The Laplace transform (defined initially on $(L^1 \cap L^2)(\mathbb{R})$, then extended by continuity) determines a linear bijection between $L^2(\mathbb{R})$ and $L^2(i\mathbb{R})$ such that $\|\mathcal{L}f\|_{L^2(i\mathbb{R})} = \sqrt{2\pi}\|f\|_{L^2(\mathbb{R})}$. Also, $L^2(0, \infty)$ is mapped onto $H^2(\mathbb{C}_+)$.

There is an inverse map defined by

$$(\mathcal{L}^{-1}G)(t) = \frac{1}{2\pi i}\int_{i\mathbb{R}} G(s)e^{st}\, ds,$$

for $G \in (L^1 \cap L^2)(i\mathbb{R})$, extended by continuity to all of $L^2(i\mathbb{R})$.

Thus,

$$\begin{array}{rcl} L^2(\mathbb{R}) & = & L^2(0,\infty) \oplus L^2(-\infty,0) \\ & \mathcal{L}\downarrow & \\ L^2(i\mathbb{R}) & = & H^2(\mathbb{C}_+) \oplus H^2(\mathbb{C}_-). \end{array}$$

Compare the discrete version:

$$\begin{array}{rcl} \ell^2(\mathbb{Z}) & = & \ell^2(\mathbb{Z}_+) \oplus \ell^2(\mathbb{Z}_-) \\ & \downarrow & \\ L^2(\mathbb{T}) & = & H^2(\mathbb{D}) \oplus H^2(\mathbb{D})^\perp, \end{array}$$

where $\mathbb{Z}_+$ is the non-negative, and $\mathbb{Z}_-$ the strictly negative, integers. Here the mapping takes a sequence (a_n) to the function $e^{i\theta} \mapsto \sum a_n e^{in\theta}$.

3.7 *Corollary*

If $f \in L^2(\mathbb{R})$, then $\mathcal{F}f \in L^2(\mathbb{R})$ (defined by (2) for $f \in L^1 \cap L^2$ and extended by continuity), and $\|\mathcal{F}f\|_2 = \sqrt{2\pi}\|f\|_2$. Moreover, the inverse map is defined by

$$(\mathcal{F}^{-1}F)(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)e^{iwt}\, dw,$$

again for $F \in L^1 \cap L^2$, then extended by continuity to $L^2(\mathbb{R})$.

Notes.

1. If $f(t)$ and $f'(t)$ are both $O((1+|t|)^{-2})$ and f is C^2, then f and $\mathcal{F}f$ are in $L^1(\mathbb{R})$, so there are no problems about convergence of integrals. Such functions form a dense subspace of $L^2(\mathbb{R})$.

2. We write $\hat{f}(w) = (\mathcal{F}f)(w)$ and $\check{F}(t) = (\mathcal{F}^{-1}F)(t)$, and note that $\hat{\hat{f}}(t) = 2\pi f(-t)$, and thus $\mathcal{F}^4 = 4\pi^2 I$.

An orthonormal basis for $L^2(0,\infty)$

A rational orthonormal basis for $H^2(\mathbb{C}_+)$ was given by Corollary 3.4. The functions transform under $\mathcal{L}^{-1}$ to functions $t \mapsto p_n(t)e^{-t}$ in $L^2(0,\infty)$, where p_n is a real polynomial of degree n; then $\sqrt{2\pi}p_n(t)e^{-t}$ form an orthonormal basis for $L^2(0,\infty)$. In fact $p_n(t) = \pm L_n(2t)/\sqrt{\pi}$, where L_n denotes the Laguerre polynomial

$$L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t}),$$

and these satisfy

$$\int_0^\infty L_n(t)L_m(t)e^{-t}\,dt = \delta_{mn}$$

(Kronecker delta). Similarly $\sqrt{2\pi}p_n(-t)e^t$ form an orthonormal basis for $L^2(-\infty,0)$.

4 Commutative Banach algebras

4.1 *Definition*

A *Banach algebra* A is a Banach space with a multiplication defined on it such that

$$\|fg\| \le \|f\|\,\|g\| \qquad (f,g \in A).$$

Also we require the usual algebraic laws: $(fg)h = f(gh)$, $f(g+h) = fg + fh$, $(f+g)h = fh + gh$, and $\alpha(fg) = f(\alpha g) = (\alpha f)g$ for $\alpha \in \mathbb{C}$ and $f,g,h \in A$.

We shall assume that A is *unital*, i.e., has a unit e such that $ef = fe = f$ for all $f \in A$, and $\|e\| = 1$. We may sometimes write the unit as 1. Further A is *commutative*, if $fg = gf$ for all $f,g \in A$.

Examples of commutative (unital) Banach algebras of functions

1. $C(K)$, where K is compact and Hausdorff, with the supremum norm. For example, $C(\mathbb{T})$.
2. $A(\mathbb{D})$, the disc algebra.
3. L^∞ spaces.
4. H^∞.

5. The *Wiener algebra*

$$W = \{f : e^{i\theta} \mapsto \sum_{n=-\infty}^{\infty} a_n e^{in\theta} : \sum_{n=-\infty}^{\infty} |a_n| < \infty\},$$

with norm $\|f\| = \sum_{n=-\infty}^{\infty} |a_n|$ (i.e., the coefficients form a sequence in $\ell^1(\mathbb{Z})$). Note that indeed $\|fg\| \leq \|f\| \, \|g\|$.
6. The subalgebra

$$W_+ = \{f : e^{i\theta} \mapsto \sum_{n=0}^{\infty} a_n e^{in\theta} : \sum_{n=0}^{\infty} |a_n| < \infty\},$$

with coefficients in $\ell^1(\mathbb{Z}_+)$. Equivalently, these are the power series $z \mapsto \sum_{n=0}^{\infty} a_n z^n$ such that $\sum_{n=0}^{\infty} |a_n| < \infty$.

We have some continuous inclusions:

$$\begin{array}{ccccccl} W_+ & \rightarrow & A(\mathbb{D}) & \rightarrow & H_\infty & & \text{analytic functions} \\ \downarrow & & \downarrow & & \downarrow & & \\ W & \rightarrow & C(\mathbb{T}) & \rightarrow & L^\infty(\mathbb{T}) & & \text{bounded functions.} \end{array}$$

We are interested now in studying sets of the form $\{f \in A : f(z_0) = 0\}$ where $z_0 \in \mathbb{D}$, and in considering mappings $f \mapsto f(z_0)$.

4.2 *Definition*

A subset I of a commutative Banach algebra A is an *ideal*, if:
(a) $I \subseteq A$ as a linear subspace;
(b) whenever $f \in A$ and $g \in I$, then $fg \in I$.
If $I \neq \{0\}, A$, then I is a *proper ideal.* If I is a proper ideal not contained in a strictly larger one, then it is a *maximal ideal.*
Note that no invertible element can lie in a proper ideal, since if $u \in I$ and $u^{-1} \in A$, then $e = u^{-1}u \in I$ and so $x = xe \in I$ for all $x \in A$.

Let Δ denote the set of all *characters* on A, the *character space*, where a character δ is a complex homomorphism $\delta : A \to \mathbb{C}$ with $\delta(e) = 1$, i.e., δ is a linear mapping such that $\delta(fg) = \delta(f)\delta(g)$ for all $f, g \in A$.

The standard example is for $A = C(K)$ and $k \in K$, when δ_k defined by $\delta_k f = f(k)$ is a character.

The *spectrum* $\sigma(f)$ of $f \in A$ is defined by

$$\sigma(f) = \{\lambda \in \mathbb{C} : f - \lambda e \text{ is not invertible in } A\},$$

analogous to the eigenvalues of a matrix. So in $C(K)$, $\sigma(f) = \{f(k) : k \in K\}$.

4.3 *Theorem*

The spectrum is a non-empty compact set contained in the closed complex disc of radius $\|f\|$.

4.4 *Corollary*

A commutative Banach algebra that is also a field is isomorphic to $\mathbb{C}$.

4.5 *Proposition*

In a commutative Banach algebra A:
(i) every proper ideal is contained in a maximal ideal;
(ii) maximal ideals are closed;
(iii) if I is a closed ideal, then A/I is a commutative Banach algebra;
(iv) if M is a maximal ideal, then A/M is a field, and hence one-dimensional.

4.6 *Theorem*

(i) Every maximal ideal M is the kernel of some character $\delta \in \Delta$;
(ii) $\lambda \in \sigma(f)$ if and only $\delta(f) = \lambda$ for some $\delta \in \Delta$;
(iii) f is invertible in A if and only if $\delta(f) \neq 0$ for all $\delta \in \Delta$.

4.7 *Proposition*

(a) For $A = C(\mathbb{T})$ or W, we have $\Delta = \{\delta_w : w \in \mathbb{T}\}$, where $\delta_w = f(w)$. So in a natural way $\Delta = \mathbb{T}$;
(b) for $A = A(\mathbb{D})$ or W_+, we have $\Delta = \overline{\mathbb{D}}$ in the same way.

For non-separable spaces such as H^∞ and L^∞, the character space is harder to describe. It is the case that $L_\infty(\mathbb{T}) \approx C(K)$ for some K compact, Hausdorff (non-metrizable), but K is not easy to describe.
For H^∞ it is clear that $f \mapsto f(w)$ is a character for $w \in \mathbb{D}$, but there are others. For example, for $w \in \mathbb{T}$, there is an ideal $\{f \in H^\infty : f(z) \to 0 \text{ as } z \to w\}$, which is contained in a maximal ideal, giving a new set of characters.
Carleson's corona theorem asserts that $\delta_{\mathbb{D}}$ is dense in $\Delta(H^\infty)$, so that all characters on H^∞ are limits of a net of characters of the form δ_w for $w \in \mathbb{D}$.

4.8 *Corollary (Wiener's theorem)*

(a) If $f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \neq 0$ for all $0 \leq \theta \leq 2\pi$ and $\sum_{n=-\infty}^{\infty} |a_n| < \infty$, then $1/f$ has a Fourier series $\sum_{n=-\infty}^{\infty} b_n e^{in\theta}$ with $\sum_{n=-\infty}^{\infty} |b_n| < \infty$.
(b) If $f(z) = \sum_{n=0}^{\infty} a_n z^n \neq 0$ for $|z| \leq 1$ and $\sum_{n=0}^{\infty} |a_n| < \infty$, then $1/f(z) = \sum_{n=0}^{\infty} b_n z^n$ with $\sum_{n=0}^{\infty} |b_n| < \infty$.

There is a stronger form of Corollary 4.8: if ϕ is a function that is analytic on a neighbourhood of $\sigma(f)$, then $\phi(f)$ can be defined with the right properties, e.g. if ϕ is the limit of a sequence (ϕ_n), then $\phi(f)$ is the limit of $(\phi_n(f))$.
Thus if $\sigma(f) \subset \mathbb{C}_+$, we can define $\log f$ and $\sqrt{f}$ in A. This is called the Wiener–Lévy theorem.

Bézout identities

We say that f and g are *coprime* in A if $fh+gk = 1$ for some $h, k \in A$. Factorizing meromorphic functions as f/g with f, g coprime is of great practical importance.

4.9 *Corollary*

Two functions f, g in $A(\mathbb{D})$ or W_+ are coprime if and only if

$$\inf_{z \in \overline{\mathbb{D}}} |f(z)| + |g(z)| > 0,$$

i.e., since the inf is attained, if and only if $|f(z)| + |g(z)| > 0$ for all $z \in \overline{\mathbb{D}}$.

Half-plane analogues

As in the L^2 case, there is a continuous (half-plane) analogue of the correspondences $W \approx \ell^1(\mathbb{Z})$ and $W_+ \approx \ell^1(\mathbb{Z}_+)$.
To do this, start with $L^1(\mathbb{R})$ and note that if $f \in L^1(\mathbb{R})$ then $\mathcal{L}f$ is continuous on $i\mathbb{R}$ and tends to zero at infinity (the Riemann–Lebesgue lemma, Lemma 5.5 below). The only difference is that there is no identity element, so we add one.
The algebra $\mathbb{C}\delta_0 \oplus L^1(\mathbb{R})$ under convolution forms a commutative Banach algebra. Here δ_0 is the identity, a Dirac mass at 0. We define

$$(\lambda\delta_0 + g) * (\mu\delta_0 + h) = \lambda\mu\delta_0 + \lambda h + \mu g + g * h.$$

Here

$$(g * h)(t) = \int_{-\infty}^{\infty} g(t-s)h(s)\, ds.$$

Note that $\mathcal{L}\delta_0 = 1$, the constant function, and $\mathcal{L}(g * h) = (\mathcal{L}g)(\mathcal{L}h)$.

5 Reproducing kernel Hilbert spaces

5.1 *Definition*

A *reproducing kernel Hilbert space (RKHS) H* is a Hilbert space of functions defined on a non-empty set S such that for all $w \in S$ the evaluation mapping

$f \mapsto f(w)$ is bounded, and thus there is a function k_w (the *reproducing kernel*) such that

$$f(w) = \langle f, k_w \rangle \qquad (f \in H).$$

Note that if ℓ_w is another such reproducing kernel, then $\ell_w(s) = \langle l_w, k_s \rangle = \overline{k_s(w)}$ for all $s, w \in S$, and hence k_w is unique and $k_w(s) = \overline{k_s(w)}$.

5.2 *Theorem (Aronszajn)*

For all $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and $w_1, \ldots, w_n \in S$ we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{\alpha}_i k_{w_j}(w_i) \alpha_j \geq 0. \tag{3}$$

Moreover, if $(w, z) \mapsto k_w(z)$ is a function on $S \times S$ satisfying (3), then there is a unique RKHS with k_w as the reproducing kernel, namely the closure of all finite linear combinations of k_w for $w \in S$, with norm

$$\left\| \sum_{j=1}^{n} \alpha_j k_{w_j} \right\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{\alpha}_i k_{w_j}(w_i) \alpha_j.$$

Note that reproducing kernels do not have to be independent (for example, if every function in the space takes the same value at two distinct points).

Examples

1. The Hardy space H^2 considered as a function space on $\mathbb{D}$, with $k_w(z) = \frac{1}{1-\overline{w}z}$, the Cauchy (or Szegö) kernel.
2. The Bergman space A^2 with $k_w(z) = \frac{1}{(1-\overline{w}z)^2}$, the Bergman kernel. An analogous space can be defined on an arbitrary open subset U of $\mathbb{C}$, the norm being $\left(\int_U |f(z)|^2 \, dA(z)\right)^{1/2}$.
3. More generally, if H is a Hilbert space of power series in the disc such that

$$\left\| \sum_{n=0}^{\infty} a_n z^n \right\|_H^2 = \sum_{n=0}^{\infty} \gamma_n |a_n|^2,$$

i.e., (z^n) is an orthogonal basis with norm $\|z^n\|^2 = \gamma_n$, then $k_w(z)$ if it exists must equal $\sum_{n=0}^{\infty} \overline{w}^n z^n / \gamma_n$, and H is a RKHS if and only if $k_w \in H$, i.e., $\sum_{n=0}^{\infty} |w|^{2n} / \gamma_n < \infty$ for all $w \in \mathbb{D}$.
4. Other examples are the Hardy–Sobolev space of functions f such that $f' \in H^2$, where $\gamma_n = 1 + n^2$, for example; and the Dirichlet space of functions f such that $f' \in A^2$, where we can take $\gamma_0 = 1$ and $\gamma_n = n$ for $n > 0$.

5.3 *Remark*

In a RKHS H the subspace $Z_w = \{f \in H : f(w) = 0\}$ is just the orthogonal complement of the span of k_w, and moreover $f(w_j) = g(w_j)$ for $j = 1, \ldots, n$ if and only if $(f - g) \perp k_{w_1}, \ldots, k_{w_n}$.

5.4 *Definition*

Let $K \subset \mathbb{R}$ be compact and nonempty. Then the *Paley–Wiener space* $PW(K)$ consists of all $f \in L^2(\mathbb{R})$ such that $\hat{f}$ is supported on K, i.e.,

$$f(t) = \frac{1}{2\pi} \int_K \hat{f}(w) e^{iwt} \, dw,$$

the integral converging pointwise absolutely.

5.5 *Lemma (Riemann–Lebesgue)*

If $\hat{f} \in L^1(\mathbb{R})$, then f lies in $C_0(\mathbb{R})$, i.e., it is continuous and tends to zero at $\pm\infty$.

5.6 *Theorem*

$PW(K)$ is a RKHS of continuous functions on $\mathbb{R}$, with

$$k_s(t) = \mathcal{F}^{-1}(\chi_k(w) e^{-iwt})(s) = \frac{1}{2\pi} \int_K e^{iw(s-t)} \, dw.$$

In the special case $K = [-b, b]$, we write $PW(b)$, and the reproducing kernel is

$$k_s(t) = \frac{1}{\pi} \frac{\sin b(s-t)}{s-t} \qquad (s \neq t),$$

with $k_s(s) = b/\pi$. Note that $k_s \perp k_t$ if $k_s(t) = 0$, i.e., if $(s - t)$ is a nonzero multiple of π.

5.7 *Proposition*

If $f \in PW(b)$, then f extends to an *entire function* (i.e., analytic on the whole of $\mathbb{C}$) by

$$f(z) = \frac{1}{2\pi} \int_{-b}^{b} \hat{f}(w) e^{iwz} \, dw,$$

and

$$|f(z)| \leq \frac{1}{2\pi} \|\hat{f}\|_{L^1} e^{b|z|},$$

that is, f is of *exponential type.*

5.8 *Theorem*

Let $b > 0$. If f is an entire function such that $f_{|\mathbb{R}} \in L^2(\mathbb{R})$ and there exists $C > 0$ such that $|f(z)| \le Ce^{b|z|}$ for all $z \in \mathbb{C}$, then $f_{|\mathbb{R}} \in PW(b)$.

5.9 *Lemma*

If $F \in L^2(-b, b)$, then

$$F = \frac{\pi}{b} \sum_{n=-\infty}^{\infty} \check{F}(n\pi/b) e_n,$$

converging in L^2 norm, where

$$e_n(w) = e^{-in\pi w/b}.$$

5.10 *Corollary (Whittaker–Kotel'nikov–Shannon sampling theorem)*

If $f \in PW(b)$, then

$$f = \frac{\pi}{b} \sum_{n=-\infty}^{\infty} f(n\pi/b) k_{n\pi/b},$$

where k denotes the reproducing kernel. The series converges in $L^2(\mathbb{R})$ norm and uniformly (i.e., in $L^\infty(\mathbb{R})$ norm). That is, f can be reconstructed from samples spaced at intervals π/b.

Notes. Note that an orthonormal basis for $PW(b)$ is the set

$$\left\{ \sqrt{\frac{\pi}{b}} k_{n\pi/b} : n \in \mathbb{Z} \right\}.$$

Since $PW(b) \subset PW(c)$ for $0 < b < c$, we can also derive a formula for f in terms of $f(n\pi/c)$, $n \in \mathbb{Z}$. This is called "oversampling".

6 Interpolating sequences

6.1 *Definition*

Let $(z_k)_1^\infty \subset \mathbb{D}$. We say that (z_k) is an *interpolating sequence*, if for every bounded sequence $(w_k)_1^\infty \subset \mathbb{C}$ there is a function $f \in H^\infty$ such that $f(z_k) = w_k$ for all k. That is, the mapping $R : H^\infty \to \ell^\infty$, defined by $f \mapsto (f(z_k))$, is surjective.

6.2 *Proposition*

If (z_k) is an interpolating sequence, then:
(i) all the points are distinct;
(ii) we have $\sum_{k=1}^\infty (1 - |z_k|) < \infty$;

(iii) there exists a $\delta > 0$ such that

$$\prod_{j \neq k} \left| \frac{z_j - z_k}{1 - \overline{z}_k z_j} \right| \geq \delta > 0 \qquad \text{for all} \quad k. \qquad (C)$$

(C) is the *Carleson condition.*

6.3 *Theorem (Shapiro–Shields)*

Suppose that (z_k) satisfy (C). Then for every sequence (λ_k) in ℓ^2 there is a $g \in H^2$ with
(i) $\|g\|_2^2 \leq \frac{2}{\delta^4}(1 - 2\log\delta) \sum_{k=1}^{\infty} |\lambda_k|^2$, and
(ii) $g(z_k) = \lambda_k/(1 - |z_k|^2)^{1/2}$ for $k = 1, 2, 3, \ldots$.

We need two preliminary lemmas.

6.4 *Lemma*

Suppose that $a_{ij} = \overline{a}_{ji}$ for $i, j = 1, 2, \ldots$ and that there exists $M > 0$ such that $\sum_{j=1}^{\infty} |a_{ij}| \leq M$ for $i = 1, 2, \ldots$. Then

$$\left| \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \lambda_i \overline{\lambda}_j \right| \leq M \sum_{k=1}^{\infty} |\lambda_k|^2$$

for all sequences (λ_k) in ℓ^2.

6.5 *Lemma*

If a sequence (z_k) satisfies (C), then

$$\sum_{j=1}^{\infty} \frac{(1 - |z_j|^2)(1 - |z_k|^2)}{|1 - \overline{z}_j z_k|^2} \leq 1 - 2\log\delta$$

for $k = 1, 2, \ldots$.

6.6 *Lemma*

Let $(z_k) \subset \mathbb{D}$ and suppose that there is a $K > 0$ such that for all (λ_k) in ℓ^2 there exists $g \in H^2$ such that:
(i) $\|g\|^2 \leq K \|(\lambda_k)\|_2^2$; and
(ii) $g(z_k) = \lambda_k/(1 - |z_k|^2)$.
Then (z_k) satisfies the *Newman condition*

$$\sum_{k=1}^{\infty} |g(z_k)|^2 (1 - |z_k|^2) \leq K \|g\|_2^2 \qquad \text{for all} \quad g \in H^2, \qquad (N)$$

and hence

$$\sum_{k=1}^{\infty} |f(z_k)|(1-|z_k|^2) < \infty \quad \text{for all} \quad f \in H^1. \qquad (N')$$

Condition (N) is sometimes expressed as saying that the measure

$$\mu = \sum_{k=1}^{\infty} (1-|z_k|)^2 \delta_{z_k}$$

is a *Carleson measure*, that is, the natural embedding $H^2 \hookrightarrow L^2(\mu)$ is bounded.

6.7 *Theorem*

Let $(z_k) \subset \mathbb{D}$, and suppose that (C) and hence (N) holds. Then (z_k) is an interpolating sequence.

6.8 *Corollary*

The following are equivalent:

(i) (C) holds;
(ii) (C) and (N) hold;
(iii) (z_k) is an interpolating sequence: for all (w_k) in ℓ^∞ there is a $g \in H^\infty$ such that $g(z_k) = w_k$ for all k;
(iv) for all (λ_k) in H^2 there is a $g \in H^2$ such that $g(z_k) = \lambda_k/(1-|z_k|^2)^{1/2}$ for all k.

In case (iii) we can find an M such that $\|g\|_\infty \le M\|w\|_\infty$; in case (iv) we can find an M such that $\|g\|_2 \le M\|\lambda\|_2$.

Example. Take z_k real such that $z_k \to 1$, and $\frac{1-z_k}{1-z_{k-1}} < c < 1$ for all k, for example $z_k = 1 - 2^{-k}$.

THE END