Real positivity and operator algebras
(work with Charles, and some recent applications)

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Charles John Read (1958-2015)

"looking forward to a city which has foundations..."

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Abstract

Subtitle: The quest for positivity in (non-selfadjoint) operator algebras

In a collaboration with Charles Read spanning many papers we studied operator algebras on Hilbert space, in particular initiating a program concerning (real) positivity in such algebras and related operator spaces. We begin this talk by surveying this work with Charles and its applications, for example to noncommutative peak sets and interpolation. In part this will be a tribute to Charles and his amazing mind. Then we turn to more recent applications of the new ideas, some in progress, for example to noncommutative Hardy spaces (focusing on one good application of noncommutative peak sets) and quantum set theory (with Louis Labuschagne and Nik Weaver), and elsewhere in operator theoretic functional analysis.

Our pattern was this: Once or twice a year Charles would arrive at the airport after or en route to diving in Florida or Mexico. He'd stay for two weeks in our home, and be part of the family. (Which I think Charles loved in some ways and found difficult in others–being an intense introvert.) When he arrived I'd go through a list of problems I thought were interesting... . One or two of them would take.

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But it was not as simple as that, as I am sure some in this room can attest to: one could never predict what problem Charles would get interested in. If he got interested he almost always solved it. In a way nobody else in the world would have been able to.

• The main theme of our collaboration:

Operator algebra: not necessarily selfadjoint subalgebra of $B(H)$ for a Hilbert space H

Notation: 'unital algebra' means has an identity of norm 1, 'approximately unital ' means has a contractive approximate identity (cai).

The first result of our discussions was Charles' answer to one of my questions to him:

Theorem (Read) Any operator algebra with a cai has a cai satisfying $||1 - 2e_t|| \leq 1$.

This appeared in a paper On the quest for positivity in operator algebras (2011).

• I asked this question because I was after a noncommutative Glicksberg peak set theorem.

If such existed it would also generalize an important result in C^*-algebra theory relating compact projections and support projections of elements, and also generalize a big part of the important theory of hereditary subalgebras (HSA's) of C^*-algebras (these satisfy $DAD \subset D$).

Let me explain these. We will mention the function algebra theory and then the C^*-algebra variant, and then Charles and my generalization of both.

Recall that a peak set for a uniform algebra $A \subset C(K)$ is a closed set $E = f^{-1}(\{1\})$ for a norm 1 function f in A. One may rechoose f such that $|f| < 1$ on E^c, in which case $f^n \to \chi_E$.

C^*-algebraic variant:

$$u(x) = \chi_{\{1\}}(x) = \mathrm{w}^*\lim_n x^n, \qquad x \in \mathrm{Ball}(A)_+.$$

Note that $u(x)^\perp$ is the support projection $s(1-x)$.

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• There is a calculus (collection of nice algebraic formulae) for these that plays a big role in C^*- and von Neumann algebra theory. This will give a matching calculus of closed ideals (even one-sided ideals), or of hereditary subalgebras (HSA's), of the C^*-algebra. This is huge, in C^*-theory.

Usually though, just as in the function algebra case, we have a fixed subalgebra A of a C^*-algebra, and are interested in peaks for elements of A, particularly for Charles and my real positive elements of A.

• We will give a good application of such noncommutative peak sets to noncommutative Hardy spaces later

Back to the function theory:

Peak interpolation: finding, or building' functions in a uniform algebra $A \subset C(K)$ which have prescribed values or behaviour on a fixed closed subset $E \subset K$ (or on several disjoint subsets).

The sets E that 'work' for this are the p-sets or generalized peak sets.

These have several characterizations e.g. the closed sets whose characteristic functions are in $A^{\perp\perp}$.

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C^*-algebraic variant: Akemann-Brown-Pedersen C^*-algebraic interpolation; Akemann's noncommutative Urysohn lemma, etc.

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C^*-algebraic variant: (Unital case) The closed projections in A^{**} (which are in bijective correspondence with the open projections, or with the closed one-sided ideals/hereditary subalgebras), are just the inf's of the $u(x)$ projections above (resp. sup's of support projections, the 'joins' of singly generated ideals/HSA's). In the separable case (or if a countable cai) these are just the $u(x)$ projections (that is, the peak projections).

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B-Read generalized Glicksberg peak set theorem: Same. So, e.g. the closed projections in $A^{\perp\perp}$ are just the inf's of peak projections $u(x)$ for $x \in A$ real positive.

Nonunital case: Similar.

Immediate applications to theory of one-sided ideals/hereditary subalgebras

Charles and I developed noncommutative peak interpolation, building operators in an operator algebra taking prescribed 'values' on 'noncommutative sets', completing the peaking theory of Hay, B-Hay-Neal.

Part II. Real positivity in operator algebras

Read's theorem again: Any operator algebra with a cai has a cai satisfying $||1 - 2e_t|| \leq 1$.

So the cai is in $B(\frac{1}{2}, \frac{1}{2})$. By taking nth roots one may make it nearly positive.

This generalizes the fact that C-algebras possess a positive contractive approximate identity.

• In [JFA 2011], Charles and I began a study of a new kind of positivity in (not necessarily selfadjoint) operator algebras

... $\mathfrak{F}_A = \{x \in A : \|1 - x\| \leq 1\}$ plays a pivotal role, and the cone $\mathbb{R}^+\mathfrak{F}_A$.

(by a theorem of Meyer, operator algebras have unique unitizations, so 1 aboveis well defined)

$\frac{1}{2}\mathfrak{F}_A$ obviously includes the conventionally positive operators of norm 1

Proposition $\overline{\mathbb{R}^+\mathfrak{F}_A} = \mathfrak{r}_A$ where the latter is the cone of elements with positive real part

These are the real positive elements (or accretives).

Purely metric descriptions, e.g. x is real positive iff $\|1 - tx\| \leq 1 + t^2\|x\|^2$ for all $t > 0$.

... again you can make these 'nearly positive' by taking roots

Charles and I used these to develope a workable theory of positivity in operator algebras. Since operator algebras often have no positive (in the usual sense) elements, it is necessary to redefine positivity by considering our somewhat larger cones to allow for natural and useful theorems.

• We shall see that A has a contractive approximate identity (cai) iff these cones are big–i.e. great abundance of 'positive elements' in new sense

• A main goal of this program is to generalize certain nice C^*-algebraic results, or nice function space results, which use positivity or positive cai's.

• In the theory of C^*-algebras, positivity and the existence of positive approximate identities is crucial.

... Run through C^*-theory, particularly where positivity and positive approximate identities are used, and also where completely positive maps appear, but for operator algebras ... the above is effective at generalizing some parts of the theory, but not others. The worst problem is that although we have a functional calculus, it is not as good. But frequently it is good enough.

• Quite often in a given C^*-subtheory this does not work. But sometimes it does work, or sometimes one has to look a little closer and work a little harder, and this can be quite interesting.

- So we are developing this new notion of positivity in operator algebras. Indeed the ideas make sense and give results in much more general spaces than operator algebras, for example unital operator spaces or Banach algebras. One current direction being pursued is how general can some of our ideas be taken.

- Simultaneously, we are developing applications, for example to noncommutative topology (eg. noncommutative Urysohn and Tietze for general operator algebras), noncommutative peak sets and related noncommutative function theory, noncommutative Hardy spaces, lifting problems, peak interpolation, comparison theory, conditional expectations and projection maps, approximate identities, and to new relations between an operator algebra and the C^*-algebra it generates.

Theorem (Kaplansky density type result) If A is an operator algebra then the ball of $\mathfrak{r}_A$ is weak* dense in the ball of $\mathfrak{r}_{A^{**}}$. Similarly for $\mathfrak{F}_A$.

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One application: to get a 'positive cai' in an algebra with cai: using this Kaplansky density to get a real positive cai by approximating $1_{A^{**}}$ in a standard way.

- This leads to another proof of Read's theorem (stated earlier)

Real positive maps

Real positive maps

Recall that $T : A \to B$ between C^*-algebras (or operator systems) is completely positive if $T(A_+) \subset B_+$, and similarly at the matrix levels

Definition A linear map $T : A \to B$ between operator algebras or unital operator spaces is *real completely positive*, or RCP, if $T(\mathfrak{r}_A) \subset \mathfrak{r}_B$ and similarly at the matrix levels. (Later variant by Bearden-B-Sharma of a notion of B-Read.)

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(Extension and Stinespring-type) Theorem A linear map $T : A \to B(H)$ on an approximately unital operator algebra or unital operator space is RCP iff T has a completely positive (usual sense) extension $\tilde{T} : C^*(A) \to B(H)$

This is equivalent to being able to write T as the restriction to A of $V^*\pi(\cdot)V$ for a $*$-representation $\pi : C^*(A) \to B(K)$, and an operator $V : H \to K$.

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Theorem (A Banach-Stone type result, B-Neal) Suppose that $T : A \to B$ is a completely isometric surjection between approximately unital operator algebras. Then T is real completely positive if and only if T is an algebra homomorphism.

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Theorem: If an approximately unital operator algebra A generates a C^*-algebra B, then A is *order cofinal* in B: given $b \in B_+$ there exists $a \in A$ with $b \preccurlyeq a$. Indeed can do this with $b \preccurlyeq a \preccurlyeq \|b\| + \epsilon$

Indeed can do this with $b \preccurlyeq C e_t \preccurlyeq \|b\| + \epsilon$, for a real positive cai (e_t) for A and scalar C

(This and the next theorem are trivial if A unital)

Order theory in the unit ball

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Here is some order theory in the unit ball of an operator algebra. A feature of the first result is that having the order theory is possible iff there is a cai around

Theorem Let A be an operator algebra which generates a C^*-algebra B, and let $\mathcal{U}_A = \{a \in A : \|a\| < 1\}$. The following are equivalent:

(1) A is approximately unital.

(2) For any positive $b \in \mathcal{U}_B$ there exists real positive a with $b \preccurlyeq a$.

(2') Same as (2), but also $a \in \frac{1}{2}\mathfrak{F}_A$.

(3) For any pair $x, y \in \mathcal{U}_A$ there exist $a \in \frac{1}{2}\mathfrak{F}_A$ with $x \preccurlyeq a$ and $y \preccurlyeq a$.

(4) For any $b \in \mathcal{U}_A$ there exist $a \in \frac{1}{2}\mathfrak{F}_A$ with $-a \preccurlyeq b \preccurlyeq a$.

(5) For any $b \in \mathcal{U}_A$ there exist $x, y \in \frac{1}{2}\mathfrak{F}_A$ with $b = x - y$.

(6) $\mathfrak{r}_A$ is a generating cone (that is, $A = \mathfrak{r}_A - \mathfrak{r}_A$).

• In an operator algebra without any kind of approximate identity there is a biggest subalgebra having good order theory:

Theorem If operator algebra A has no cai then $D = \mathfrak{r}_A - \mathfrak{r}_A$ is the biggest subalgebra with a cai. It is a HSA (*hereditary subalgebra*, that is, $DAD \subset D$).

• We recall that the positive part of the open unit ball $\mathcal{U}_B$ of a C^*-algebra B is a directed set, and indeed is a net which is a positive cai for B. The following generalizes this to operator algebras:

Corollary If A is an approximately unital operator algebra, then $\mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A$ is a directed set in the $\preccurlyeq$ ordering, and with this ordering $\mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A$ is an increasing cai for A.

Corollary If B is a C^*-algebra generated by approximately unital operator algebra A, and $b \in B_+$ with $\|b\| < 1$ then there is a 'nearly positive' increasing cai for A in $\frac{1}{2}\mathfrak{F}_A$, every term of which dominates b (in the $\preccurlyeq$ ordering).

There is a nonselfadjoint 'Tietze' extension theorem, a noncommutative version of:

Theorem Suppose that A is a function algebra on a compact Hausdorff space K, and E is a peak (or p-) set for A. If $f \in A$ with $f(E) \subset F$, where F is closed convex set F in the plane, then there exists a function $g \in A$ which agrees with f on E, which has norm $\|g\|_K = \|f_{|E}\|_E$, and which has range $g(K) \subset F$

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- Essentially a result of Smith et al, and this one generalizes to the case A is a Banach algebra.

Corollary Can lift real positives in quotients A/J to real positives in A (if J is nice)

- Just like in C^*-algebras

- With Ozawa we generalized some of the results above to Banach algebras.

• In addition to the last Tietze theorem, we have a 'real positive version' of the Urysohn lemma.

B-Neal-Read noncommutative Urysohn lemma Let A be an operator algebra (unital for simplicity). Given p, q closed projections in A^{**}, with $pq = 0$ there exists $f \in \mathrm{Ball}(A)$ almost positive and $fp = 0$ and $fq = q$.

• Can also do this with q closed in B^{**}, where B is the containing C^*-algebra, but now need an $\epsilon > 0$ (i.e. f 'close to zero' on p; that is $||fp|| < \epsilon$).

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B-Read Strict noncommutative Urysohn lemma This is the variant where you want f above with also $0 < f < 1$ 'on' $q - p$.

• Generalizes both the topology strict Urysohn lemma, and the Brown-Pedersen strict noncommutative Urysohn lemma.

Part IV. In extremis on Mount Doom

We will gets them in the dead marshes, precious ... Follow Sméagol! He can take you through the marshes, through the mists ... and you may go a long way, quite a long way, before He catches you, yes perhaps.

–The Two Towers, Tolkein

This was basically Charles' email response when I once complained by email that checking his proof so far was like a trip to Mordor, in extremis amongst the perils of Mount Doom.

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Here come the large integers... His mathematical trademark I would think are numbers larger than you would ever believe and growing faster than anyone else would ever dream of.

• One such monster, lurking inside c_0, may be found in our paper on whether, roughly speaking, weak compactness of an operator algebra, or the lack of it, can be seen in the spectra of its elements. It is a singly generated, semisimple commutative operator algebra with a contractive approximate identity, such that the spectrum of the generator is a null sequence and zero, but the algebra is not the closed linear span of the idempotents associated with the null sequence and obtained from the analytic functional calculus. Moreover the multiplication on the algebra is not weakly compact. This is a 'large' operator algebra of orthogonal idempotents, which may be viewed as a dense subalgebra of c_0. In particular a semisimple commutative approximately unital operator algebra with discrete spectrum need not be weakly compact.

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• A story from my time at Leeds (Charles developing very detailed and complicated math ideas internally while externally teaching an undergraduate class). This became his paper THE BIDUAL OF A RADICAL OPERATOR ALGEBRA CAN BE SEMISIMPLE, in which he exhibits a fearsome such example.

Part V. Recent applications

- We are continuing to systematically use the real positive elements in an operator algebra (or other more general spaces) in place of the positives in a C^*-algebra

- For example, one can try to generalize C^*-results which use completely positive maps on C^*-algebras.

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For example: applications to contractive projections on operator algebras (with Matt Neal, 2015, 2016)

Main idea here: Study completely contractive projections P (that is, idempotent linear maps), bicontractive projections, and conditional expectations on operator algebras to find operator algebra generalizations of certain deep results of Choi and Effros, Tomiyama, Størmer, Friedman and Russo, Effros and Størmer, Robertson and Youngson, Youngson, and others, concerning projections and their ranges, assuming in addition that the map is real completely positive

Recent applications (cont.d)

(Arveson's) noncommutative H^∞–for general von Neumann algebras (joint with Louis Labuschagne, ArXiV 2016).

- In several papers B-Labuschagne extended much of the theory of generalized H^p spaces for function algebras from the 1960s to the von Neumann algebraic setting of Arveson's subdiagonal algebras, a.k.a. noncomm. H^∞.

- Subdiagonal algebras are certain unital weak* closed subalgebras A of a von Neumann algebra M, such that there exists a normal conditional expectation $M \to A \cap A^*$ which is multiplicative on A.

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- Earlier, we worked in the setting that M possesses a faithful normal tracial state, as Arveson mostly did too.

• Ueda followed our work by removing a hypothesis involving a dimensional restriction on $A \cap A^*$ in four or five of our results (e.g. F. & M. Riesz and Gleason-Whitney theorems), and also establishing several other beautiful theorems such as the fact that such an A has a unique predual, all of which followed from his very impressive noncommutative peak set type theorem.

• The part of the noncomm. H^∞ theory we focus on in this talk will be generalizing these four or five results (Ueda's theorems plus the improved F. & M. Riesz and Gleason-Whitney theorems) to subalgebras of general von Neumann algebras.

• As before we state the function algebra case of these results first, then the matching von Neumann algebra results, then generalize both.

• $H^\infty(\mathbb{D})$ has a unique predual (Ando-Wojtaszczyk)/von Neumann algebras have unique predual (Dixmier-Sakai)

Related to: Functionals on a von Neumann algebra have a normal plus singular decomposition with $\|\varphi\| = \|\varphi_n\| + \|\varphi_s\|$.

F. & M. Riesz reformulation For any functional φ on $L^\infty(\mathbb{T})$ annihilating $H^\infty(\mathbb{D})$, we have $\varphi_n, \varphi_s \perp H^\infty(\mathbb{D})$.

Gleason-Whitney type theorem Suppose that A is a weak* closed subalgebra of $M = L^\infty(\mathbb{T})$ satisfying the last result. Then $A + A^*$ is weak* dense in M if and only if every normal functional on A has a unique Hahn-Banach extension to M, and if and only if every normal functional on A has a unique normal Hahn-Banach extension to M.

• The main ingredient one may use to prove these is a theorem about peak sets, in the classical case due to Amar and Lederer: 'Any closed set of measure zero is contained in a peak set of measure zero'.

Ueda's (nc Amar-Lederer) peak set result may be phrased as saying that any singular support projection (i.e. the support of any singular state on M), is dominated by a peak projection p for A with p in the 'singular part' of M^{**} (that is, p annihilates all normal functionals on M).

Lemma (Characterization of peak projections for subalgebras of a von Neumann algebra M) A projection q in M^{**} is a peak projection for A if and only if $q \in A^{\perp\perp}$ and $q = \wedge_n q_n$, the infimum in M^{**} of a decreasing sequence (q_n) of projections in M.

- Ueda proved this peak set result in the case that M has a faithful normal tracial state.

(1) We generalize this, and hence all the consequences above, to von Neumann algebras with a faithful state. (2) We also dashed hopes of being able to prove the result in ZFC for all von Neumann algebras (or even commutative ones).

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- We discuss (1) first.

Lemma (Kaplansky density type) Let M be a unital operator space or operator system. Let σ be any linear topology on M weaker than the norm topology. Let X be a subspace of M for which $\mathrm{Ball}(X)$ is dense in $\mathrm{Ball}(M)$ in the topology σ. Then the real positive elements in X are dense in the real positive elements in M in the topology σ.

Theorem (Kaplansky density type) If A is a maximal subdiagonal algebra in a von Neumann algebra M with a faithful state, then $\mathrm{Ball}(A + A^*)$ is weak* dense in $\mathrm{Ball}(M)$. Also, $(A + A^*)_+$ is weak* dense in M_+.

- Uses Haagerup's reduction theory; as does most of the rest of our paper. It becomes very technical.

Ueda's strategy for proving his peak set theorem: A tale of two transforms:

Let φ be a singular state. Then there exist an increasing sequence of projections (q_n) in Ker(φ) with supremum 1. Replacing by a subsequence if necessary, $g = \sum_n n\, q_n^{\perp} \in L^2(M)_+$.

Take the Hilbert transform of g to get an accretive element of noncommutative H^2 with real part g.

The Cayley transform of this gives an element b of A with an (unbounded) inverse, and $a = \frac{1}{2}(1 + b)$ peaks at the desired peak projection p dominating the support projection of φ.

Then p in the singular part of M^{**} since $a^n \to 0$ WOT, which follows because there is no nontrivial subspace on which a acts isometrically. In turn this follows easily because, as we said, b has an (unbounded) inverse.

Theorem Let A be a subdiagonal subalgebra of a von Neumann algebra M with a faithful state. Any singular support projection (i.e. the support of any singular state), is dominated by a peak projection p for A with p in the 'singular part' of M^{**} (that is, p annihilates all normal functionals on M).

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- We then get the same consequences as before (unique predual, F. & M. Riesz and Gleason-Whitney theorems, etc)

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- There is a set theoretic/cardinality obstruction to the Ueda peak set result being true for 'large' von Neumann algebras.

• Henceforth we take $A = M$; but note that if the Ueda peak set result fails then it fails for all subalgebras A of M.

• We noticed that this peak set result being true for all commutative (atomic) von Neumann algebras, implies a solution to a notorious problem in set theory that nobody believes is solvable in ZFC.

Theorem (B-Weaver, 2016) For a von Neumann M TFAE:

(i) Ueda's peak set result holds for M.

(ii) For all singular states φ of M, there is a sequence (q_n) of projections in Ker(φ) with $\vee_n q_n = 1$.

(iii) Every collection of mutually orthogonal projections in M has cardinality bounded by a fixed cardinal κ (such that ...).

Corollary Ueda's theorem fails for $M = l^\infty(\kappa)$ if there exists a countably additive singular state on M.

• Weaver and I also looked at other natural continuity properties for singular states. Similar conditions on states were studied in the context of axiomatic von Neumann algebra quantum mechanics by e.g. L. Bunce and J. Hamhalter.

Quantum cardinals

In set theory there is an elaborate hierarchy of large cardinal properties, some of which involve various notions of measurability. These are related to natural continuity properties for singular measures, which can be viewed as states on $l^\infty(\kappa)$ for the cardinal κ.

It is natural to consider the analogous properties for states on $B(l^2(\kappa))$, and other von Neumann algebras. This is what Weaver and I do. E.g:

Theorem There exists a singular countably additive pure state on $B(l^2(\kappa))$ if and only if κ is (Ulam) measurable.

Some of the proofs make use of a variant of the recent Kadison-Singer solution, and Farah and Weaver's theory of quantum filters, which is also used to prove results such as:

Theorem Every countably additive (on projections) pure state is sequentially weak* continuous.

Adieu, Charles!

As a mathematician Charles' mathematical trademark I would think are huge numbers doing things one would never believe and growing faster than anyone else would ever dream of, you see this in the glorious architecture of so many of his papers. Basically to summarize what I saw about Charles' mathematics in one sentence: he showed us whether to believe a certain thing was true or not, by showing that there are much much bigger things there than our minds had space for in that investigation.

He was not imperfect like all of us, but when not lost in his thoughts he was a kind, warm, and considerate person. In earlier talks you may have caught a glimpse of what his heart was like; and also some of the hurtful treatment he received, particularly as a very very young graduate.

Charles did not favor fine clothing in life, but perhaps now it “will dazzle your eyes to look on them. (I am now excerpting from John Bunyan describing heaven.) There also you shall meet with thousands and ten thousands that have gone before us to that place; none of them are hurtful, but loving and holy; every one walking in the sight of God, and standing in His presence with acceptance for ever. In a word, there we shall see the [...] men, that by the world were cut in pieces [] for the love they bare to the Lord of the place; all well, and clothed with immortality as with a garment.” I think of Charles.

In Toronto a couple of days before his death Charles was excited in his work and in the mathematical atmosphere, seemingly in great spirits and health. We had many meals and coffees together, and happy companionship, the three of us. And he was the same old Charles.

Charles John Read (1958-2015)