Infinite-dimensional systems
Part 1: Operators, invariant subspaces and H^∞

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Plan of course

1. The input/output (operator theory) approach to systems.

Operators, graphs and causality.

Links with invariant subspaces, H^∞, delay systems.

2. Semigroup systems, state space.

Admissibility, controllability and observability.

Links with Hankel operators and Carleson embeddings.

Discrete-time linear systems

Informally. Systems have inputs $u(0)$, $u(1)$, $u(2), \ldots$, often vector-valued, and outputs $y(0)$, $y(1)$, $y(2), \ldots$, also vector-valued.

More formally. Look at operators T defined on an input space $\mathcal{U}$, such as $\ell^2(\mathbb{Z}_+, H)$ mapping into an output space $\mathcal{Y}$, such as $\ell^2(\mathbb{Z}_+, K)$.

Here H and K are Hilbert spaces, usually finite-dimensional in practice, say $H = \mathbb{C}^m$ and $K = \mathbb{C}^p$.

Sometimes work with SISO (single-input, single-output) systems, $m = p = 1$.

Physically we would expect inputs and outputs to be real, i.e., expect $\ell^2(\mathbb{Z}_+, \mathbb{R}^m)$ to map into $\ell^2(\mathbb{Z}_+, \mathbb{R}^p)$.

Our operators may also be unbounded, and defined on a domain $\mathcal{D}(T)$, a proper subspace of $\ell^2(\mathbb{Z}_+, \mathbb{C}^m)$.

Example.

Let $y(t) = \sum_{k=0}^t u(k)$, a discrete integrator or "summer".

Causality. If $u \in \mathcal{D}(T)$ and $u(t) = 0$ for $t \le n$, then $y(t) = 0$ for $t \le n$. The past cannot depend on the future.

Algebraically, $P_n T P_n u = P_n T u$, where

$$P_n u = (u(0), \ldots, u(n), 0, 0, \ldots).$$

The example above is causal, and has dense domain.

Causality corresponds to a lower triangular (block) matrix representation using the standard orthonormal basis of ℓ^2.

Shift invariance.

Let S be the right shift on $\mathcal{U}$, so

$$S(u_0, u_1, u_2, \ldots) = (0, u_0, u_1, \ldots).$$

We also use S for the analogous operator on $\mathcal{Y}$.

Shift-invariance: if $y = Tu$, then $Su \in \mathcal{D}(T)$, and

$Sy = T(Su)$.

Consequences.

Automatic continuity: if T is shift-invariant and $\mathcal{D}(T) = \mathcal{U}$ then T is a bounded operator, at least for $\mathcal{U} = \ell^2(\mathbb{Z}_+, \mathbb{C}^m)$.

Causality: Shift-invariant operators with $\mathcal{D}(T) = \mathcal{U}$ will also be causal (easy).

Transfer functions

Shift-invariant operators have a representation as multiplication operators (Hartman–Winter and Fourés–Segal, 1954/1955) using the theory of Hardy spaces.

We'll work with $H^2(\mathbb{D}, \mathbb{C}^m)$, analytic vector-valued functions

$$U(z) = \sum_{k=0}^{\infty} u(k) z^k,$$

with

$$\|U\|_2^2 = \sum_{k=0}^{\infty} \|u(k)\|^2 < \infty.$$

Can be regarded as power series in the disc $\mathbb{D}$, extending to give L^2 vector-valued functions on the circle $\mathbb{T}$.

Likewise $H^\infty(\mathbb{D}, \mathcal{L}(\mathbb{C}^m, \mathbb{C}^p))$, bounded analytic matrix-valued functions in $\mathbb{D}$, extending also to L^∞ functions on $\mathbb{T}$; here

$$\|G\|_\infty = \sup_{|z|<1} \|G(z)\|.$$

Using the obvious unitary equivalence between $\ell^2(\mathbb{Z}_+)$ and H^2, the shift-invariant operators become multiplications

$$Y(z) = G(z)U(z) \qquad \text{and}$$

$$\|T\| = \|G\|_\infty.$$

On $\ell^2(\mathbb{Z}_+)$ they look like convolutions

$$(Tu)(t) = \sum_{k=0}^{t} h(k)u(t-k),$$

where $h(0), h(1), \ldots$ are the Fourier coefficients of an H^∞ **transfer function**.

Finite-dimensional systems

These correspond to rational (matrix-valued) functions.

Convenient from a computational point of view.

These can be realized using finite state matrices,

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

where $x(t) \in \mathbb{C}^n$ denotes the *state* of the system.

If $x(0) = 0$, then the associated transfer function is

$$D + Cz(I - zA)^{-1}B.$$

Many infinite-dimensional systems can be realized using operators A, B, C, D, rather than matrices (see later).

Continuous-time systems

We work with operators

$$T : L^2(0, \infty; \mathbb{C}^m) \to L^2(0, \infty; \mathbb{C}^p).$$

Again notions such as causality and shift-invariance make sense.

For shift-invariance (i.e., time-invariance) we suppose that T commutes with all right shifts S_τ.

If T just commutes with S_τ for $\tau = n\tau_0$, then it is a *periodic* system, and can be handled using methods from discrete-time systems.

To translate this into function theory, use the Laplace transform

$$\mathcal{L} : L^2(0, \infty; \mathbb{C}^m) \to H^2(\mathbb{C}_+; \mathbb{C}^m),$$

$$(\mathcal{L}u)(s) = \int_0^\infty e^{-st} u(t)\, dt,$$

giving an isometry (up to a constant) between $L^2(0,\infty;\mathbb{C}^m)$ and a Hardy space of analytic vector-valued functions on the right half-plane $\mathbb{C}_+$ (Paley–Wiener).

This is a closed subspace of $L^2(i\mathbb{R};\mathbb{C}^m)$.

Again the causal, bounded, everywhere-defined, shift-invariant operators correspond to **transfer functions**, i.e., multiplication by functions in $H^\infty(\mathbb{C}_+,\mathcal{L}(\mathbb{C}^m,\mathbb{C}^p))$.

These are bounded analytic matrix-valued functions in $\mathbb{C}_+$, extending also to L^∞ functions on $i\mathbb{R}$.

Note that a shift by T in $L^2(0,\infty)$ (the **time domain**)

corresponds to a multiplication by e^{-sT} on $H^2(\mathbb{C}_+)$ (the **frequency domain**).

We may define a continuous-time linear system in state form by the equations

$$\begin{aligned}\frac{dx(t)}{dt} &= Ax(t)+Bu(t),\\ y(t) &= Cx(t)+Du(t).\end{aligned}$$

In the finite-dimensional case, these are matrices; more generally, they are operators (more details later).

The associated transfer function is

$$C(sI-A)^{-1}B+D,$$

supposed to be matrix-valued and analytic in some right half-plane.

Examples (all with zero initial conditions)

$$\frac{dy(t)}{dt}+ay(t)=u(t), \qquad G(s)=1/(s+a)$$

this is $H^\infty(\mathbb{C}_+)$ stable only if $a>0$.

$$\frac{dy(t)}{dt}+ay(t-1)=u(t), \qquad G(s)=1/(s+ae^{-s})$$

this is a delay system (Körner's shower bath) and is $H^\infty(\mathbb{C}_+)$ stable only if $0<a<\pi/2$.

Graphs and invariant subspaces.

We deal now with operators

$$T:\mathcal{D}(T)\to H^2(\mathbb{C}^p)$$

that have closed shift-invariant graphs.

Why closed? There are results which say that systems stabilizable by feedback (i.e., useful ones) will be closable.

We try not to specify whether we are in discrete or continuous time (i.e., $\mathbb{D}$ or $\mathbb{C}_+$).

Note that the graph $\mathcal{G}(T)$ is defined to be

$$\left\{ \begin{pmatrix} u \\ Tu \end{pmatrix} : u \in \mathcal{D}(T) \right\} \subset H^2(\mathbb{C}^m) \times H^2(\mathbb{C}^p) = H^2(\mathbb{C}^{m+p}).$$

We can use the Beurling–Lax theorems on shift-invariant subspaces of $H^2(\mathbb{C}^N)$ to classify the closed shift-invariant operators, by means of their graphs.

Theorem (Georgiou–Smith, 1993). Let $T : \mathcal{D}(T) \to H^2(\mathbb{C}^p)$ be closed, shift-invariant, with $\mathcal{D}(T) \subseteq H^2(\mathbb{C}^m)$. Then there exist $r \le m$, $M \in H^\infty(\mathcal{L}(\mathbb{C}^r, \mathbb{C}^m))$ nonsingular and $N \in H^\infty(\mathcal{L}(\mathbb{C}^r, \mathbb{C}^p))$ such that

$$\mathcal{G}(T) = \begin{pmatrix} M \\ N \end{pmatrix} H^2(\mathbb{C}^r) = \Theta H^2(\mathbb{C}^r),$$

where Θ is inner in the sense that

$\|\Theta u\| = \|u\|$ for all $u \in H^2(\mathbb{C}^r)$.

(If M is allowed to be singular, it's not a graph!)

What this means for SISO systems

Take $m = p = 1$. Then

$$\mathcal{G}(T) = \begin{pmatrix} M \\ N \end{pmatrix} H^2,$$

with M, $N \in H^\infty$ and $|M(z)|^2 + |N(z)|^2 = 1$ a.e. on $\mathbb{T}$ or $i\mathbb{R}$ (as appropriate).

This means that T acts as multiplication by N/M.

The domain $\mathcal{D}(T)$ is MH^2, dense provided that M is outer.

Causality can be characterized in terms of inner divisors of M and N.

Example

An unstable delay system

$$\frac{dy(t)}{dt} - y(t) = u(t-1),$$

$$G(s) = \frac{e^{-s}}{s-1}.$$

Take

$$N(s) = \frac{e^{-s}}{s+\sqrt{2}}, \qquad M(s) = \frac{s-1}{s+\sqrt{2}}.$$

This is a **normalized coprime factorization**.

We can solve a Bézout identity (cf. the corona theorem) $XN + YM = 1$ over H^∞, e.g. take

$$X(s) = e(1+\sqrt{2}), \qquad Y(s) = \frac{s + \sqrt{2} - e^{-s}X(s)}{s-1}.$$

Interlude: using the whole time-axis

We used $\ell^2(\mathbb{Z}_+)$ and $L^2(0,\infty)$ as our input/output spaces.

Why not use $\ell^2(\mathbb{Z})$ and $L^2(\mathbb{R})$?

We can use Fourier/Laplace transforms and work, equivalently, with operators on $L^2(\mathbb{T})$ or $L^2(i\mathbb{R})$.

Beurling–Lax theorems on shift-invariant subspaces are replaced by Wiener theorems.

Typical theorem (the discrete case)

If $T : \mathcal{D}(T) \to L^2(\mathbb{T})$ is a closed shift-invariant system with $\mathcal{D}(T) \subseteq L^2(\mathbb{T})$ then

$$\mathcal{G}(T) = \begin{pmatrix} M \\ N \end{pmatrix} L^2(\mathbb{T}),$$

where M, $N \in L^\infty(\mathbb{T})$ and $|M(e^{i\omega})|^2 + |N(e^{i\omega})|^2 \in \{0,1\}$ a.e. Also $|N(e^{i\omega})| \neq 1$ a.e.

Thus again T "is" multiplication by N/M (even though $L^\infty(\mathbb{T})$ is not an integral domain so we need to be careful!)

Multi-input multi-output (MIMO) case similar, but requires measurable projection-valued functions (omitted here).

Transfer functions. Again we can interpret N/M as a transfer function. In the full-axis case it may only make sense on the circle, however.

Digression: how to upset an engineer.

Physically this seems rather implausible, but the transfer function

$$G(z) = \exp(-((1-z)/(1+z))^2)$$

defines a linear system with nontrivial domain in $\ell^2(\mathbb{Z})$ but trivial domain in $\ell^2(\mathbb{Z}_+)$.

Reason: $G = N/M$ with N and M in $L^\infty(\mathbb{T})$, but not if we want $M \in H^\infty$.

Causality.

Unlike in the $\mathbb{Z}_+$ case, shift-invariant operators defined on $\ell^2(\mathbb{Z})$ are not automatically causal (e.g. the backward shift!)

To characterize causality, we may assume that $\mathcal{D}(T)$ actually contains some inputs u with $u(k) = 0$ for $k < 0$.

If so, then T (with closed graph) is causal if and only if N/M lies in the **Smirnoff class** of analytic functions on the disc.

The Smirnoff class is the set of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ which can be written as $f = f_1/f_2$ with both functions in H^∞ and f_2 outer.

This is bigger than H^∞, for example $1/(z-1)$ is in the class.

Closability.

Recall that we need closed operators to be able to handle feedback stability.

Theorem Causal convolution operators

$$(Tu)(t) = \sum_{n=0}^{\infty} g(n)u(t-n)$$

on $\ell^2(\mathbb{Z})$ are closable whenever their domain of definition is dense; this happens, for example, if we can find some nonzero element of $\ell^2(\mathbb{Z}_+)$ that is in $\mathcal{D}(T)$.

The bad news is contained in the *Georgiou–Smith paradox* of 1995.

Here is a discrete-time version of their example.

Define the operator T on $\ell^2(\mathbb{Z})$ by

$$(Tu)(t) = \sum_{n=0}^{\infty} 2^n u(t-n) = u(t) + 2u(t-1) + 4u(t-2) + \dots.$$

This makes sense with

$$\mathcal{D}(T) = \{u \in \ell^2(\mathbb{Z}) : Tu \in \ell^2(\mathbb{Z})\}.$$

Then T is causal, and it will be closable, by the above theorem, since the sequence $u = (\dots, 0, 0, 1, -2, 0, 0, \dots)$ is in $\mathcal{D}(T)$. Indeed $Tu = (\dots, 0, 0, 1, 0, 0, \dots)$.

HOWEVER, the extended system is no longer causal.

Let $u_n = 2^{-n}e_{-n} - e_0$. This means that

$$u_n(t) = \begin{cases} 2^{-n} & \text{for } t = -n, \\ -1 & \text{for } t = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$u_n = (\dots, 2^{-n}, 0, 0, \dots, 0, 0, -1, 0, 0, \dots).$$

Now

$$Tu_n = \sum_{j=-n}^{-1} 2^j u_j = (\ldots, 0, 0, 2^{-n}, \ldots, 2^{-2}, 2^{-1}, 0, 0, \ldots).$$

In the limit, $n \to \infty$, we have a non-casual extension:

$$u_n \to -e_0 \qquad \text{and} \qquad y_n \to \sum_{j=-\infty}^{-1} 2^j e_j.$$

When can this paradox occur?

Answer: if

$$(Tu)(t) = \sum_{n=0}^{\infty} g(n)u(t-n)$$

defines a causal convolution system on $\ell^2(\mathbb{Z})$ whose domain contains nonzero sequences in $\ell^2(\mathbb{Z}_+)$, then the closure of P is also causal if and only if

$$G(z) = \sum_{n=0}^{\infty} g(n)z^n \qquad \text{lies in the Smirnoff class.}$$

In our case $G(z) = 1/(1-2z)$, which is not Smirnoff.

Contrast the example

$$G(z) = 1/(1-z)^2, \qquad \text{with } g(n) = n+1, \quad n \in \mathbb{Z}_+,$$

which has causal closure since G is Smirnoff.

Related results and open problems.

If we work with the space $s(\mathbb{Z})$ of all sequences, we can still characterise all the systems with closed graphs – they turn out to have finite-degree.

How about closed graphs using the *extended signal space*, $\ell^2_e(\mathbb{Z})$, sequences $(u(n))_{n\in\mathbb{Z}}$ such that

$$\sum_{-\infty}^{-1} |u(n)|^2 < \infty,$$

i.e., ℓ^2 in the past, arbitrary in the future?

How about L^2_{loc}, functions which are square-integrable on any finite interval? Here the continuous-time is genuinely harder.

Feedback control

Return to the case of $L^2(0,\infty)$ or $\ell^2(\mathbb{Z}_+)$.

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} (I-KP)^{-1} & K(I-PK)^{-1} \\ P(I-KP)^{-1} & (I-PK)^{-1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Internal stability means that this mapping has a bounded extension to $H^2(\mathbb{C}^{m+p})$.

In our situation, this means that each of the four entries lies in H^∞.

We say that the "plant" P is **stabilized** by the "controller" K.

Assuming that everything lies in the field of fractions of H^∞, then stabilizability implies

* plant and controller have coprime factorizations;

* their graphs are closable.

We suppose from now on that we have closed graphs.

Beautiful identity

K stabilizes P if and only if their operator graphs satisfy

$$\mathcal{G}(P) \oplus \mathcal{G}'(K) = H^2(\mathbb{C}^{m+p}),$$

where $\mathcal{G}'$ denotes the transpose of the graph, i.e., vectors $\begin{pmatrix} Ky \\ y \end{pmatrix}$.

Youla parametrization

A **doubly coprime factorization** of a matrix-valued function P with entries in the field of fractions of H^∞ is an identity

$$P = \widetilde{M}^{-1}\widetilde{N} = NM^{-1},$$

where we have the Bézout identities

$$\begin{aligned} \begin{pmatrix} \widetilde{X} & -\widetilde{Y} \\ -\widetilde{N} & \widetilde{M} \end{pmatrix} \begin{pmatrix} M & Y \\ N & X \end{pmatrix} &= \begin{pmatrix} M & Y \\ N & X \end{pmatrix} \begin{pmatrix} \widetilde{X} & -\widetilde{Y} \\ -\widetilde{N} & \widetilde{M} \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \end{aligned}$$

We can then parametrize all the K stabilizing P by right coprime factorizations

$$K = (Y + MQ)(X + NQ)^{-1},$$

and by left coprime factorizations

$$K = (\widetilde{X} + Q\widetilde{N})^{-1}(\widetilde{Y} + Q\widetilde{M}),$$

where in each case Q is an arbitrary matrix of the appropriate size with entries in H^∞.

The gap metric

The gap metric between closed subspaces of a Hilbert space is given by

$$\delta(V, W) = \|P_V - P_W\|,$$

where P denotes the orthogonal projection.

The gap metric for closed operators with domains contained in the same Hilbert space is

$$\delta(A, B) = \delta(\mathcal{G}(A), \mathcal{G}(B)),$$

i.e., the gap between their graphs.

For bounded operators this metric gives the norm topology.

Returning to linear systems, suppose that

$\mathcal{G}(P_1) = \Theta_1 H^2(\mathbb{C}^r)$ and $\mathcal{G}(P_2) = \Theta_2 H^2(\mathbb{C}^r)$.

Then

$$\delta(P_1, P_2) = \max\left\{\inf_{Q\in H^\infty} \|\Theta_1 - \Theta_2 Q\|_\infty, \inf_{Q\in H^\infty} \|\Theta_2 - \Theta_1 Q\|_\infty\right\}.$$

This is a typical H^∞ optimization problem.

We can also talk about "best" controllers. Suppose that K stabilizes P. How big a neighbourhood of P (in the gap metric) does it stabilize?

Let's restrict to the SISO case for simplicity.

Answer: if $P = NM^{-1}$ and

$$K = VU^{-1} = (Y + MQ)(X + NQ)^{-1}$$

à la Youla, then the stability radius is b where

$$\left\|\begin{pmatrix} Y + MQ \\ X + NQ \end{pmatrix}\right\|_\infty = 1/b.$$

Thus we obtain another H^∞ optimization problem, to find

$$\inf_{Q\in H^\infty} \left\|\begin{pmatrix} Y \\ X \end{pmatrix} + \begin{pmatrix} M \\ N \end{pmatrix} Q\right\|_\infty$$

This equals

$$(1 + \|\Gamma_R\|^2)^{1/2}$$

where $\Gamma_R : H^2(\mathbb{C}_+) \to H^2(\mathbb{C}_-)$ is the Hankel operator given by

$$u \mapsto P_{H^2(\mathbb{C}_-)}(R.u),$$

for $u \in H^2(\mathbb{C}_+)$, with

$$R = M^*Y + N^*X \in L^\infty(i\mathbb{R}).$$

Other H^∞ optimization problems

*** Model matching**

Given P choose a stabilizing K to minimize

$$\|W_1(P(I-KP)^{-1} - R)W_2\|_\infty.$$

Here W_1, $W_2 \in H^\infty$ are (optional) frequency-weighting functions, e.g. $W(s) = 1/(s+1)$ in the scalar continuous-time case.

Also $R \in H^\infty$ is the desired "reference function".

*** Sensitivity minimization**

Again, given P (and W_1, W_2) choose K to minimize

$$\|W_1(I-PK)^{-1}W_2\|_\infty.$$

Such problems can generally be reduced, via Youla, to problems of the form

$$\inf\{\|T_1 - T_2QT_3\|_\infty : \; Q \in H^\infty\},$$

where T_1, T_2, T_3 are in H^∞ and matrix-valued.

Many methods of solution now available.

Shameless advertisement

BOOK: Jonathan R. Partington, Linear Operators and Linear Systems, LMS Student Texts, Cambridge University Press, 2004.

Notes for the course (free)

See my web site (linked from Network's home page) and follow the link from *Teaching.*

Infinite-dimensional systems
Part 2: Semigroups, Carleson measures, Hankel operators

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Linear systems associated with semigroups

H a complex Hilbert space, $(T_t)_{t\geq 0}$ a strongly continuous semigroup of bounded operators,

i.e., $T_{t+u} = T_t T_u$ and $t \mapsto T_t x$ is continuous.

A the infinitesimal generator, defined on domain $\mathcal{D}(A)$.

$$Ax = \lim_{t\to 0} \frac{1}{t}(T_t - I)x.$$

A continuous-time linear system in state form:

$$\begin{aligned}\frac{dx(t)}{dt} &= Ax(t) + Bu(t),\\ y(t) &= Cx(t) + Du(t),\end{aligned}$$

with $x(0) = x_0$, say.

Often we take $D = 0$. Sometimes B and C (the control and observation operators) are bounded.

Note that

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0$$

has mild solution

$$x(t) = T_t x_0.$$

Examples

1. A **delay equation** (matrix-valued).

$$\frac{dx(t)}{dt} = A_0 x(t) + \sum_{j=1}^{p} A_j x(t - jh), \qquad t \geq 0.$$

With $x(0) = x_0$ and $x(t) = f_0(t)$ for $-ph \leq t < 0$.

Define the Hilbert space

$$H = \mathbb{C}^n \oplus L^2([-ph, 0]; \mathbb{C}^n),$$

and

$$A \begin{pmatrix} w \\ f \end{pmatrix} = \begin{pmatrix} A_0 w + \sum_{j=1}^{p} A_j f(-jh) \\ f' \end{pmatrix},$$

with domain

$$\mathcal{D}(A) = \{ \begin{pmatrix} w \\ f \end{pmatrix} \in H : f \text{ abs. cont.}, \\ f' \in L^2([-ph, 0]; \mathbb{C}^n),\ f(0) = w \}.$$

This delay equation is equivalent to

$$\frac{dz}{dt} = Az, \qquad \text{with} \qquad z_0 = \begin{pmatrix} x_0 \\ f_0 \end{pmatrix}.$$

2. **PDE example** (equation of an undamped beam)

$$\frac{\partial^2 y}{\partial t^2} = -\frac{\partial^4 y}{\partial x^4}, \qquad 0 \le x \le 1, \quad t \ge 0,$$

with initial conditions on the position and velocity,

$$y(x, 0) = y_1(x) \qquad \text{and} \quad y_t(x, 0) = y_2(x),$$

given, and boundary conditions

$$y(0, t) = y(1, t) = y_{xx}(0, t) = y_{xx}(1, t) = 0,$$

i.e., the beam is fixed at the endpoints.

Let

$$B = -\frac{d^2}{dx^2}$$

with domain

$$\mathcal{D}(B) = \{ z \in L^2(0, 1) : z, \frac{dz}{dx} \text{ abs. cont.}, \\ \frac{d^2 z}{dx^2} \in L^2(0, 1), z(0) = z(1) = 0 \}.$$

We can rewrite the equation as

$$\frac{dz}{dt} = Az,$$

with

$$z = \begin{pmatrix} y \\ \frac{dy}{dt} \end{pmatrix}, \qquad \text{and} \qquad A = \begin{pmatrix} 0 & I \\ -B^2 & 0 \end{pmatrix},$$

where z lies in $\mathcal{D}(A)$, a subspace of the Hilbert space

$$H = \mathcal{D}(B) \oplus L^2(0,1),$$

equipped with the norm

$$\|(z_1, z_2)\|^2 = \|Bz\|^2 + \|z_2\|^2.$$

(Infinite-time) admissibility

There is a **duality** here between control and observation.

Admissibility of control operators.

Consider

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t), \qquad x(0) = x_0, \quad t \geq 0,$$

where $u(t) \in \mathcal{U}$ is the input at time t,

$\mathcal{U}$ is a separable Hilbert space, and

$B : \mathcal{D}(B) \to H$ is usually unbounded, with $\mathcal{D}(B) \subseteq \mathcal{U}$.

How can we guarantee that the state $x(t)$ lies in H?

Sufficient that $B \in \mathcal{L}(\mathcal{U}, \mathcal{D}(A^*)')$ and $\exists m_0 > 0$ such that

$$\left\| \int_0^\infty T_t B u(t)\, dt \right\|_H \leq m_0 \|u\|_{L^2(0,\infty;\mathcal{U})}$$

(the admissibility condition for B).

Admissibility of observation operators.

Consider

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t), \\ y(t) &= Cx(t), \end{aligned}$$

with $x(0) = x_0$, say.

Let $C : \mathcal{D}(A) \to \mathcal{Y}$, Hilbert, be an A-bounded 'observation operator', i.e.,

$$\|Cz\| \leq m_1 \|z\| + m_2 \|Az\|$$

for some m_1, $m_2 > 0$.

C is **admissible**, if $\exists m_0 > 0$ such that $y \in L^2(0,\infty;\mathcal{Y})$ and

$$\|y\|_2 \leq m_0 \|x_0\|.$$

Note $y(t) = CT_t x_0$.

The duality

B is an admissible control operator for $(T_t)_{t\geq 0}$ if and only if B^* is an admissible observation operator for the dual semigroup $(T_t^*)_{t\geq 0}$.

The Weiss conjecture

Suppose C admissible, take Laplace transforms,

$$\begin{aligned}\hat{y}(s) &= \int_0^\infty e^{-st} y(t)\,dt,\\ &= C(sI - A)^{-1} x_0.\end{aligned}$$

Now if $y \in L^2(0,\infty;\mathcal{Y})$, then $\hat{y} \in H^2(\mathbb{C}_+,\mathcal{Y})$, Hardy space on RHP, and

$$\|\hat{y}(s)\| = \left\| \int_0^\infty e^{-st} y(t)\,dt \right\| \leq \frac{\|y\|_2}{\sqrt{2\operatorname{Re} s}},$$

by Cauchy–Schwarz.

Thus admissibility, i.e.,

$$\|CT_t x_0\|_{L^2(0,\infty;\mathcal{Y})} \leq m_0 \|x_0\|,$$

implies the **resolvent condition**: $\exists m_1 > 0$ such that

$$\|C(sI - A)^{-1}\| \leq \frac{m_1}{\sqrt{\operatorname{Re} s}}, \qquad \forall s \in \mathbb{C}_+.$$

George Weiss (1991) conjectured that the two conditions are equivalent.

This would imply several big theorems in function theory in an elementary way.

Easy examples show it could only be valid on Hilbert spaces.

1. The case $\dim \mathcal{Y} < \infty$**.**

Weiss proved it for normal semigroups and right-invertible semigroups.

A decade later, other special cases were considered.

JRP–Weiss (2000). The right-shift semigroup.

Jacob–JRP (2001). All contraction semigroups (see later).

Le Merdy (2003). Bounded analytic semigroups.

Jacob–Zwart (to appear, 2004). Not true for all semigroups. They won \$ 100.

Example 1

$$H = L^2(\mathbb{C}_+, \mu).$$

$$(T_t(x))(\lambda) = e^{-\lambda t} x(\lambda).$$

$$(Ax)(\lambda) = -\lambda x(\lambda).$$

For which Borel measures μ on $\mathbb{C}_+$ does C defined by

$$Cf = \int_{\mathbb{C}_+} f(\lambda)\, d\mu(\lambda)$$

satisfy the resolvent condition?

Answer: if and only if

$$\int_{\mathbb{C}_+} \frac{d\mu(\lambda)}{|s+\lambda|^2} \le \frac{M}{\operatorname{Re} s} \quad \forall s \in \mathbb{C}_+.$$

This actually means that μ is a Carleson measure: μ-measure of square $[0, 2h] \times [a - h, a + h]$ is $O(h)$.

So when is C admissible?

Precisely when there is a continuous embedding

$$H^2(\mathbb{C}_+) \to L^2(\mathbb{C}_+, \mu).$$

This equivalence is the **Carleson–Vinogradov embedding theorem**: if

$$\|1/(s+\lambda)\|_{L^2(\mathbb{C}_+,\mu)} \le M \|1/(s+\lambda)\|_{H^2},$$

for each $\lambda \in \mathbb{C}_+$, then a similar inequality holds for all H^2 functions.

Thus the Weiss conjecture for the above semigroup is equivalent to the above embedding theorem (Weiss).

Example 2

Take the right shift semigroup on $H = L^2(0, \infty)$:

$$(T_t x)(\tau) = x(\tau - t), \qquad \tau \ge t.$$

Equivalently,

$$H = H^2(\mathbb{C}_+).$$

$$(T_t(x))(\lambda) = e^{-\lambda t} x(\lambda).$$

$$(Ax)(\lambda) = -\lambda x(\lambda).$$

Now $C : \mathcal{D}(A) \to \mathbb{C}$ is A-bounded iff it has the form

$$Cx = \int_{-\infty}^{\infty} \overline{c(i\omega)} x(i\omega)\, d\omega,$$

where $c(z)/(1+z) \in H^2(\mathbb{C}_+)$ (easy).

Then C is admissible, i.e.,

$$\int_0^\infty |CT_t x_0|^2\, dt \le m_0 \|x_0\|^2,$$

if and only if the following Hankel operator is bounded:

$$\Gamma_c : H^2(\mathbb{C}_-) \to H^2(\mathbb{C}_+), \qquad \Gamma_c u = \Pi_+(c.u),$$

where Π_+ is the orthogonal projection from

$$L^2(i\mathbb{R}) = H^2(\mathbb{C}_+) \oplus H^2(\mathbb{C}_-)$$

onto $H^2(\mathbb{C}_+)$.

So $c(z) \in BMOA$, Bounded Mean Oscillation Analytic.

Resolvent Condition equivalent to

$$\left\| \Gamma_c \left(\frac{1}{s-a} \right) \right\| \le m' \|1/(s-a)\| \qquad \forall a \in \mathbb{C}_+.$$

Bonsall (1984): same idea. Hankel operator Γ_c bounded if and only if bounded on normalized rationals of degree 1 (reproducing kernels).

Hence the Weiss conjecture for the shift semigroup is equivalent to Bonsall's theorem (which requires Fefferman's theorem, etc.)

General contraction semigroups

Use Sz.-Nagy–Foias model theory for contraction semigroups.

A contraction semigroup T_t is equivalent to

$$T'_{t\,|H_1} \oplus T''_{t}{}_{|H_2},$$

where T'_t is a unitary semigroup on H_1, and T''_t is completely non-unitary on H_2.

Result for T'_t equivalent to Carleson embedding theorem.

For T''_t, equivalent to the following model:

E, F Hilbert spaces,

$\Theta : \mathbb{C}_+ \to \mathcal{L}(E, F)$, holomorphic, such that

$$\|\Theta(s)\|_\infty \leq 1.$$

$$X = H^2(\mathbb{C}_+, F) \oplus \overline{\Delta L^2(i\mathbb{R}, E)},$$

where

$$\Delta(i\omega) = [I_E - \Theta(i\omega)^* \Theta(i\omega)]^{1/2},$$

for $\omega \in \mathbb{R}$, and

$$U = X \ominus \{(\Theta f, \Delta f) : f \in H^2(\mathbb{C}_+, E)\},$$

$$T_t u = P_U[e^{-i\omega t} u(i\omega)].$$

An earlier example: $E = 0$, $F = \mathbb{C}$, $\Theta = 0$, $\Delta = 1$, right shift on $L^2(0, \infty)$.

Another interesting one is $E = F = \mathbb{C}$, $\Theta(s) = e^{-s}$,

$U = H^2 \ominus \Theta H^2$, gives right shift on $L^2(0, 1)$.

Proof of Weiss conjecture for contraction semigroups uses results for unitary semigroups, a stronger form of the result for the shift, and some auxiliary results.

Combining all the 'smaller' theorems gives the big theorem, from which the smaller theorems can be deduced.

2. The case $\dim \mathcal{Y} = \infty$.

Jacob–JRP–Pott (2002) (using results of Gillespie–Pott–Treil–Volberg).

Bonsall's theorem fails for Hankel operators on $H^2(\mathbb{C}_+, \mathcal{Y})$.

This in turn provides a counterexample to the 'extended' Weiss conjecture—for C an operator of infinite rank—using the shift semigroup on $L^2(0, \infty)$.

Other counterexamples for analytic semigroups, Jacob–Staffans–Zwart also date from IWOTA 2000 in Bordeaux.

Jacob–JRP–Pott (2003). Stronger resolvent condition with Hilbert–Schmidt norm:

$$\|C(sI - A)^{-1}\|_{HS} \leq \frac{m_1}{\sqrt{\operatorname{Re} s}}, \qquad \forall s \in \mathbb{C}_+.$$

For shift on $L^2(0, \infty; K)$ and C mapping into $\mathcal{Y}$, this is NSC for admissibility if at least one of K and $\mathcal{Y}$ is finite-dimensional.

HS condition is sufficient for admissibility even if both K and $\mathcal{Y}$ are infinite-dimensional.

Links with conditions for boundedness of vector-valued Hankel operators.

Indeed, HS condition is sufficient for ANY contraction semigroup, and ANY $\mathcal{Y}$.

Proof uses dilation to an isometric semigroup, and the Sz.-Nagy–Foias model theory again.

Controllability

Assume an exponentially stable semigroup $(T_t)_{t\geq 0}$, i.e.,

$$\|T_t\| \leq Me^{-\lambda t}, \qquad (t \geq 0),$$

for some $M > 0$ and $\lambda > 0$.

Look at the equation

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t),$$

with solution

$$x(t) = T_t x_0 + \int_0^t T_{t-s} Bu(s)\, ds,$$

suitably interpreted.

Let's assume B admissible (an easier case to describe).

Then we have a bounded operator $\mathcal{B}_\infty : L^2(0,\infty;\mathcal{U}) \to H$, defined by

$$\mathcal{B}_\infty u = \int_0^\infty T_t Bu(t)\, dt.$$

The system is **exactly controllable**, if $\operatorname{Im} \mathcal{B}_\infty = H$, i.e., we can steer the system where we like, using the input u.

Alternatively, it is **approximately controllable**, if $\operatorname{Im} \mathcal{B}_\infty$ is dense.

There are dual notions of exact and approximate observability (omitted).

Controllability involves more links with the theory of interpolation, as follows.

Diagonal semigroups

An important special case where most things are known (includes some heat equations, vibrating structures, etc.)

Suppose that

$$A\phi_n = \lambda_n \phi_n,$$

with (ϕ_n) normalized eigenvectors forming a Riesz basis.

Let (ψ_n) be the dual basis.

So every $x \in H$ can be written

$$x = \sum_{n=1}^{\infty} \langle x, \psi_n \rangle \phi_n$$

with

$$K_1 \sum |\langle x, \psi_n \rangle|^2 \leq \|x\|^2 \leq K_2 \sum |\langle x, \psi_n \rangle|^2.$$

Note that

$$T_t \sum_{n=1}^{\infty} c_n \phi_n = \sum_{n=1}^{\infty} c_n e^{\lambda_n t} \phi_n.$$

By exponential stability $\sup_n \operatorname{Re} \lambda_n < 0$.

We look at the case $\dim \mathcal{U} = 1$: interesting (and simpler).

Finite-dimensional $\mathcal{U}$ can be handled similarly.

If $\dim \mathcal{U} = 1$ then $B : \mathbb{C} \to H$ so is a vector,

$$b = \sum_{n=1}^{\infty} b_n \phi_n.$$

Let's calculate $\mathcal{B}_\infty$:

$$\mathcal{B}_\infty u = \int_0^\infty T_t B u(t)\, dt, \qquad \text{so}$$

$$\mathcal{B}_\infty u = \sum_{n=1}^{\infty} b_n \int_0^\infty e^{\lambda_n t} u(t)\, dt\ \phi_n.$$

This is just

$$\sum_{n=1}^{\infty} b_n \hat{u}(-\lambda_n) \phi_n.$$

(Laplace Transform!)

Since $\mathcal{L} : L^2(0,\infty) \to H^2(\mathbb{C}_+)$ is an isomorphism,

exact controllability is equivalent to:

for every $(c_n) \in \ell^2$ there is a function $g \in H^2(\mathbb{C}_+)$ such that

$$b_n g(-\lambda_n) = c_n.$$

This brings us back directly to Carleson interpolation problems (exact result needed due to McPhail (1990)).

The NSC is that

$$\nu = \sum_{n=1}^{\infty} \frac{|\mathrm{Re}\,\lambda_n|^2}{|b_n|^2} \prod_{k\neq n} \frac{|\overline{\lambda_n}+\lambda_k|^2}{|\lambda_k-\lambda_n|^2}\delta_{-\lambda_n}$$

is a Carleson measure on $\mathbb{C}_+$,

i.e., ν-measure of square $[0,2h]\times[a-h,a+h]$ is $O(h)$.

Here δ_λ denotes a Dirac (point mass) at λ.

We don't need $(-\lambda_n)$ to be a Carleson sequence, but it does need to be a Blaschke sequence.

However $\lambda_n = -n^\beta$ with $0<\beta\leq 1$ is never exactly controllable (in the finite-dimensional case).

Approximate controllability (dense range) is much easier and just requires $b_n \neq 0$ for all n and distinct eigenvalues (λ_n).

An intermediate concept is **null controllability**. We just require $\mathrm{Im}\,\mathcal{B}_\infty$ to contain $T_{t_1}H$ for some $t_1 \geq 0$.

The NSC now becomes that

$$\nu = \sum_{n=1}^{\infty} \frac{|\mathrm{Re}\,\lambda_n|^2}{|b_n|^2} e^{2t_1\,\mathrm{Re}\,\lambda_n} \prod_{k\neq n} \frac{|\overline{\lambda_n}+\lambda_k|^2}{|\lambda_k-\lambda_n|^2}\delta_{-\lambda_n}$$

is a Carleson measure on $\mathbb{C}_+$.

An open problem

For finite-dimensional systems exact/approximate controllability coincide.

The **Popov–Hautus test** for exact controllability asserts that a system

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t),$$

with $A\in\mathbb{C}^{n\times n}$ and $B\in\mathbb{C}^{n\times m}$ is exactly controllable if and only if

$$\mathrm{rank}\,[\, sI-A \quad B\,] = n \qquad \text{for all } s\in\mathbb{C}.$$

Infinite-dimensional **conjecture**

(Russell, Weiss, Jacob, Zwart).

$(T_t)_{t\geq 0}$ exponentially stable on a Hilbert space H.

B an admissible control operator.

Then the system is exactly controllable if and only if

(C1) $(T(t))_{t\geq 0}$ is similar to a contraction semigroup;

(C2) there exists $m>0$ such that

$$\|(sI - A^*)x\|^2 + |\operatorname{Re} s| \, \|B^*x\|^2 \geq m|\operatorname{Re} s|^2 \, \|x\|^2$$

for all $x \in D(A^*)$, $s \in \mathbb{C}_-$.

Without condition (C1) the conjecture fails to hold.

The conjecture holds for diagonal systems and finite-rank B.

In the rank-1 case (with admissibility), it is equivalent to

$$\inf_{n \in \mathbb{N}} \frac{|b_n|^2}{|\operatorname{Re} \lambda_n|} > 0 \qquad \text{and} \qquad \inf_{k,n \in \mathbb{N}, k \neq n} \frac{|\lambda_k - \lambda_n|}{|\operatorname{Re} \lambda_n|} > 0.$$

We'd like to know the answer for an exponentially stable semigroup such as the damped shift on $L^2(0, \infty)$, i.e.,

$$(T_t f)(\tau) = e^{-at} f(\tau - t), \qquad (t \geq 0)$$

for some fixed $a > 0$.

The answer would tell us more about Hankel operators (just as admissibility is the same as a boundedness condition).

Volterra systems and composition operators

$$\begin{aligned} x(t) &= x_0 + \int_0^t a(t-s) A x(s) \, ds, \quad t \geq 0, \\ y(t) &= Cx(t), \end{aligned}$$

where $a \in L^1_{loc}(0, \infty) \backslash \{0\}$ has a Laplace transform in some right half-plane.

Note that for the choice $a(t) \equiv 1$ we obtain the Cauchy problem

$$\begin{aligned} \dot{x}(t) &= Ax(t), \quad x(0) = x_0, \\ y(t) &= Cx(t), \quad t \geq 0. \end{aligned}$$

If we write

$$x(t) = S(t) x_0, \quad t \geq 0,$$

then $\|S(t)\|$ can grow faster than exponentially, unlike in the semigroup case.

Suppose instead *la vie est belle* and

$$\|S(t)\| \leq M e^{\omega t}, \quad t \geq 0.$$

Then

$$H(s) x_0 := \hat{S}(s) x_0 = \frac{1}{s} (I - \hat{a}(s) A)^{-1} x_0, \qquad \operatorname{Re} s > \omega.$$

In fact

$$H(s) = \frac{1}{s\hat{a}(s)} \left(\frac{1}{\hat{a}(s)} I - A \right)^{-1}, \qquad \operatorname{Re} s > \omega.$$

By Paley–Wiener, C is admissible if and only if there exists a constant $M > 0$ such that

$$\|CH(\cdot)x_0\|_{H^2(\mathbb{C}_+,Y)} \leq M\|x_0\|, \qquad x_0 \in D(A).$$

If we know that C is admissible for the standard Cauchy problem $a \equiv 1$, then we deduce admissibility for the Volterra system whenever the following weighted composition operator on $H^2(\mathbb{C}_+)$ is bounded:

$$(QF)(s) = \frac{1}{s\hat{a}(s)} F\left(\frac{1}{\hat{a}(s)}\right).$$

A new Weiss-type conjecture: is it enough to check a resolvent-type condition in this case?