Approximate amenability and Charles' role in its advancement

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Derivations

$$A = \text{a Banach algebra}$$

$$X = \text{a Banach } A\text{-bimodule}$$

Derivation $D : A \longrightarrow X$ is a linear mapping that satisfies

$$D(ab) = D(a) \cdot b + a \cdot D(b), \quad (a, b \in A).$$

D is **inner** if $\exists x \in X$ such that

$$D(a) = a \cdot x - x \cdot a, \quad (a \in A), \quad \text{i.e. } D = \mathrm{ad}_x$$

amenable Banach algebras

A is amenable if every continuous $D : A \longrightarrow X^*$ is inner, for all X.

Concept founded by **Barry Johnson**, in late 1960's.

Motivated by his attempts in answering the following question:

Suppose that G is a locally compact group, $L^1(G)$ is the group algebra and $M(G)$ is the measure algebra of G. Given $\mu \in M(G)$, the mapping

$$D_\mu : f \mapsto f \star \mu - \mu \star f, \quad (f \in L^1(G))$$

is a derivation.

Question Is every derivation D on $L^1(G)$ a D_μ for some $\mu \in M(G)$? (Barry Johnson's Ph.D. thesis, Cambridge 1962).

contractible Banach algebras

Defn.(A. Helemskii) A is **contractible** if every continuous $D \to X$ is inner, for all X.

It is not known whether there exists an infinite-dimensional contractible Banach algebra.

approximately inner derivation

Continuous derivation $D : A \longrightarrow X$ is **approximately inner** if

$$D(a) = \lim_i \mathrm{ad}_{x_i}(a), \quad (a \in A).$$

All derivations will be assumed to be **continuous**.

Approximately amenable Banach algebra

Defn. (R.J. Loy, F. Gh.) A is **approximately amenable** if every $D : A \longrightarrow X^*$ is approximately inner for all X.

Boundedly approximately amenable:

$D = \lim \mathrm{ad}_{x_i^*}$ (SO), for some operator-norm bounded net $(\mathrm{ad}_{x_i^*})$.

Approximately contractible

$D : A \longrightarrow X, \ldots$

Boundedly approximating contractible ...

Uniform variants

uniformly approximating amenable (contractible):

Approximation is in operator-norm topology - instead of strong-operator topology.

Weak* approximate amenability

Approximation is in weak*-topology.

Equivalences

1. (**B. E. Johnson**, 2000) uniformly approx. contractible $\Leftrightarrow$ contractible.
2. (**A. Pirkowskii**, and independenly, **R. J. Loy, Y. Zhang, F. Gh.**, 2005)

   unif. approx. amen. $\Leftrightarrow$ amen.
3. (**R. J. Loy, Y. Zhang, F. Gh.**, 2004)

   Approx. amen. $\Leftrightarrow$ Approx. contract. $\Leftrightarrow$ Weak*-approx. amen.

Intrinsic characterization of amenability

Barry Johnson A is amenable iff there exists a **bounded** net $(M_i) \subset A\hat{\otimes}A$

such that:

1. $a \cdot M_i - M_i \cdot a \to 0,$
2. $\pi(M_i) \cdot a \to a, \quad (a \in A),$

Where $\pi : A\hat{\otimes}A \longrightarrow A$ is the map specified by

$$\pi(a \otimes b) = ab, \quad (a, b \in A).$$

Interinsic characterization of approximate amenability

A is approximately amenable iff there exists in net $(M_i) \subset (A^\sharp \hat{\otimes} A^\sharp)^{**}$ such that

$$a \cdot M_i - M_i \cdot a \to 0,$$

$$\pi^{**}(M_i) = 1, \forall i.$$

(**Loy, Gh.**, 2001)

A is approx. amen. if and only if, there exist nets $(M_i) \subset (A\hat{\otimes}A)^{**}$, $(F_i), (G_i) \subset A^{**}$, such that

(i) $a.M_i - M_i.a + F \otimes a - a \otimes G \longrightarrow 0 \qquad (a \in A)$

(ii) $a.F_i \longrightarrow a, G_i.a \longrightarrow a \qquad (a \in A)$

(iii) $\pi^{**}(M_i) = F_i + G_i \qquad \forall i.$

$\pi(a \otimes b) = ab$

For bounded approximate amenability, additionally, $\exists C > 0$

(i′) $\|a.M_i - M_i.a + F_i \otimes a - a \otimes G_i\| \leq C\|a\|, \qquad \forall a, \forall i;$

(ii′) $\|a.F_i\| \leq C\|a\|, \ \|G_i.a\| \leq C\|a\| \qquad \forall a \in A, \forall i.$

For bounded approximate contractibility

$$(F_i), (G_i) \subset A,\ (M_i) \subset A\hat{\otimes}A,\ \exists C > 0$$

$$\|.\| \leq C\|a\|.$$

Approximate identities

1. (**R. J. Loy, F. Gh.**, 2000) A approx. amen $\Rightarrow$ A has a right approx. identity and a left approx. identity.

2. (**Y. Choi, Y. Zhang, F. Gh.**, 2007) Suppose that A is bddly. approx. amen. and has a multiplier-bounded right approx. identity and a multiplier-bounded left approx. identity. $\Rightarrow$ A has a bounded approximate identity.

[**Defn:** (e_i) is multiplier-bounded if $\exists K > 0$, such that $\|ae_i\| \leq K\|a\|$, $\forall a \in A$, $\forall i$].

(**Y. Choi, Y. Zhang, F. Gh.**, 2007) Suppose that A is boundedly approximately contractible. Then A has a bounded approximate identity.

Does the result extend to bddly. approx. amen. algebras?

(**C. J. Read, F. Gh.**, 2010) There exists a boundedly approximately amenable Banach algebra, that has a bounded left approximate identity but no bounded right approximate identity.

Some details:

$K(l^1)$ = compact operators on l^1

$K(l^1)$ is known to be amenable (Barry Johnson)

(e_k) = the standard basis for l^1.

For $N = 1, 2, \ldots,$ let

$$|||T|||_N = \|T\|_{\text{op}} + N \limsup_{k\to\infty} \|Te_k\|, \qquad (T \in K(l^1))$$

$$A_N = \left(K(l^1), |||.|||_N\right)$$

Detour

Defn.:

$$c_R(A) = \inf\{M \ : \ A \text{ has a bounded right approximate identity of bound } M\}.$$

$$P_n = \text{projection onto span}\{e_1, e_2, \ldots, e_n\}, \qquad n = 1, 2, \ldots$$

(P_n) is a bounded left approximate identity for each A_N of bound 1, but $c_R(A_N) \geq N + 1$. So if we set

$$A = c_0 - \oplus_N A_N$$

A has a bounded left approximate identity of bound 1, but has no bounded right approximate identity.

And the following is needed:

(**C. J. Read, F. Gh.**, 2010). Suppose that (B_n) is a sequence of amenable Banach algebras, and there is an $M > 0$ such that each B_n has a bounded left approximate identity of bound M. Then $\mathcal{B} = c_0 - \oplus_n B_n$ is boundedly approximately amenable.

So

Bdd. approx. amen. is not bdd. approx. contract.

The algebra A defined as above has the property : $A \oplus A^{op}$ is not approximately amenable.

Whereas $A^{\sharp} \oplus A^{op}$ is approximately amenable. So an approximately amenable algebra may contain an ideal of co-dimension 1, that is not approximately amenable.

How about bdd. approx. amen. compared to approx. amen?

(**C. J. Read, F. Gh.**, 2013). There exists an approximately amenable Banach algebra which is not boundedly approximately amenable.

Ideas of proof

Step 1. Let A be a Banach algebra, $(e_i)_{i\in I}$ a bounded left approximate identity for A, and $(f_j)_{j\in J}$ a r.a.i for A (not necessarily bounded). Suppose that for every $i \in I$ and $j \in J$ there exists $d_{i,j} \in A\hat{\otimes}A$ with

$$\pi(d_{i,j}) = e_i + f_j - f_j e_i, \qquad\qquad (\pi(a_1 \otimes a_2) = a_1 a_2)$$

and for each $a \in A$ we have $\lim_j \limsup_i \|a.d_{i,j} - d_{i,j}.a\| = 0$.

Then A is approximately amenable.

Step 2. Suppose that $(A_n)_{n=1}^{\infty}$ are Banach algebras each of which satisfies the conditions of Step 1, and suppose further that the norms of the b.l.a.i's involved are uniformly bounded. Then the c_0-direct sum $\oplus_n A_n$ is approximately amenable.

Detour

Defn. The **bounded approximate amenability constant** (baac) of a boundedly approximately amenable Banach algebra is the infimum of all K, such that

$$\|a.M_i - M_i.a\| \leq K\|a\|, \qquad (a \in A)$$

extended over all $(M_i) \subset (A^{\#}\hat{\otimes}A^{\#})^{**}$ acting as an approximate diagonal for A.

We find a sequence (A_n) of boundedly approximately amenable Banach algebras that satisfies $\mathrm{baac}(A_n) \geq \frac{n}{2}$, and the sequence (A_n) satisfies the hypothesis of the above corollary. Then $A = c_0 - \oplus_n A_n$ is the desired algebra.

Open question

Is there an approximately amenable Banach algebra with no two-sided approximate identity?

Approximate amenability in abstract harmonic analysis

$$
\begin{aligned}
G &= \text{a locally compact group} \\
L^1(G) &= \text{the group algebra of } G \\
M(G) &= \text{the measure algebra of } G
\end{aligned}
$$

(**R. J. Loy, F. Gh.**, 2000)

(i) $L^1(G)$ is approximately amenable if and only if G is amenable.

(ii) $M(G)$ is approximately amenable if and only if G is discrete and amenable.

(Using a result of H. Dales, A. Helemeskii, F. Gh., 2000)

(iii) $L^1(G)^{**}$ (with 1st or 2nd Arens product) is approximately amenable if and only if G is finite.

The Fourier algebra $A(G)$

$$A(G) = \{f * \check{g} \ : \ f, g \in L^2(G)\}$$

$$\check{g}(x) = g(x^{-1}) \qquad\qquad (x \in G)$$

And for $h \in A(G)$,

$$\|h\|_{A(G)} = \inf\{\|f\|_2\|g\|_2 \ : \ h = f * \check{g},\ f \in L^2(G),\ g \in L^2(G)\}$$

(**B. Forrest, V. Runde**, 2004) $A(G)$ is amenable if and only if G has an amenable subgroup of finite index.

(**R. Stokke, F. Gh.**, 2005) Suppose that G is amenable and contains an open abelian subgroup. Then $A(G)$ is approximately amenable.

(**Y. Choi, F. Gh.**, 2009) Suppose that G has an open abelian subgroup. Then $A(G)$ is boundedly approximately amenable, if and only if G is amenable.

Corollary. Suppose that G is discrete and amenable. Then $A(G)$ is approximately amenable.

Example

Let

$$H = \left\{ \begin{bmatrix} 1 & m & n \\ 0 & 1 & p \\ 0 & 0 & 1 \end{bmatrix} \; : \; m, n, p \in \mathbb{Z} \right\};$$

the Heisenberg group.

$A(H)$ is approx. amen., but not amen.

Open question

Suppose that G is a discrete group. Does approximate amenability of $A(G) \Rightarrow$ amenability of G?

(**Y. Choi, Y. Zhang, F. Gh.**, 2007) If $\mathbb{F}_2$ is a closed subgroup of G, then $A(G)$ is not approximately amenable.

(**Y. Choi, F. Gh**, 2010) $A(G)^{**}$ (with an Arens product) is boundedly approximately amenable if and only if G is finite.

This in particular extends a result of E. Granirer, who proved that $A(G)^{**}$ is amenable if and only if G is finite

Open question: When is $A(G)^{**}$ approximately amenable?

Beurling algebras

weight:

$$w : G \longrightarrow (0, \infty)$$

$$w(xy) \leq w(x)w(y) \qquad\qquad (x, y \in G)$$

w continuous.

$$L^1(G, w) = \{f \ : \ fw \in L^1(G)\}$$

$$\|f\|_w = \|wf\|_1$$

$$(f * g)(x) = \int_G f(xy)g(y^{-1})\, d(y),$$

$$(f, g \in L^1(G, w), \text{ a.e.x}).$$

Amenability of Beurling algebras

(**Niels Gronbaek** 1990). Suppose that $w(e) = 1$. Then $L^1(G, w)$ is amenable

if and only if

(i) G is amenable;

(ii) $x \mapsto w(x)w(x^{-1})$ is bounded on G (w is diagonally bounded).

An alternative proof of this result in general case (not assuming $w(e) = 1$)

was given by (R. loy, Y. Zhang and F. Gh), in 2004.

Defn. A weight on G is *symmetric* if $w(x^{-1}) = w(x) \;\; \forall x \in G$.

(**E. Samei, Y. Zhang, F. Gh.**, 2008) Suppose that w is symmetric. Then $L^1(G, w)$ is boundedly approximately contractible if and only if G is amenable and w is bounded.

Segal algebras

$(S(G), \|.\|_S)$ is a Segal subalgebra of $L^1(G)$ if:

(i) $S(G)$ is a dense linear subspace of $L^1(G)$;

(ii) $\|.\|_S \geq \|.\|_1$;

(iii) $S(G)$ is left translation invariant and $x \mapsto l_x f : G \to S(G)$ is continuous;

(iv) $\|l_x f\|_S = \|f\|_S \qquad (f \in S(G))$.

S(G) is symmetric if, additionally, the above conditions hold on the right.

Example

$$S_{1,p} = L^1(G) \cap L^p(G)$$

$$\|f\| = \|f\|_1 + \|f\|_p \quad (f \in S_{1,p})$$

and convolution product.

If G is unimodular, $S_{1,p}$ is a symmetric Segal algebra.

(**Y. Choi, Y. Zhang, F. Gh.**, 2007) A proper symmetric Segal subalgebra of $L^1(G)$ can never be boundedly approximately amenable.

(**R. Loy, H.G. Dales**, 2010) Specific Segal algebras on $\mathbb{R}^n, \mathbb{T}^n$ are not approximately amenable.

(**Y. Choi, F. Gh.**, 2009) No non-trivial Segal subalgebra of $\mathbb{R}^n$ or $\mathbb{T}^n$ is approximately amenable.

(**M. Alagamandan**, 2015) If G is a SIN group, then no non-trivial symmetric Segal algebra $S(G)$ can be approximately amenable.

Conjecture: Non-trivial Segal algebras are never approximately amenable.

Operator algebras

Open question: When is a C^*- algebra approximately amenable?

(**Y. Choi, Y. Zhang, F. Gh.**, 2008) Let Γ be a discrete group. Then the following are equivalent:

(i) The full group C^*-algebra $C^*(\Gamma)$ of Γ is approximately amenable.

(ii) The reduced C^*-algebra $C_r^*(\Gamma)$ is approximately amenable.

(iii) Γ is amenable.

non-self adjoint operator algebras

(**Y. Choi**, and independently, **C. J. Read, F. Ghahramani**). There exists a non-self-adjoint operator algebra which is approximately amenable but not amenable.

Approximate amenability of $K(X)$

$K(X) =$ Compact operators on X

Amenability of $K(X)$, for $X = C(\Omega)$, where Ω is uncountable, compcat metrizable space and $X = l^p,\ \ 1 < p < \infty$ was established by Barry Johnson, Mem. AMS, 1972.

(**N. Grønbaek, B. E. Johnson, G. A. Willis**, 1994). Suppose that $X = X_1 \oplus X_2$ and $K(X)$ is amenable. Then at least one of the maps

$$\begin{aligned} \pi_1 \quad &: \quad K(X_1, X)\hat{\otimes}K(X, X_1) \to K(X) \\ &\quad \pi_1(T_1 \otimes T_2) = T_1 \circ T_2 \qquad (T_1 \in K(X_1, X),\ T_2 \in K(X, X_1)) \end{aligned}$$

or

$$\pi_2 \ : \ K(X_2, X)\hat{\otimes}K(X, X_2) \to K(X) \tag{1}$$

$$\pi_2(T_1 \otimes T_2) = T_1 \circ T_2, \quad (T_1 \in K(X_2, K), T_2 \in K(X, X_2))$$

is surjective.

A question which was open since 2000:

Is there a Banach space X for which $K(X)$ is approximately amenable, but not amenable?

Yes, **C. J. Read, F. Gh.** (2014).

Theorem

If X is **fairly close to Hilbert space**, then $K(X)$ is approximately amenable.

Then, we define X_1 and X_2 such that $X = X_1 \oplus X_2$ fails the

[Grønback-Johnson-Willis] necessary condition for amenability of $K(X)$, and

yet X is fairly close to Hilbert space.