In memoriam Charles John Read – mathematician, gentleman, and friend

Approximate amenability of tensor products of Banach algebras

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Outline

Basic reminders from Fereidoun's talk

Throughout A will be a Banach algebra, X a Banach A-bimodule.

Definition

- A *derivation* $D : A \to X$ is a linear map such that

$$D(ab) = D(a) \cdot b + a \cdot D(b) \qquad (a, b \in A) .$$

  All derivations are assumed to be continuous.

- A derivation $D : A \to X$ is *inner* if, for some $\xi \in X$, it is of the form

$$\mathrm{ad}_\xi : a \mapsto a \cdot \xi - \xi \cdot a \qquad (a \in A) ,$$

  and

- it is *approximately inner* if, for some net $(\xi_i) \subset X$,

$$D(a) = \lim \mathrm{ad}_{\xi_i}(a) \qquad (a \in A) .$$

More reminders

Definition

- A is *amenable* if for every A-bimodule X any derivation $D : A \to X^*$ is inner.
- A is *approximate amenable* if for every A-bimodule X any derivation $D : A \to X^*$ is approximately inner.

Fact [G, L, Read & Zhang]

Approximate amenability $\iff$ weak* approximate amenability $\iff$ derivations into any Banach bimodule are approximately inner.

The last property here is *approximate contractibility*.

Last reminders

Definition

A is *boundedly approximate amenable* if for every A-bimodule X any derivation $D : A \to X^*$ is approximately inner with a bounded net of approximating inner derivations.

NB. This is not the same if one allows any A-bimodule.

NB. It is the net of approximating inner derivations that is required to be bounded, not the net of implementing elements.

Boundedness of the implementing net is a much stronger condition:

Theorem (Gourdeau)

The Banach algebra A is amenable if and only if any derivation $D : A \to X^*$ into a dual bimodule is approximable by a net (ad_{ξ_i}) with the net (ξ_i) bounded.

Some examples

- Any finite dimensional approximately amenable algebra is amenable.
- For any locally compact space X, $C(X)$ is amenable. (Note that an amenable uniform algebra A must be $C(\Phi_A)$, whether the same holds for approximate amenability is unknown.)
- For any locally compact group G, $L^1(G)$ is approximately amenable if and only if it boundedly approximately amenable if and only if it is amenable, if and only if G is amenable as a group.
- $c_0((\ell_n^1)^\#)$ is approximately amenable but not amenable.
- $\ell^1(\mathbb{N})$, $c_0(\ell_n^1)$ are neither.

  Note that $c_0((\ell_n^1)^\#)$ has a bounded approximate identity, $c_0(\ell_n^1)$ does not.

c_0-sums

Much more work is required to split boundedly approximately amenable and approximately amenable.

Theorem (G & Read)

There exists a sequence (A_n) of boundedly approximately amenable algebras such that $c_0(A_n)$ is approximately amenable but not boundedly approximately amenable.

The approximate amenability of $c_0(A_n)$ here is a non-trivial part of the result – cf. $c_0(\ell^1_n)$ above.

In fact things can go ‘awry’ with finite sums.

More on sums

Theorem (G & Read)

There exists a boundedly approximately amenable algebra A such that $A \oplus A^{\mathrm{op}}$ is not approximately amenable.

(Here $A = c_0(A_n)$ with (A_n) amenable, whence $A \oplus A^{\mathrm{op}}$ has this same form. Also, $A \oplus A$ is boundedly approximately amenable.)

There is one way to avoid this 'pathology' for finite direct sums:

Theorem (G, L & Zhang)

If A and B are approximately amenable, and one of them has a bounded approximate identity, then $A \oplus B$ is approximately amenable.

So again, the presence of a bounded approximate identity facilitates 'good behaviour'.

A little building

The above examples/results with sums are more than just useful indicators as we now turn to the topic at hand, namely tensor products.

Theorem (Johnson)

The tensor product of amenable algebras is again amenable.

Barry's argument can be used more generally:

Theorem (Choi, G & L)

Suppose that A is approximately amenable with a bounded approximate identity, that B is amenable, and let D be a derivation from $A \widehat{\otimes} B$ to a dual bimodule. Then D is approximately inner on $A \otimes B$.

If A is boundedly approximately amenable then $A \widehat{\otimes} B$ is boundedly approximately amenable.

A little building

So, for instance, with the strong additional hypotheses:

- A has a bounded approximate identity,
- B is finite dimensional amenable,

A approximately amenable implies $A \widehat{\otimes} B$ is approximately amenable.

Overkill?

Well, as you probably suspect by now . . .

- The tensor product of boundedly approximately amenable algebras need not be approximately amenable.

Example

Let A be the Banach algebra of G & R such that A is boundedly approximately amenable yet $A \oplus A^{\mathrm{op}}$ is not approximately amenable.

Write $B = A^{\mathrm{op}}$, adjoin identities 1_A to A and 1_B to B, and set $\mathcal{A} = A^{\#} \widehat{\otimes} B^{\#}$.

Then $\mathcal{A}$ decomposes into closed subspaces:

$$\mathcal{A} = \mathbb{C}(1_A \otimes 1_B) + (\mathbb{C}1_A \otimes B) + (A \otimes \mathbb{C}1_B) + (A \widehat{\otimes} B)\,.$$

Here $A \widehat{\otimes} B$ is a closed two-sided ideal; consider $\mathcal{A}/(A \widehat{\otimes} B)$.

Example continued

If $\mathcal{A}$ is approximately amenable, then so is the quotient algebra

$$\mathbb{C}(1_A \otimes 1_B) + (\mathbb{C}1_A \otimes B) \oplus (A \otimes \mathbb{C}1_B) \simeq \Big((\mathbb{C}1_A \otimes B) \oplus (A \otimes \mathbb{C}1_B)\Big)^{\#},$$

whence so is $(\mathbb{C}1_A \otimes B) \oplus (A \otimes \mathbb{C}1_B) \simeq A \oplus B$.

But by the specific choice of A and B, $A \oplus B$ is not approximately amenable.

So $\mathcal{A}$ cannot be approximately amenable.

Unitizations

A similar argument (with same A and B) shows that for the boundedly approximately amenable algebra $\mathcal{A} = A^{\#} \oplus B^{\#}$, $\mathcal{A} \widehat{\otimes} \mathcal{A}$ is not approximately amenable.

Unitizations are used here to obtain counterexamples. They are also useful in giving positive results.

Theorem

Suppose that $A^{\#} \widehat{\otimes} B^{\#}$ is approximately amenable. Then A, B and $A \oplus B$ are approximately amenable.

Proof.

Both $A^{\#}$ and $B^{\#}$ are homomorphic images. For $A \oplus B$ use another decomposition argument. □

Something missing here?

Bounded approximate identities

Theorem

Suppose that $A^{\#} \widehat{\otimes} B^{\#}$ is (boundedly) approximately amenable, and that A and B have bounded approximate identities. Then $A \widehat{\otimes} B$ is (boundedly) approximately amenable.

Proof.

Thanks to the bounded approximate identities, it suffices to consider neo-unital $(A \widehat{\otimes} B)$-bimodules.

Thanks to the bounded approximate identities again, any derivation $D : A \widehat{\otimes} B \to X^*$ lifts to the double centralizer algebra, then restricts to $A^{\#} \widehat{\otimes} B^{\#}$, where it is suitably approximately inner. □

More bounded approximate identities

Theorem

Let A and B be Banach function algebras on their respective carrier spaces Φ_A and Φ_B, and suppose that A and B have bounded approximate identities consisting of elements of finite support. Then $A \widehat{\otimes} B$ is approximately amenable.

Proof 1.

The carrier space of $A \widehat{\otimes} B$ is $\Phi_A \times \Phi_B$, which is discrete.

The bounded approximate identities having finite support means that A and B have the (bounded) approximation property, so that the natural map $A \widehat{\otimes} B \to A \check{\otimes} B$ is injective.

It follows that $A \widehat{\otimes} B$ is semisimple, and so is a Banach function algebra on $\Phi_A \times \Phi_B$.

More bounded approximate identities

Proof 2.

Further, $A \widehat{\otimes} B$ has a bounded approximate identity consisting of elements of finite support, built from those of A and B.

But it is well known that a Banach function algebra having a bounded approximate identity consisting of elements of finite support is approximately amenable.

(There are several published proofs; all essentially the same except for their generality.)

So A, B and $A \widehat{\otimes} B$ are approximately amenable. $\square$

Semi-inner derivations

A new name for something well-known!

Definition

Let A be a algebra, X an A-bimodule. A map $D : A \to X$ is *semi-inner* if there are $m, n \in X$ such that

$$D(a) = a \cdot m - n \cdot a \qquad (a \in A)\,.$$

Such maps, with B a superalgebra of A, are commonly known as 'generalized inner derivations' .

When D is also a derivation, then m and n are highly constrained:

$$a \cdot (m - n) \cdot b = 0 \qquad (a, b \in A)\,.$$

In the Banach case, with $D : A \to X^*$ with X neo-unital, then necessarily $m = n$ and D is inner.

Semi-inner derivations

To us 'approximately generalized' is an oxymoron, so we use 'semi-inner', and only use it for derivations:

Definition

For A a Banach algebra, X a Banach A-bimodule, a derivation $D : A \to X$ is *approximately semi-inner* if there are nets $(m_i), (n_i)$ in X with

$$D(a) = \lim_i (a \cdot m_i - n_i \cdot a) \qquad (a \in A)\,.$$

In this case, for X neo-unital, and $D : A \to X^*$, then

$$\lim_i \left(a \cdot (m_i - n_i) \cdot b \right) = 0 \qquad (a, b \in A)\,,$$

whence $m_i - n_i \to 0$ weak*, so that D is in fact weak* approximately inner, and hence approximately inner by the Fact.

Example

Take $A = \ell^2$ under pointwise operations, $D : \ell^2 \to X$ a derivation into an A-bimodule, (E_n) the standard (unbounded) approximate identity of ℓ^2: $E_n = (\underbrace{1, 1, \ldots, 1}_{n \text{ times}}, 0, 0, \ldots)$.

The map $D_n : E_n\ell^2 \to X$ is a derivation from a finite-dimensional semisimple algebra and hence is inner, say implemented by $\xi_n \in X$.

Thus for $a \in \ell^2$,

$$
\begin{aligned}
D(a) &= \lim_n D(E_n a) = \lim_n (E_n a \cdot \xi_n - \xi_n \cdot E_n a) \\
&= \lim_n \left(a \cdot (E_n \cdot \xi_n) - (\xi_n \cdot E_n) \cdot a \right),
\end{aligned}
$$

So D is approximately semi-inner.

Note that ℓ^2 is not approximately amenable.

A simple construction

Let A and B be Banach algebras, $D : A \to X$ a derivation into an A-bimodule X.

Make $X \widehat{\otimes} B$ into an $A \widehat{\otimes} B$-bimodule as follows: for $a \in A$, $b_1, b_2 \in B$ and $x \in X$, set

$$(a \otimes b_1) \cdot (x \otimes b_2) = a \cdot x \otimes b_1 b_2 \,, \quad (x \otimes b_2) \cdot (a \otimes b_1) = x \cdot a \otimes b_2 b_1 \,.$$

The map $\Delta : A \widehat{\otimes} B \to X \widehat{\otimes} B$ defined by

$$\Delta(a \otimes b) = D(a) \otimes b \qquad (a \in A, b \in B) \,.$$

is a derivation.

Fix $b_0 \in B$, $b_0^* \in B^*$ with $\langle b_0^*, b_0 \rangle = 1$ and define the operator

$$T : X \widehat{\otimes} B \to X : x \otimes b \mapsto \langle b_0^*, b \rangle x \,.$$

Lemma (A)

Lemma (A)

Suppose that $A \widehat{\otimes} B$ is approximately amenable. Then any derivation from A or B into any Banach bimodule is approximately semi-inner.

Proof. (For the algebra A)

Given a derivation $D : A \to X$, take $\Delta : A \widehat{\otimes} B \to X \widehat{\otimes} B$ as above. By approximate amenability Δ is approximately inner.

(This uses the Fact that approximate amenability is the same as approximate contractibility.)

Lemma (A)

Proof cont.

So there is a net $\left(\sum_{k=1}^{\infty} x_{k,i} \otimes b_{k,i}\right)_i$ in $X \widehat{\otimes} B$ with

$$\Delta(a \otimes b) = \lim_i \left(\sum_k (a \cdot x_{k,i}) \otimes bb_{k,i} - \sum_k (x_{k,i} \cdot a) \otimes b_{k,i} b \right) .$$

Applying T to both sides,

$$D(a) = \lim_i (a \cdot m_i' - n_i' \cdot a) ,$$

where $m_i' = \sum_k \langle b_0^*, b_0 b_{k,i} \rangle x_{k,i}$, $n_i' = \sum_k \langle b_0^*, b_{k,i} b_0 \rangle x_{k,i}$. □

Lemma (B)

Lemma (B)

Suppose that $A \widehat{\otimes} B$ is boundedly approximately amenable. Then any derivation from A or B into any dual bimodule is boundedly approximately semi-inner.

NB Conclusion is both weaker and stronger than Lemma (A).

Proof 1.

We cannot argue as before! And $X^* \widehat{\otimes} B$ is unlikely to be dual.

We start with a more sophisticated version of the map T.

Lemma (B)

Proof 2.

Fix $b_0 \in B$, $b_0^* \in B^*$ with $b_0^*(b_0) = 1$, and let $S : X \to (X^* \widehat{\otimes} B)^*$ be specified by

$$\langle S(x), x^* \otimes b\rangle = \langle x^*, x\rangle\langle b_0^*, b\rangle\,, \qquad (x \in X, b \in B)\,,$$

and set $T = S^* : (X^* \widehat{\otimes} B)^{**} \to X^*$.

For $m = \sum_k x_k^* \otimes b_k \in X^* \widehat{\otimes} B$, a straightforward calculation yields

$$T((a \otimes b_0) \cdot m) = \sum_k \langle b_0^*, b_0 b_k\rangle a \cdot x_k^* = a \cdot \underbrace{\sum_k \langle b_0^*, b_0 b_k\rangle x_k^*}_{=:x^*(m)}\,,$$

with the estimate

$$\|x^*(m)\| \leq \|b_0\|\,\|b_0^*\|\,\|m\|\,.$$

Lemma (B)

Proof 4.

For a general $m \in (X^* \widehat{\otimes} B)^{**}$, a weak*-density and compactness argument, using the weak* to weak*-continuity of T, now shows that there is $\xi^* \in X^*$, bounded by a multiple of $\|m\|$, which satisfies

$$T((a \otimes b_0) \cdot m) = a \cdot \xi^* \qquad (a \in A)\,.$$

Similarly, there is $\eta^* \in X^*$, bounded by a multiple of $\|m\|$, with

$$T(m \cdot (a \otimes b_0)) = \eta^* \cdot a \qquad (a \in A)\,.$$

Lemma (B)

Proof 5.

Now back to derivations.

Given a derivation $D : A \to X$, take $\Delta : A \widehat{\otimes} B \to X \widehat{\otimes} B$ as above, viewed as mapping into $(X^* \widehat{\otimes} B)^{**}$.

Then there is a net (m_i) in $(X^* \widehat{\otimes} B)^{**}$ and a constant $K > 0$ such that for $a \in A, b \in B$,

$$D(a) \otimes b = \Delta(a \otimes b) = \lim_i \Big((a \otimes b) \cdot m_i - m_i \cdot (a \otimes b) \Big) ,$$

$$\|(a \otimes b) \cdot m_i - m_i \cdot (a \otimes b)\| \leq K \|a\| \, \|b\| .$$

Lemma (B)

Proof 6.

Setting $b = b_0$, and applying T gives nets (m_i') and (n_i') in X^* with

$$D(a) = \lim_i (a \cdot m_i' - n_i' \cdot a) \qquad (a \in A)\,,$$

$$\|a \cdot m_i' - n_i' \cdot a\| \leq K\|T\|\,\|b_0\|\|a\| \qquad (a \in A)\,.$$

□

Since D is a derivation one also gets the bounds

$$\|a \cdot (m_i' - n_i') \cdot b\| \leq 3K\|T\|\|b_0\|\|a\|\|b\|\,, \qquad (a, b \in A)\,,$$

which we will need later.

When $m_i' = n_i'$

Theorem

Suppose that $A \widehat{\otimes} B$ is (boundedly) approximately amenable. If B has an element b_0 with $b_0 \notin \overline{\{b_0 b - b b_0 : b \in B\}}$, then A is (boundedly) approximately amenable.

Proof.

Choose the functional b_0^* in the proof of the Lemmas to vanish on $\{b_0 b - b b_0 : b \in B\}$. Then the resulting nets (m_i') and (n_i') are the same. □

The commutator condition here is due to Barry Johnson.

Note there is no conclusion about B.

When $m_i' - n_i' \to 0$

Theorem

Suppose that $A \widehat{\otimes} B$ is boundedly approximately amenable. Suppose that one of A or B has an identity. Then A and B are boundedly approximately amenable.

Proof 1.

Suppose that B has an identity e. Previous result gives A is approximately amenable. But what about B?

When $m_i' - n_i' \to 0$

Proof 2.

Let X be a Banach B-bimodule. By the usual reduction, we may suppose that $e \cdot x = x = x \cdot e$ for $x \in X$.

Let $D : B \to X^*$ be a derivation, and consider the nets given by Lemma (B). Since

$$\lim_i \left(b \cdot (m_i' - n_i') \cdot c \right) = 0 \qquad (b, c \in B) ,$$

putting $b = c = e$ gives $m_i' - n_i' \to 0$, so that D is boundedly approximately inner.

□

Use of bounded approximate identities

Theorem

Suppose that $A \widehat{\otimes} B$ is boundedly approximately amenable and that A has a bounded approximate identity. Then A is approximately amenable.

Proof 1.

Let $D : A \to X^*$ be a derivation into the dual of a neo-unital bimodule X. From Lemma (B), we have nets (m_i'), (n_i') in X^*, and $K \geq 0$ such that for $a, a_1, a_2 \in A$

$$D(a) = \lim_i (a \cdot m_i' - n_i' \cdot a)\,, \quad \|a \cdot m_i' - n_i' \cdot a\| \leq K\|a\|\,, \tag{1}$$

$$\lim_i (a_1 \cdot (m_i' - n_i') \cdot a_2) = 0\,, \quad \|a_1 \cdot (m_1' - n_i') \cdot a_2\| \leq 3K\|a_1\| \cdot \|a_2\|\,. \tag{2}$$

Use of bounded approximate identities

Proof 2.

For a given $x \in X$, (2) gives

$$\langle m_i' - n_i', a_2 x a_1 \rangle \to 0, \quad |\langle m_i' - n_i', a_2 x a_1 \rangle| \leq 3K \|a_1\| \cdot \|a_2\| \cdot \|x\| .$$

Since X is neo-unital, it follows that

- $m_i' - n_i' \to 0$ weak* ,

and letting a_1, a_2 range over an approximate identity with bound M ,

- $\|m_i' - n_i'\| \leq 3KM^2$.

Use of bounded approximate identities

Proof 3.

Together with (1), these give for $a \in A$,

$$D(a) = \mathrm{weak}^* - \lim_i (a \cdot m_i' - m_i' \cdot a)\,, \quad \|a \cdot m_i' - m_i' \cdot a\| \leq 4M^2 \|a\|\,,$$

and we have derivations from A into duals of neo-unital bimodules are boundedly weak*- approximately inner.

It remains to remove the neo-unital assumption.

This is a standard decomposition argument for approximation in norm.

The same type of argument also works here, but only because of the boundedness.

Lemma (C)

Lemma (C)

Let A have a bounded approximate identity. Suppose that any derivation from A into the dual of a neo-unital bimodule is boundedly weak*-approximately inner.

Then A is (boundedly) weak*-approximately amenable, and so approximately amenable.

Proof 1.

Let (e_α) be a bounded approximate identity for A. Let E be a weak*-limit point of the left multiplication operators on X^* by the elements of (e_α), F similarly for right multiplication.

Lemma (C)

Proof 1.

Then E and F are commuting projections on X^*, and give a decomposition

$$X^* = EFX^* \oplus E(I - F)X^* \oplus (I - E)X^* .$$

For a derivation $D : A \to X^*$, set

$$D_1 = EFD, D_2 = E(I - F)D, D_3 = (I - E)D .$$

Then D_1, D_2, D_3 are derivations into the corresponding summands of X^*.

Lemma (C)

Proof 2.

The actions of A on the right of $E(I-F)X^*$, and on the left of $(I-E)X^*$, are trivial, and A has a bounded approximate identity, so that D_2 and D_3 are boundedly approximately inner.

For D_1, note that EFX^* is isomorphic to $(X_{ess})^*$, where $X_{ess} = A \cdot X \cdot A$ is a neo-unital A-bimodule. So by hypothesis, $D_1 : A \to EFX^*$ is boundedly weak*-approximately inner.

Lemma (C)

Proof 3.

A consideration of the weak* topologies on the bounded sets of EFX^* and X^* shows that $D_1 : A \to X^*$ is boundedly weak*-approximately inner.

Adding D_1, D_2 and D_3, D is boundedly weak*-approximately inner, and hence approximately inner by Fact.

Central bounded approximate identities

Theorem

Suppose that $A \hat{\otimes} B$ is approximately amenable and that one of A or B has a central bounded approximate identity. Then A and B are approximately amenable.

Proof 1.

Suppose that (e_α) is a central bounded approximate identity in B. Let $D : B \to X^*$ be a derivation into the dual of a neo-unital bimodule X.

From Lemma (A), we have a nets (m_i') and (n_i') in X^* such that

$$D(b) = \lim_i (b \cdot m_i' - n_i' \cdot b) \qquad (b \in B),$$
$$\lim_i (b_1 \cdot (m_i' - n_i') \cdot b_2) = 0 \qquad (b_1, b_2 \in B).$$

Central bounded approximate identities

Proof 2.

Now follow Lemma (C) to get D_1, D_2 and D_3. Then for $b \in B$,

$$\begin{aligned} D_1(b) &= (w^* - \lim_\alpha)(w^* - \lim_\beta) e_\alpha D(b) e_\beta \\ &= (w^* - \lim_\alpha)(w^* - \lim_\beta) \lim_i [e_\alpha(b \cdot m_i' - n_i' \cdot b) e_\beta]\,. \end{aligned}$$

Using centrality of (e_α),

$$D_1(b) = (w^* - \lim_\alpha)(w^* - \lim_\beta) \lim_i [b \cdot \overbrace{(e_\alpha \cdot m_i' \cdot e_\beta)} - \overbrace{(e_\alpha \cdot n_i' \cdot e_\beta)} \cdot b]\,.$$

Central bounded approximate identities

Proof 3.

Thus the standard method of considering finite subsets of B and X, gives a net $(x_\gamma^*) \subset X^*$ such that

$$D_1(b) = \text{weak}^* - \lim_\gamma (b \cdot x_\gamma^* - x_\gamma^* \cdot b)\,, \qquad (b \in B)\,.$$

Since D_2 and D_3 are approximately inner we finally deduce that D is weak*-approximately inner. Thus B is approximately amenable.

Since B has a non-zero central element, A is approximately amenable by the Johnson criterion. □

Extension of Barry's result

Theorem

Suppose that $A \widehat{\otimes} B$ is amenable. Then A and B are amenable.

Proof 1.

Amenability of $A \widehat{\otimes} B$ implies it has a bounded approximate identity, whence so do A and B.

Let $D : A \to X^*$ be a derivation into the dual of a neo-unital bimodule X.

Take Δ as in Lemma (A) and use the necessary part of Gourdeau's theorem to obtain a bounded net (m_i). This then gives a bounded nets (m_i') and (n_i') with

$$D(a) = \lim_i (a \cdot m_i' - n_i' \cdot a) \qquad (a \in A) .$$

Extension of Barry's result

Proof 2.

The argument of a previous theorem gives D boundedly weak*-approximately inner, with implementing net bounded.

Now use the argument of Lemma (C) to see that derivations from A into a dual module are weak*-approximately inner, with a bounded net of implementing elements. The argument behind the Fact now shows that any derivation into any A-bimodule is approximately inner with a bounded net of implementing elements, that is, A is amenable by the sufficient part of Gourdeau's theorem. □

Three questions

These obvious questions remain :

1. Does $A^{\#} \widehat{\otimes} B^{\#}$ (boundedly) approximately amenable imply $A \widehat{\otimes} B$ (boundedly) approximately amenable? (Yes, with a bounded approximate identity in each factor)
2. Does $A \widehat{\otimes} B$ (boundedly) approximately amenable imply A and B are (boundedly) approximately amenable? (Yes, with a central bounded approximate identity in either factor)
3. Does $A \widehat{\otimes} B$ (boundedly) approximately amenable imply $A^{\#} \widehat{\otimes} B^{\#}$ (boundedly) approximately amenable?

THANK YOU FOR YOUR ATTENTION