Some papers of Charles Read

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Invariant subspaces and Banach algebras:
A meeting in memory of Charles Read

Dedicated to Charles, 1958–2015

Leeds, 2 September 2016

Some references

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[R2] C. J. Read, Different forms of the approximation property, typewritten manuscript, about 1986.

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Preliminary notation - algebra

Let A be a (complex, associative) algebra. A **character** on A is a homomorphism φ from A onto $\mathbb{C}$, and the space of these is the **character space** Φ_A. We set

$$A^{[2]} = \{ab : a, b \in A\}, \quad A^2 = \text{lin } A^{[2]}.$$

The algebra A **factors** if $A = A^{[2]}$ and **factors weakly** if $A = A^2$.

Let X be an A-bimodule. Then a **derivation** from A into X is a linear map $D : A \to X$ such that

$$D(ab) = a \cdot Db + Da \cdot b \quad (a, b \in A).$$

Take $\varphi \in \Phi_A$. Then a **point derivation** at φ is a linear functional d such that

$$d(ab) = \varphi(a)d(b) + \varphi(b)d(a) \quad (a, b \in A).$$

Set $M_\varphi = \ker \varphi$. Then point derivations at φ correspond to linear functionals d on M_φ such that $d \mid M_\varphi^2 = 0$.

Preliminary notation - Banach spaces

Let E be a Banach space. Then:

$\mathcal{B}(E)$ is the Banach algebra of all bounded linear operators on E;

$\mathcal{F}(E)$ is the ideal of finite-rank operators;

$\mathcal{A}(E) = \overline{\mathcal{F}(E)}$ is the closed ideal of approximable operators;

$\mathcal{K}(E)$ is the closed ideal of compact operators;

$\mathcal{W}(E)$ is the closed ideal of weakly compact operators.

Clearly $\mathcal{F}(E) \subset \mathcal{A}(E) \subset \mathcal{K}(E) \subset \mathcal{W}(E) \subset \mathcal{B}(E)$.

The ideal $\mathcal{F}(E)$ is the minimum non-zero ideal in $\mathcal{B}(E)$, and $\mathcal{A}(E)$ is the minimum non-zero, closed ideal in $\mathcal{B}(E)$.

Automatic continuity questions

Let A be a Banach algebra. Then:

all derivations from A are continuous
means that all derivations from A into any Banach A-bimodule are automatically continuous;

all homomorphisms from A are continuous
means that all homomorphisms from A into any other Banach algebra are automatically continuous.

Assume that there is a discontinuous derivation from A. Then it is easy to construct a discontinuous homomorphism from A (but the converse does not hold).

See [D].

Semester in Leeds 1987

There was a semester on **Banach algebras and automatic continuity** in Leeds from March to early July, 1987. (Funded by SERC – total cost £18k.)

Participants included Bill Bade, Phil Curtis, Peter Dixon, Jean Esterle, Sandy Grabiner, Niels Gronbaek, Barry Johnson, Rick Loy, Tom Ransford, Allan Sinclair, and Marc Thomas.

It culminated in the **6th Banach algebra conference,** 23 June to 3 July, 1987.

Charles Read came from Cambridge for the conference, and lectured on 'Invariant subspaces'.

He met some people, learnt about Banach algebras and automatic continuity, and was asked some questions.

Derivations from $\mathcal{B}(E)$

A question discussed was: *Are all derivations from* $\mathcal{B}(E)$ *continuous?* (for a Banach space E).

Johnson (1972): Yes, if E has a 'continued bisection of the identity', which is true of most standard Banach spaces, including all those with $E \sim E \times E$.

I asked about other spaces.

Loy and Willis (1989): Also 'yes' for $E = J_2$ and $E = C([0, \omega_1])$, which do not have a continued bisection of the identity.

Similar results about the automatic continuity of homomorphisms from $B(E)$ for these spaces.

The answer of Charles

Charles grasped the question, and soon [R1] produced a remarkable counter-example.

Theorem There is a Banach space E_R and a character φ on $\mathcal{B}(E_R)$ such that

$$\mathcal{W}(E_R) \subset M_\varphi \subset \mathcal{B}(E_R)\,,$$

such that $\mathcal{W}(E_R)$ has infinite codimension in $\mathcal{B}(E_R)$, and $M_\varphi^2 \subset \mathcal{W}(E_R)$. $\square$

It follows immediately that there are discontinuous (and non-zero, continuous) point derivations on $\mathcal{B}(E_R)$ at φ.

The construction

[Taken from the thesis of Richard Skillicorn, Lancaster, 2016.]

Idea: We know that $\mathcal{W}(J_2)$ has codimension 1 in $\mathcal{B}(J_2)$; try taking E to be an infinite direct sum of spaces J_2. But this does not work; we need more complicated spaces than J_2, and different spaces in the direct sum. The spaces used are 'James-type spaces based on different Lorentz sequence spaces'.

Then from the direct sum, take a quotient that makes the coordinates wrap around in a complicated way. This involves an inductive, combinatorial proof.

Full exposition in the thesis of Skillicorn.

Here E_R''/E_R is a separable, infinite-dimensional Hilbert space with an explicit orthonormal basis.

Further results

Theorem (D, Loy, Willis, 1994) There is a Banach space X such that all derivations from $\mathcal{B}(X)$ are continuous, but (with (CH)), there are discontinuous homomorphisms from $\mathcal{B}(X)$. $\square$

The structure of (maximal) ideals in $\mathcal{B}(E)$ for related spaces, including $C([0, \sigma])$, has been investigated by Niels Laustsen, Rick Loy, Andras Zsak, et al. - including Charles.

Extensions of Banach algebras

Let A be a Banach algebra. Then an **extension** of A is a short exact sequence

$$0 \to I \stackrel{\iota}{\to} \mathfrak{A} \stackrel{q}{\to} A \to \{0\}\,,$$

where $\mathfrak{A}$ is a Banach algebra, I is a closed ideal in $\mathfrak{A}$, and ι and q are continuous homomorphisms.

The sequence is **admissible** or **splits** or **splits strongly** if q has a right inverse that is a bounded linear operator, an algebra homomorphism, or a continuous algebra homomorphism, respectively.

The theory of extensions is related to that of Wedderburn decomposition of Banach algebras (when $I = J(\mathfrak{A})$) and of Hochschild cohomology; see [D].

Questions on extensions

There is an AMS memoir on this of Bade, D, Lykova in 1999. One of several questions raised is: *For which Banach algebras A is it true that every extension that splits also splits strongly?*

This is a form of an automatic continuity question. Many positive and negative answers are given in the memoir, but it was left open whether this is true for $\mathcal{B}(E)$ for every Banach space E; it is true for many such E, including those with a continued bisection. Now this is solved:

The **homological bidimension** of a Banach algebra A is $\mathrm{db}\, A$: $\mathrm{db} A \geq 2$ if and only if $\mathcal{H}^2(A, X) \neq \{0\}$ for a Banach A-bimodule X.

Theorem (Laustsen and Skillicorn, 2016) There is an extension of $\mathcal{B}(E_R)$ that splits algebraically, but which is not even admissible, and so does not split strongly. Further, $\mathrm{db}\, \mathcal{B}(E_R) \geq 2$. $\square$

Approximation properties

A Banach space E has the **approximation property** (AP) if, for each compact subset K in E and each $\varepsilon > 0$, there is $T \in \mathcal{F}(E)$ with

$$\|Tx - x\| < \varepsilon \quad (x \in K)\,.$$

The space E has the **bounded approximation property** (BAP) if there exists $m > 0$ such that T can always be chosen with $\|T\| \leq m$.

There are many equivalently formulations of (AP), (BAP), (CAP), (BCAP), etc., many relations between them, some counter-examples, some open questions. Not all spaces have (AP) (Enflo, 1972) or even (CAP). See a survey of Casazza.

E.g., Willis (1992): (CAP) does not imply (AP).

Suppose that E has (AP). Then $\mathcal{A}(E) = \mathcal{K}(E)$; the converse is open. But there are spaces E with $\mathcal{A}(E) \subsetneq \mathcal{K}(E)$, and these were of special interest to Charles. [I would like to know more about the radical Banach algebra $\mathcal{K}(E)/\mathcal{A}(E)$.]

Further properties

We see that a Banach space E has (BAP) if and only if there is a bounded net (T_α) in $\mathcal{F}(E)$ such that $T_\alpha x \to x$ for each $x \in E$.

Definition The space E has the **commuting bounded approximation property** (CBAP) if there is such a net with $T_\alpha T_\beta = T_\beta T_\alpha$ for all α, β.

Definition The space E has the π-**property** if there is such a net with each T_α being a projection.

Every space with a basis has the π-property; an example of Szarek (1987) shows the converse fails.

These properties are important in the theory of decompositions of Banach spaces.

The contribution of Charles

The following theorem is in [R2], from 1986.

Theorem There is a reflexive Banach space with (CBAP) (and hence with (BAP), even (MAP)) which fails the π-property. $\square$

[History of the paper.]

Idea: The proof uses random Banach spaces and ‘Gluskin’ spaces. Start with a sequence (E_n) of finite-dimensional spaces such that the basis constant of $E_n \oplus \ell_2$ is at least n. Think about $(\bigoplus E_n)_{\ell^2}$. The spaces E_n have to ‘overlap’ and be glued together in such a way that there are uniformly bounded finite-rank operators passing through the overlaps, but no good projections. A space exists because the probability that there is such a space is > 0.

As Casazza says: ‘This is a very delicate operation and the complexity of the proof reflects this.’

Factorization in $\mathcal{K}(E)$

Recall that a **bounded left approximate identity** (BLAI) in a Banach algebra A is a bounded net (e_α) in A such that $\lim_\alpha e_\alpha a = a \ (a \in A)$.

Theorem (Cohen 1959) Let A be a Banach algebra with a BLAI. Then A factors. □

Theorem (Dixon 1986) Let E be a Banach space. Then $\mathcal{K}(E)$ has a BLAI if and only if E has (BCAP). So there are spaces E such that $\mathcal{K}(E)$ does not have a BLAI. □

Obvious questions Does $\mathcal{K}(E)$ always factor? Does $\mathcal{K}(E)$ always factor weakly? Does weak factorization imply factorization?

I asked Charles these questions around 1993, and suggested that modifications of his earlier constructions might give counter-examples.

Remark A modification (D, Jarchow, 1994) of an example of Pisier shows that $\mathcal{A}(E)^2$ can have infinite codimension in $\mathcal{A}(E)$. □

The answer of Charles - 1

The following theorem is in [R3].

Theorem There is a Banach space E such that E is super-reflexive and such that $\mathcal{A}(E) \not\subset \mathcal{K}(E)^2$. In particular, $\mathcal{K}(E)$ does not factor weakly. Further, $\mathcal{K}(E)^2$ has infinite codimension in $\mathcal{K}(E)$, and so there are discontinuous point derivations on $\mathcal{K}(E)$. □

[History]

Idea Charles introduced the notion of an **approximate character** (with constants $C > 0$ and $\varepsilon > 0$) on a Banach algebra A. This is a map $\varphi : A \to \mathbb{C}$ such that:

$$\varphi(\alpha a) = \alpha\varphi a \ \ (\alpha \in \mathbb{C},\, a \in A)\,;$$

$$|\varphi(a)| \leq C\,\|a\| \ \ (a \in A) \ :$$

$$|\varphi(a+b) - \varphi(a) - \varphi(b)| < \varepsilon(\|a\|+\|b\|) \ \ (a, b \in A)\,;$$

$$|\varphi(ab) - \varphi(a)\varphi(b)| < \varepsilon(\|a\|\ \|b\|) \ \ (a, b \in A)\,.$$

The answer of Charles - 2

Idea, continued There is a constant $C > 0$ such that, for each $n \in \mathbb{N}$, there are 'Gluskin spaces' E_n of dimension n that have approximate characters with surprisingly good constants C and $C/\sqrt{n}$.

These spaces can be combined in a 'similar way' to that in [R2] to produce an infinite-dimensional, super-reflexive Banach space E on which there is a sequence (φ_i) of approximate characters on $\mathcal{B}(E)$, such that φ_i has constants C and C/i^3 and φ_i and φ_{i+1} are 'close'. Any space E with such a sequence has the properties specified in the theorem.

A further question and possible answer

I raised a further question with Charles: *Can the above construction be modified to produce a Banach space E such that $\mathcal{K}(E)^2$ is not even dense in $\mathcal{K}(E)$?*

For such a space E there would be non-zero, continuous point derivations on $\mathcal{K}(E)$.

Paper [R4], found in Charles' estate, states the following theorem:

Theorem There is a Banach space E such that $\mathcal{K}(E)^2$ is not dense in $\mathcal{K}(E)$. Indeed, $\mathcal{K}(E)/\overline{\mathcal{K}(E)^2}$ is non-separable because it contains an image of the Banach space ℓ^∞/c_0. $\Box$

The paper also claims that the space E has CBAP without the π-property, and gives a 'neater' proof of this result than that in [R2].

A main problem is that we found only 16 pages of the text; an unknown number of pages are missing.

Amenable Banach algebras

Let A be a Banach algebra, and let E be a Banach A-bimodule. Then the dual space E' is also a Banach A-bimodule:

$$\langle b, a \cdot \lambda \rangle = \langle ba, \lambda \rangle , \quad \langle b, \lambda \cdot a \rangle = \langle ab, \lambda \rangle ,$$

for $a, b \in A$ and $\lambda \in E'$.

Definition (Johnson, 1972) A Banach algebra A is **amenable** if $\mathcal{H}^1(A, E') = \{0\}$ for each Banach A-bimodule E.

Thus we need all continuous derivations from A into all E' to be inner.

Theorem (Johnson, 1972) A group algebra $L^1(G)$ is amenable if and only if the locally compact group G is an amenable group. $\square$

Charles studied amenable and approximately amenable Banach algebras intently - see later talks.

An intrinsic characterization

Let A be a Banach algebra. The projective tensor norm on $A \otimes A$ is denoted by $\|\cdot\|_\pi$, and its completion $(A \widehat{\otimes} A, \|\cdot\|_\pi)$ is a Banach algebra and a Banach A-bimodule. The canonical map $\pi_A : A \widehat{\otimes} A \to A$ satisfies

$$\pi_A(a \otimes b) = ab \quad (a, b \in A)\,.$$

An **approximate diagonal** for A is a net (u_α) in $A \widehat{\otimes} A$ such that $(\pi_A(u_\alpha))$ is a BLAI in A and

$$\lim_\alpha (u_\alpha \cdot a - a \cdot u_\alpha) = 0 \quad (a \in A)\,.$$

Theorem (Johnson, Helemskii) A Banach algebra A is amenable if and only if it has an approximate diagonal □

In my opinion, Charles had a deep intuitive understanding of approximate diagonals and used them to resolve many questions.

[There is a delightful proof by Charles of the fact that $\mathcal{B}(\ell^1)$ is not amenable in 2006. More general results are due to Volker Runde.]

CRBAs

The (Jacobson) **radical** of an algebra is $J(A)$. For a CBA = commutative Banach algebra A, $J(A)$ is equal to the set of **quasi-nilpotents** of A; these are the elements $a \in A$ such that $\lim_{n\to\infty} \|a^n\|^{1/n} = 0$.

Question It seemed very hard for an amenable Banach algebra to have a large radical. Was there a CBA that was both radical and amenable?

This was open from about 1972 to about 1999.

Pomona and Leeds

The 14th **Banach algebra conference** took place at Pomona, Claremont, California, 25 July to 7 August, 1999; it was organised by Sandy Grabiner and Marc Thomas.

Charles attended and gave a penetrating and lucid lecture that established the following theorem.

Theorem There is a commutative, amenable radical Banach algebra. $\square$

[History; HoD 2000–2003; Chairman: Malcolm Bloor. Charles was appointed at Leeds from September 2000.]

The construction of Charles

Definition Let A be a closed subalgebra of a CBA B, and take $\delta > 0$. An element $b \in B$ is a **metric approximate unit** for A with constant δ if $\|b\| \leq 1$ and $\|ab - a\| \leq \delta \|a\| \quad (a \in A)$.

An **FDNC** algebra is a finite-dimensional, nilpotent, commutative algebra

Lemma Let A be a FDNC Banach algebra, and take $\delta > 0$. Then there is an 'Arens–Hoffman extension' B of A that is also a FDNC algebra and that contains a metric approximate unit for A with constant δ. $\square$

Lemma A similar, but much more complicated result involving $A \otimes A$. $\square$

Proof of the theorem Using the lemmas, Charles builds a sequence of FDNC algebras, keeping clever track of the constants, to build a radical CBA that has an approximate diagonal, and so is amenable. $\square$

The algebra of formal power series

We shall consider subalgebras of the algebras of all formal power series in one and several variables over $\mathbb{C}$.

First

$$\mathfrak{F} = \mathbb{C}\,[[X]] = \left\{ \sum_{k=0}^{\infty} \alpha_k X^k : \alpha_0, \alpha_1, \dots \in \mathbb{C} \right\},$$

with product defined by $X^m \cdot X^n = X^{m+n}$. Thus

$$\left(\sum_{j=0}^{\infty} \alpha_j X^j \right) \cdot \left(\sum_{k=0}^{\infty} \beta_k X^k \right) = \sum_{\ell=0}^{\infty} \gamma_\ell X^\ell,$$

where $\gamma_\ell = \sum \{ \alpha_j \beta_k : j + k = \ell \}$.

There is one maximal ideal and all other ideals are closed and they are the standard ones.

The n-dimensional version is $\mathfrak{F}_n$. For $n \geq 2$, the ideal structure is complicated; there are many prime ideals. But $\mathfrak{F}_n$ is Noetherian (Hilbert).

Formal power series in infinitely many variables

We regard $\mathfrak{F} = \mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \cdots$, with obvious embeddings, and so $\bigcup \mathfrak{F}_n$ is an algebra.

Now let $\mathfrak{F}_\infty$ consist of elements of the form

$$\sum \left\{ \alpha_{(r_1,\ldots,r_n)} X_1^{r_1} \cdots X_n^{r_n} : (r_1, \ldots, r_n) \in (\mathbb{Z}^+)^n \right\},$$

where $n \in \mathbb{N}$. Then $\mathfrak{F}_\infty$ is an algebra, with $\bigcup \mathfrak{F}_n \subset \mathfrak{F}_\infty$. But $\mathfrak{F}_\infty$ contains elements not in $\bigcup \mathfrak{F}_n$, such as

$$X_1 + \frac{1}{2} X_2 + \cdots + \frac{1}{n} X_n + \cdots, \qquad (*)$$

and $\mathfrak{F}_\infty$ is not Noetherian.

We can say that $\mathfrak{F}_n$ and $\mathfrak{F}_\infty$ are the **free semigroup algebras** over $(\mathbb{Z}^+)^n$ and $(\mathbb{Z}^+)^{<\omega}$, respectively.

Fréchet algebras

Taken from [DPR] and a lecture of Charles at the 19th **Banach algebra conference** in Bedlewo, Poland, July 2009.

Let A be an algebra, with a sequence (p_k) of algebra seminorms such that $\bigcap \ker p_k = \{0\}$ and the metric specified by

$$d(a,b) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{p_k(a-b)}{1 + p_k(a-b)}$$

for $a, b \in A$ is complete. Then A is a **Fréchet algebra.**

We can suppose that $p_k(a) \leq p_{k+1}(a)$ always.

Examples of Fréchet algebras

Take $n \in \mathbb{N}$. Define p_k on $\mathfrak{F}_n$ by

$$p_k\left(\sum\{\alpha_r X^r : r \in (\mathbb{Z}^+)^n\}\right)$$

$$= \sum\left\{|\alpha_r| : r \in (\mathbb{Z}^+)^n,\ |r| \le k\right\}$$

Then we obtain a Fréchet algebra. The topology is that of **coordinatewise convergence**, called τ_c.

Similarly, $(\mathfrak{F}_\infty, \tau_c)$ is a commutative Fréchet algebra. The seminorms are now given by

$$p_k\left(\sum\{\alpha_r X^r : r \in (\mathbb{Z}^+)^n\}\right)$$

$$= \sum\left\{|\alpha_r| : r \in (\mathbb{Z}^+)^k,\ |r| \le k\right\}.$$

The continuity of characters

Let A be a Fréchet algebra. Then A is **functionally continuous** if every character on A is continuous.

Michael's problem, 1952 Is every commutative, Fréchet algebra functionally continuous? Also known to Mazur about 1937. Still wide open!

1) Almost trivially, every Banach algebra is functionally continuous.

2) (Arens, 1958) Suppose that A is (topologically) finitely generated as a Fréchet algebra. Then A is functionally continuous.

3) If there are only countably many characters, they are all continuous. For example, $\mathfrak{F}_\infty$ is functionally continuous.

Major work by Dixon and Esterle relate Michael's problem to analogues of Picard's theorem in several variables.

A test case

We return to the Fréchet algebra $\mathfrak{F}_\infty$.

For $m \in \mathbb{N}$, take q_m to be

$$q_m\left(\sum \alpha_r X^r\right) = \sum |\alpha_r|\, m^{|r|}$$

and then take $\mathcal{U}_m$ to be

$$\{f \in \mathfrak{F}_\infty : q_m(f) < \infty\}\ .$$

Clearly $(\mathcal{U}_m, q_m)$ is a unital Banach algebra continuously embedded in $\mathfrak{F}_\infty$.

Then set $\mathcal{U} = \bigcap\{\mathcal{U}_m : m \in \mathbb{N}\}$.

We see that $\mathcal{U}$ is a commutative, unital Fréchet algebra for the sequence (q_m), and that $\mathcal{U} \supset \bigcup \mathbb{C}[X_1, \ldots, X_n]$.

Theorem (Clayton 1975) Assume that every character on $\mathcal{U}$ is continuous. Then every commutative Fréchet algebra is functionally continuous. $\square$

Functional continuity of Fréchet algebras of power series

Weaker question - raised by Patel Is at least every Fréchet algebra of power series (in 1 or n variables) functionally continuous?

Theorem There is a continuous embedding θ of $\ell^1((\mathbb{Z}^+)^{<\omega})$ into $(\mathfrak{F}, \tau_c)$ with $\theta(X_1) = X$, and so the Banach algebra $\ell^1((\mathbb{Z}^+)^{<\omega})$ is (isometrically isomorphic to) a Banach algebra of power series.

Corollary The Fréchet algebra $\mathcal{U}$ is (isomorphic to) a Fréchet algebra of power series, and so the 'weaker question' is the same as Michael's problem. □

Remarks on the proof of the theorem

Set $A = \ell^1((\mathbb{Z}^+)^{<\omega})$.

Step 1 Let $(g_i : i \in \mathbb{N})$ be a sequence in $\mathfrak{F}$ with $g_1 = X$ such that $\mathbf{o}(g_i) \geq i$ $(i \in \mathbb{N})$. Then clearly there is a unique continuous, unital homomorphism $\theta : (\mathfrak{F}_\infty, \tau_c) \to (\mathfrak{F}, \tau_c)$ with $\theta(X_i) = g_i$ $(i \in \mathbb{N})$.

Since $\theta(X_1) = X$, we have $\theta(\mathcal{U}) \supset \mathbb{C}[X]$, and so all the required conditions are satisfied save perhaps for the fact that $\theta \mid A$ is an injection.

(Of course, θ cannot be an embedding of $\mathfrak{F}_\infty$ itself because there is not even an embedding of $\mathfrak{F}_2$ in $\mathfrak{F}$.)

We claim that we can choose elements g_i for $i \in \mathbb{N}$ so that $\theta \mid A$ is indeed an injection.

Continuation of the proof

Step 2 Start with a function $\gamma : \mathbb{N} \to \mathbb{N}$ such that $\gamma_i \le i$ $(i \in \mathbb{N})$ and, for each $n \in \mathbb{N}$ and $r = (r_1, \dots, r_n) \in \mathbb{N}^n$, there exists $k \in \mathbb{N}$ with

$$(\gamma(k+1), \dots, \gamma(k+n)) = (r_1, \dots, r_n)\,.$$

For each $i \in \mathbb{N}$ with $i \ge 2$, define

$$E_i = \{j \in \mathbb{N} \setminus \{1\} : \gamma(j) = i\}\,,$$

and take $E_1 = \{1\}$, so that $\{E_i : i \in \mathbb{N}\}$ is a partition of $\mathbb{N}$, and each E_i save for E_1 is infinite.

Now take a 'very rapidly increasing sequence' (a_i, b_i) with $1 = a_1 < b_1 < a_2 < b_2 < \cdots$ and $a_{i+1} > ib_i$ $(i \in \mathbb{N})$ and [Hall-mark of Charles!]

$$b_i > i \cdot (i(1+a_i))! \cdot i^{i(1+a_i)} \cdot b_{i-1}^{i(1+a_i)} \quad (i \ge 2)\,.$$

Then define $g_i = \sum\{b_j X^{a_j} : j \in E_i\}$.

Continuation of the proof

The claim will follow easily from the following lemma, which gives the combinatorial flavour of the proof.

Step 3 Let $m \in \mathbb{N}$. Let $(r_1, \ldots, r_m)$ be such that $r_1 \leq r_2 \leq \cdots \leq r_m$, and choose $k \in \mathbb{N}$ with $k > m$ and

$$(\gamma(k+1), \ldots, \gamma(k+m)) = (r_1, \ldots, r_m)\,.$$

Set $P = \sum_{i=1}^{m} a_{k+i}$ and $Q = \prod_{i=1}^{m} b_{k+i}$.

Then $\pi_P(g_{r_1} \cdots g_{r_m}) \geq Q$ and

$$\pi_P(g_{s_1} \cdots g_{s_n}) \leq Q/k$$

whenever $\{s_1, \ldots, s_n\} \neq \{r_1, \ldots, r_m\}$.

Step 4 Now suppose that $f = \sum \beta_r X^r \neq 0$ in A, say $\sum |\beta_r| = 1$. Choose r with $\beta_r \neq 0$, and then choose k with $k\,|\beta_r| > 1$, with corresponding P and Q. Then

$$|\pi_P(\theta(f))| > Q \cdot (|\beta_r| - 1/k) > 0\,,$$

and so $\pi_P(\theta(f)) \neq 0$. Hence θ is injective. $\square$

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