Analysis of Radical Banach Algebras

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Radical Banach algebras

Definition

A commutative Banach algebra $\mathcal{R}$ is *radical* if it satisfies any one of the following equivalent conditions:

- every homomorphism $\phi : \mathcal{R} \to \mathbb{C}$ is zero;
- $\lambda \mathbf{1} - a$ is invertible in $\mathcal{R}^\sharp$ for every $a \in \mathcal{R}$ and $\lambda \in \mathbb{C} \setminus \{0\}$;
- the only maximal proper ideal in $\mathcal{R}^\sharp$ is $\mathcal{R}$ itself; and
- $\lim_{n\to\infty} \|a^n\|^{1/n} = 0$ for every $a \in \mathcal{R}$.

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Radical Banach Algebras and Automatic Continuity, Proceedings of conference at Long Beach 1981. Springer Lecture Notes **975**.

- Let $w : \mathbb{N} \to \mathbb{R}^+$ be a *submultiplicative radical weight*, *i.e.*, $w_{m+n} \leq w_m w_n$ and $\lim_{n\to\infty} w_n^{1/n} = 0$. Then

  $$\ell^1(\mathbb{N}, w) = \left\{ f = (f_n) \in \mathbb{C}^{\mathbb{N}} : \|f\| = \sum_{n=1}^{\infty} |f_n| w_n < \infty \right\}$$

  is a radical Banach algebra under the convolution product

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- The *Volterra algebra* is

  $V = L^1(0, 1)$ with the product $(f * g)(x) = \int_0^x f(x-t) g(t) \, dt$.

A Question

H.G. Dales: Suppose that $w = (w_n)$ is a radical weight.
Is there $f \in L^1(0,1)$ such that $w_n = \|f^{*n}\|_1$ for every $n \in \mathbb{N}$?

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Proposition

The set of compact elements of $\ell^1(\mathbb{N}, w)$ is the *standard ideal*

$$I_k = \{f \in \ell^1(\mathbb{N}, w) : f_n = 0 \text{ if } n < k\},$$

where k is the smallest number at which w is regulated.

Answers

Proposition (W.)

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Both arguments use algebraic properties of V to establish an analytic statement about $w = (\|f^{*n}\|_1)$.

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Proposition

Let $\mathcal{A}$ be a Banach algebra in which multiplication by every $a \in \mathcal{A}$ is compact and such that there is $s \in \mathcal{A}$ with $\mathbb{C}[s]^{-} = \mathcal{A}$ and a subsequence of $\left(\frac{s^n}{\|s^n\|}\right)$ is a b.a.i. for $\mathcal{A}$.
Then $\mathcal{A}$ is either isomorphic to $\mathbb{C}^n$ for some n or is radical.

A construction

Idea for construction: define a norm on $s\mathbb{C}[s]$ such that

- $\|\cdot\|$ is an algebra norm for polynomial multiplication;
- multiplication by a is compact for every a in the completion; and
- $\left\| s^{n_k+1}/\|s^{n_k}\| - s \right\| \to 0$ as $k \to \infty$ for some increasing sequence (n_k) of integers.

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More details: choose a basis $(p_i(s))_{i=1}^{\infty}$ for $s\mathbb{C}[s]$ and, defining $\|\cdot\|$ to be the ℓ^1-norm for this basis, arrange that

- $\|p_i(s)p_j(s)\| \leq 1$ for all $i, j \geq 1$;
- there are numbers $\lambda_i \in [0,1]$ such that $\|sp_i(s) - \lambda_i s\| \to 0$ as $i \to \infty$; and
- there is an increasing sequence (n_k) of integers such that $\left\| s^{n_k+1}/\|s^{n_k}\| - s \right\| \to 0$ as $k \to \infty$.

Further properties of $\mathcal{R}$

Charles used rapidly increasing sequences to construct the example, which was included in a joint paper with R. Loy and V. Runde.
In the same paper it is shown that the algebra $\mathcal{R}$ constructed is

- weakly amenable and
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In a subsequent paper, joint with F. Ghahramani, it was shown that

- the lattice of closed ideals in $\mathcal{R}$ is isomorphic to $[0, 1]$.

More rapidly increasing sequences are introduced into the construction.

The idea is to approximate the nilpotent finite shift on $\mathbb{C}^n$ within $\mathcal{R}$ and use that the invariant subspaces of this shift are linearly ordered. Operators whose invariant subspaces are linearly ordered are called *unicellular*.