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Abstract: We study band-limited versions of classical interpolation issues in the unit disk $\mathbb{D}$ of the complex plane, that generalize Nevanlinna–Pick and Carathéodory–Fejér problems in Hardy spaces H^p of $\mathbb{D}$, $1 \leq p \leq \infty$. We show how they can be linked to a family of bounded extremal problems about which we provide some results.

Key-words: Nevanlinna–Pick and Carathéodory–Fejér interpolation problems, bounded extremal problems, approximation in Hardy spaces

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Interpolation contrainte dans H^p sur des sous–ensembles du cercle

Résumé : Nous étudions des questions d'interpolation de fonctions dans le disque unité $\mathbb{D}$ du plan complexe du type Nevanlinna–Pick et Carathéodory–Fejér mais sur un arc de cercle, dans les espaces de Hardy H^p de $\mathbb{D}$, $1 \leq p \leq \infty$. Nous montrons qu'elles sont reliées à une famille de problèmes extrémaux bornés au sujet desquels nous donnons quelques résultats.

Mots-clés : Problèmes d'interpolation de Nevanlinna–Pick et Carathéodory–Fejér, problèmes extrémaux bornés, approximation dans les espaces de Hardy

1 Introduction

We here consider band-limited versions of Nevanlinna–Pick and Carathéodory–Fejér problems in Hardy spaces H^p of the unit disk $\mathbb{D}$ of the complex plane, $1 \leq p \leq \infty$. They are classically handled in the uniform norm on the whole unit circle $\mathbb{T}$ where questions of approximation by analytic and meromorphic functions are linked to the study of Hankel operators, as is well known to follow from Adamjan-Arov-Krein theory, see [1, 18, 19, 25, 26, 28] among others for further discussions of this. These interpolation issues can also be tackled on the Hilbert space H^2 with the aid of the reproducing kernel [3, 15].

The subject has found many applications in approximation, interpolation, control theory and signal processing. Nehari's theorem, which can be regarded as a special case of the AAK theory, together with associated interpolation theorems such as the Nevanlinna–Pick results, has been a cornerstone of H^∞ control, being linked with model matching problem and similar issues, as described in the book of Francis [16] for instance. Recently further applications have been found in system modelling, including questions of robust identification and model validation, for which we refer to [23] and the bibliography therein.

It is therefore natural that solutions to a generalized version on an arbitrary (measurable) subset of $\mathbb{T}$ of the classical (dual) extremal problem on H^p [14, 17], which we shall refer to as the bounded extremal problem, still provide answers to the band-limited extensions of classical interpolation issues in $\mathbb{D}$ we are interested in.

After stating those problems in section 3, we explain in section 4 how the results of [6, 8], that are stated and whose proofs are briefly sketched in section 5, can be used both to ensure existence and to compute band-limited interpolants. We discuss in section 6 some related bounded completion problems in H^p for $p = 2, \infty$, see [9, 11].

2 Notations and preliminaries

In the following, μ denotes the Lebesgue measure on $\mathbb{T}$. When $E \subset \mathbb{T}$, we write $C(E)$ for the space of continuous complex–valued functions on E while $L^p(E)$ designates the familiar Lebesgue space for $1 \leq p \leq \infty$. The norm on $L^p(E)$ will be the natural one, denoted by $\| \, \|_{L^p(E)}$.

We let $H^p \subset L^p(\mathbb{T})$ be the Hardy space with exponent p on $\mathbb{D}$, consisting of functions with vanishing Fourier coefficients of negative index, and the disc algebra $\mathcal{A} \subset C(\mathbb{T})$ is defined analogously. When $p = 2$, we also introduce the conjugate Hardy space $\bar{H}_0^2$ which is the orthogonal complement to H^2 in $L^2(\mathbb{T})$, that is to say the subspace of functions with vanishing Fourier coefficients of non-negative index.

In a normed space, we write $d(\phi, S)$ for the distance of the element ϕ to the subset S; we use the same notation regardless of which space we are working in, but the context will keep the meaning clear. The subscript $|_E$ applied to a function or to a set of functions indicates restriction to E; for instance, $H^p_{|_E}$ is the space of traces on E of H^p functions. Whenever f is defined on E and h is defined on its complement $\mathbb{T} \setminus E$, then $f \vee h$ stands for the concatenated function which is defined on all of $\mathbb{T}$.

3 Band-limited interpolation and approximation problems

Let K be a measurable subset of $\mathbb{T}$, and let $1 \leq p \leq \infty$ be fixed. We are interested in the following version of the Nevanlinna–Pick problem:

Problem 3.1 *Given $M \geq 0$, points $z_1, \ldots, z_n \in \mathbb{D}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, find $F \in H^p$ such that $\|F\|_{L^p(\mathbb{T}\setminus K)} \leq M$, $F(z_k) = \alpha_k$ for $k = 1, 2, \ldots, n$, and $\|F\|_{L^p(K)}$ is minimized.*

We also consider a band-limited Carathéodory–Fejér problem:

Problem 3.2 *Given $M \geq 0$ and coefficients $a_0, \ldots, a_{n-1} \in \mathbb{C}$, find $F \in H^p$ such that*

$$F(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + O(z^n),$$

$\|F\|_{L^p(\mathbb{T}\setminus K)} \leq M$, and $\|F\|_{L^p(K)}$ is minimized.

Solutions to problems 3.1 and 3.2 will be deduced from that of the following bounded extremal problem (BEP) where we are given an L^p function on K which we want to approximate by traces of H^p functions that meet some gauge outside K:

Problem 3.3 *For $\psi \in L^p(\mathbb{T} \setminus K)$, $M \geq 0$, define*

$$\mathcal{B}_{M,\psi} := \{g \in H^p\,,\ \|g - \psi\|_{L^p(\mathbb{T}\setminus K)} \leq M\}\,.$$

Given $f \in L^p(K)$, we seek $g_0 \in \mathcal{B}_{M,\psi}$ such that

$$\|f - g_0\|_{L^p(K)} = \min_{g \in \mathcal{B}_{M,\psi}} \|f - g\|_{L^p(K)} := \beta(f, \psi, M). \tag{1}$$

For simplicity, we will note $\beta = \beta(f, \psi, M)$, the dependence being kept clear from the context.

If $K = \mathbb{T}$, then $\mathcal{B}_{M,\psi} = H^p$ and the bounded extremal problem reduces to a standard (dual) extremal approximation (see [14, chap. 8], [17, chap. IV], [21, chap. VII]); it always admits a solution which is unique if $1 \leq p < \infty$ or if $f \in H^\infty + C(\mathbb{T})$ (in the latter case, the error is circular: $|f - g_0| = \|f - g_0\|_{L^\infty(\mathbb{T})}$ a.e. on $\mathbb{T}$.

For $K \subset \mathbb{T}$, BEP was solved for the case $p = \infty$ in [8]; for $1 \leq p < \infty$ this problem was discussed in [6], and an explicit solution given for $p = 2$. Special cases of the L^2 problem (where either f or ψ is identically zero) were solved in [4] and [22]. We describe in section 5 some of those results that essentially extend those that are available on the whole $\mathbb{T}$.

4 Extension of some classical interpolation results

In this section we apply our results of [4, 6, 8] about problem 3.3 (see section 5) in order to solve for problems 3.1 and 3.2.

Theorem 4.1 *Both problems* 3.1 *and* 3.2 *can be reduced to the bounded extremal problem* 3.3 *and admit a solution (unique up to a multiplicative constant of modulus 1) which saturates the constraint:*

$$\|F\|_{L^p(\mathbb{T}\setminus K)} = M\,.$$

Moreover, when $p=\infty$, it satisfies

$$|F(e^{it})| = \begin{cases} \beta & a.e.\ for\ e^{it} \in K, \\ M & a.e.\ for\ e^{it} \in \mathbb{T}\setminus K, \end{cases}$$

where β is the minimum achievable value of $\|F\|_{L^\infty(K)}$.

Proof: the set of all functions F satisfying the conditions of problem 3.1 is easily seen to be parametrized as

$$F = p + Bg\,,$$

where p is any polynomial such that $p(z_k) = \alpha_k$ for each k, B is a finite Blaschke product with zeroes at $z_1, \ldots, z_n$, and $g \in H^p$.

Now to minimize $\|p+Bg\|_{L^p(K)}$ under the constraint that $\|p+Bg\|_{L^p(\mathbb{T}\setminus K)} \le M$ is equivalent to minimizing $\|B^{-1}p+g\|_{L^\infty(K)}$ under the constraint that $\|B^{-1}p+g\|_{L^\infty(\mathbb{T}\setminus K)} \le M$. This is just bounded extremal problem 3.3, with the conditions $f = \psi = -B^{-1}p$. Since $f \vee \psi = -B^{-1}p$ belongs to $C(\mathbb{T})$ but does not belong to H^p (and then not to any $\mathcal{B}_{M,\psi}$), the assertions now follow from theorems 5.1 and 5.4 below.

Concerning problem 3.2, the proof is the same if we note that the set of solutions to the interpolation condition is $F = p + z^n g$, where $p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}$ and $g \in H^\infty$ and then take $f = \psi = -z^{-n}p$ in problem 3.3. ■

When $p = 2, \infty$, constructive solutions to problems 3.1 and 3.2 are thus provided by theorems 5.5 and 5.3 since F can be computed from the solution to some BEP. In both cases, the relation of F to the constraint outside K (saturation) will be implicitely given, via a Lagrange-type parameter. Our resolution algorithm when $p=2$ is based on iterative computations of the resolvant of some Toeplitz operator while, for $p=\infty$, it goes through a singular value decomposition of some Hankel operator, see more details in section 5.

More simple is the following solution to an extended form of Pick's problem. For each $\rho > 0$ let w_ρ be the outer factor whose modulus is ρ on K and 1 on $\mathbb{T}\setminus K$:

$$w_\rho(z) = \exp\left\{\frac{1}{2\pi}\log\rho \int_K \frac{e^{it}+z}{e^{it}-z}\,dt\right\}. \tag{2}$$

Proposition 4.1 *Given $M \geq 0$, points $z_1, \ldots, z_n \in \mathbb{D}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, a necessary and sufficient condition for the existence of a function $F \in H^\infty$ such that $F(z_k) = \alpha_k$ for $1 \leq k \leq n$, $\|F\|_{L^\infty(K)} \leq 1$ and $\|F\|_{L^\infty(\mathbb{T}\setminus K)} \leq M$ is that the matrix*

$$S = \left(\frac{1 - \bar{\beta}_k \beta_k}{1 - \bar{z}_j z_k} \right)_{j,k=1}^{n}$$

be positive semi-definite, where $\beta_j = \alpha_j w_M(z_j)/M$.

Proof: this reduces to the classical Pick problem (i.e. the version with $M = 1$) for the function $G(z) = F(z) w_M(z)/M$. The solution to this problem [23, 26] gives a solution to the original problem. ■

Finally the following result is also a straightforward extension of a known result.

Proposition 4.2 *Given $M \geq 0$ and coefficients $a_0, \ldots, a_{n-1} \in \mathbb{C}$, there is a function $F \in H^\infty$ such that*

$$F(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + O(z^n),$$

with $\|F\|_{L^\infty(K)} \leq 1$ and $\|F\|_{L^\infty(\mathbb{T}\setminus K)} \leq M$, if and only if $I - B^* B \geq 0$, where B is the matrix

$$B = \frac{1}{M} \begin{pmatrix} w_0 & 0 & \ldots & 0 \\ w_1 & w_0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ w_{n-1} & w_{n-2} & \ldots & w_0 \end{pmatrix} \begin{pmatrix} a_0 & 0 & \ldots & 0 \\ a_1 & a_0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \ldots & a_0 \end{pmatrix},$$

and $w_0, \ldots, w_{n-1}$ are the first n Taylor coefficients of $w_M(z)$.

The proof goes as the one of proposition 4.1.

Concerning the problems of this section, criteria and constraints of the more general form $\|F - \chi\|_{L^p(K)}$ and $\|F - \phi\|_{L^p(\mathbb{T}\setminus K)} \leq M$ for $\chi \in L^p(K)$, $\phi \in L^p(\mathbb{T} \setminus K)$ can be handled as well. The details are similar.

5 Bounded extremal problems

Let us now review the solutions to problem 3.3. Although we provide here almost no details about the proofs, let us mention that they involved arguments that are based on weak-* compactness of balls in H^p and on convexity properties of the considered classes of approximants $\mathcal{B}_{M,\psi|_K}$. As usual, they different according to the value of p. For $1 < p < \infty$, the uniform convexity of L^p plays the role of the duality mapping between extremal problems on the whole $\mathbb{T}$ [17, ch. IV] since it provides, even when $K \subset \mathbb{T}$, a best approximation projection onto $\mathcal{B}_{M,\psi|_K}$ [13, 3.II.1].

5.1 When $p = \infty$

We begin with the L^∞ case and summarize some results. The first one answers existence and uniqueness issues about BEP.

Theorem 5.1 ([8, thm 2]) *Suppose that $\mu(\mathbb{T} \setminus K) > 0$ and that f is not the trace of a function in $\mathcal{B}_{M,\psi}$ (so that $\beta = \beta(f, \psi, M) > 0$). Then:*
(i) A solution g_0 exists, provided that $\mathcal{B}_{M,\psi} \neq \emptyset$.
(ii) Unless $\beta = d(f, H^\infty_{|K})$, then any solution g_0 satisfies $\|\psi - g_0\|_{L^\infty(\mathbb{T}\setminus K)} = M$.
(iii) The solution g_0 is unique at least when $f \vee \psi$ lies in $H^\infty + C(\mathbb{T})$, and in this case the functions $f - g_0$ and $\psi - g_0$ have constant modulus β and M a.e. on K and $\mathbb{T} \setminus K$ respectively.

Observe that the set $\mathcal{B}_{M,\psi}$ of approximants could indeed be empty; this is the case for example if ψ is taken to be the inverse of some H^∞ function whose zeros accumulate at some interior point of $\mathbb{T} \setminus K$ and if M is large enough, see [8, lem. 1].

However, the following density result – which is of importance by itself – provides sufficient conditions in order to ensure $\mathcal{B}_{M,\psi} \neq \emptyset$:

Theorem 5.2 ([8, thm 1]) *Let K be a subset of $\mathbb{T}$ such that $\mu(K) > 0$. Then:*
(i) $H^\infty_{|K}$ is not dense in $L^\infty(K)$.
(ii) If K is open, the closure of $H^\infty_{|K}$ in $L^\infty(K)$ is contained in $H^\infty_{|K} + C(K))$.
(iii) If K is a proper closed subset of $\mathbb{T}$, then $\mathcal{A}_{|K}$ is dense in $C(K)$, and the closure of $H^\infty_{|K}$ in $L^\infty(K)$ contains $(H^\infty + C(\mathbb{T}))_{|K}$.

Thus, for continuous ψ and arbitrary $M > 0$, $\mathcal{B}_{M,\psi} \neq \emptyset$.

Moreover, for continuous f (although not in general), we may then find analytic functions g_n approximating f arbitrarily closely on K, although in this case $\|g_n\|_{L^\infty(\mathbb{T})} \to \infty$ if f is not already the trace of an analytic function. In this case, this amounts to the fact that $\beta \to 0$ when $M \to \infty$ and shows that problem 3.3 is ill-posed without a constraint on $\mathbb{T} \setminus K$, since the infimum (equal to zero) is not achieved in $H^\infty_{|K}$.

Observe that this provides a partial answer to the issue, unsolved to our knowledge, of characterizing the closure of $H^\infty_{|K}$ in $L^\infty(K)$: it follows from points (ii), (iii) of theorem 5.2 that its restriction to every compact subset of K is equal to the restriction of $H^\infty + C(\mathbb{T})$.

A constructive way of solving for problem 3.3 for $p = \infty$ is given by the following result which also serves in establishing uniqueness and error circularity properties (iii) in theorem 5.1. Recall that w_ρ is defined by (2) for $\rho > 0$.

Theorem 5.3 ([8, thm 4]) *Under the hypotheses of theorem 5.1, let $v_0 \in H^\infty$ solve the Nehari problem*

$$\|(f \vee \psi) w_{M/\beta} - v_0\|_{L^\infty(\mathbb{T})} = \min_{v \in H^\infty} \|(f \vee \psi) w_{M/\beta} - v\|_{L^\infty(\mathbb{T})}, \qquad (3)$$

Then $g_0 = v_0 w_{\beta/M}$ *is a solution to problem* 3.3*. Conversely any solution to problem* 3.3 *gives rise to a solution* $v_0 = g_0 w_{M/\beta}$ *to* (3)*.*

Whenever $f \vee \psi \in C(\mathbb{T})$, this provides us with an implicit scheme (recall that β is equal to the error in BEP); however, the right value for β is the one that makes the value of (3) equal to 1. Now, the error in this Nehari problem is well-known to coincide with the largest singular value of the Hankel operator $\mathcal{H}$ with symbol $(f \vee \psi)\, w_{M/\beta} \in H^\infty + C(\mathbb{T})$:

$$\begin{array}{rcl} \mathcal{H} : H^2 & \to & \bar{H}_0^2 \\ h & \mapsto & P_{\bar{H}_0^2}((f \vee \psi)\, w_{M/\beta}\, h)\,, \end{array}$$

where $P_{\bar{H}_0^2}$ is the usual orthogonal projection, and this permits us to compute β by dichotomy. The solution v_0 is then given by the associated Schmidt pair. Precise convergence properties of these approximation schemes are established in [9] for more general problems.

5.2 For $1 \le p < \infty$

In the L^p case for $1 \le p < \infty$, the following results about problem 3.3 are available.

Theorem 5.4 ([6, thm 2], [8, prop. 1]) *Let* p *be fixed with* $1 \le p < \infty$*, and let* K *be a subset of* $\mathbb{T}$ *such that* $\mu(\mathbb{T} \setminus K) > 0$*. Then* $H^p_{|K}$ *is dense in* $L^p(K)$*. Hence* $\mathcal{B}_{M,\psi} \neq \emptyset$ *if* $\mu(K) > 0$*. Thus, BEP admits a unique solution* g_0 *which satisfies*

$$\|\psi - g_0\|_{L^p(\mathbb{T}\setminus K)} = M\,,$$

provided that f *is not the trace of a function in* $\mathcal{B}_{M,\psi}$ *(so that* $\beta = \beta(f, \psi, M) > 0$*).*

The above density result is a consequence of the F. and M. Riesz theorem [20, ch. 4] and of the injective property on H^p of the restriction map on subsets of $\mathbb{T}$ of positive measure. Observe the contrast with the *non-density* result stated in theorem 5.2.

For $p = 2$ it is possible to provide an "explicit" solution to problem 3.3, of which special cases were treated in [4, 22] and that we now describe. Let T denote the Toeplitz operator with symbol $\chi_{\mathbb{T}\setminus K}$, that is

$$\begin{array}{rcl} T : H^2 & \to & H^2 \\ h & \mapsto & P_{H^2}(\chi_{\mathbb{T}\setminus K}\, h), \end{array} \tag{4}$$

where P_{H^2} is the usual orthogonal projection. Note that the spectrum of T is contained in $[0, 1]$.

Theorem 5.5 ([6, thm 4, cor. 1]) *For* $p = 2$ *and* f *not the trace of a function in* $\mathcal{B}_{M,\psi}$*, the solution to BEP is given by the implicit equation*

$$g_0 = (1 + \lambda\, T)^{-1}\, P_{H^2}(f \vee (\lambda + 1)\, \psi), \tag{5}$$

where $\lambda \in (-1, +\infty)$ is the unique real number such that

$$\|g_0 - \psi\|_{L^2(\mathbb{T}\setminus K)} = M\,. \tag{6}$$

It can also be expressed as

$$g_0(z) = \frac{1}{2\pi i}\int_K \left(\frac{w_\rho(\xi)}{w_\rho(z)}\right)^\alpha (f \vee \psi)(\xi)\,\frac{d\xi}{\xi - z}, \qquad (z \in \mathbb{D}),$$

where w_ρ is the "quenching" function defined in (2) and real valued parameters $\rho > 0$, $\alpha \in \mathbb{R}$ linked by $\rho^{2\alpha} = 1/(\lambda+1)$.

The last Carleman-type integral is closely related to the recovery formula of Patil [24] for H^p functions, see also [2].

A consequence of theorems 5.4 and 5.5 is that the error β smoothly decreases to 0 as $\lambda \to -1$ while M grows to ∞; we are then able to solve for BEP using a procedure that iterates computations of the resolvent $(1+\lambda T)^{-1}$ until we get the value of λ that ensures (6).

Finally, when $2 < p < \infty$, an algorithm for solving extremal problems on H^p is proposed in [12, 27] and may be used to compute solutions to BEP in these cases. Indeed, it easily turns out that they can always be equivalently formulated as problems on the whole $\mathbb{T}$ for functions that involve f, ψ and M (this is asserted by theorem 5.3 for $p = \infty$).

6 Bounded completion problems

Companion to problem 3.3 is the bounded completion problem (BCP) 6.1 that was studied in [9, 11], for $p = \infty$ and, with $\psi = 0$, for $p = 2$.

Problem 6.1 *For $\psi \in L^p(\mathbb{T}\setminus K)$ and $L \geq 0$, let $\mathcal{D}_{L,\psi}$ be the ball of radius L centred at ψ:*

$$\mathcal{D}_{L,\psi} := \{h \in L^p(\mathbb{T}\setminus K),\ \|h - \psi\|_{L^p(\mathbb{T}\setminus K)} \leq L\}\,.$$

Given $f \in L^p(K)$, we seek $h_0 \in \mathcal{D}_{L,\psi}$ such that

$$d(f \vee h_0, H^p) = \min_{h \in \mathcal{D}_{L,\psi}} d(f \vee h, H^p) := \gamma(f, \psi, L)\,.$$

The bounded completion problem 6.1 is close in spirit to the bounded extremal problem 3.3: in words, we are given an L^p function on a subset K of the circle and we seek an extended definition to the whole circle that meets some gauge outside K, and makes the global function as close to an analytic function as possible. A solution to problem 6.1 can be used as a suitable behaviour when solving for problem 3.3.

Again, we simplify the notation into $\gamma(f, \psi, L) = \gamma(L)$ and we begin with the L^∞ case.

Theorem 6.1 ([9, thm 3]) *Let $f \in L^\infty(K)$, $\psi \in L^\infty(\mathbb{T} \setminus K)$, and $L \geq 0$. Set*

$$M = L + \gamma(f, \psi, L).$$

Then $\mathcal{B}_{M,\psi} \neq \emptyset$ and $\beta(M) \leq \gamma(L)$. If $f \notin \mathcal{B}_{L,\psi}$ (so that $\gamma(L) > 0$ and $M > 0$) and if g_0 is a solution to problem 3.3, then

$$h_0 = \frac{L}{M} g_0 + \left(1 - \frac{L}{M}\right) \psi \tag{7}$$

is a solution to problem 6.1, and

$$\|f \vee h_0 - g_0\|_{L^\infty(\mathbb{T})} = \gamma(f, \psi, L).$$

Thus the bounded completion problem 6.1 in L^∞ reduces to the bounded extremal problem 3.3, which in turn reduces to the Nehari problem 3.

Moreover, if K has an interior point, and $f \vee \psi \in H^\infty + C(\mathbb{T})$, then the solution is unique, and, if f is not the trace of a function in $\mathcal{B}_{L,\psi}$, then $|\psi - h| = L$ a.e. on $\mathbb{T} \setminus K$.

An L^2 version of this problem was solved in [11], for the case $\psi = 0$, but a similar proof extends to general ψ and we establish the more general result now. We use the notation of problem 6.1.

Theorem 6.2 *Let $f \in L^2(K)$ and $\psi \in L^2(\mathbb{T} \setminus K)$. Then for every $L > 0$ there is a unique function $h_0 \in \mathcal{D}_{L,\psi}$ such that*

$$d(f \vee h_0, H^2) = \min_{h \in \mathcal{D}_{L,\psi}} d(f \vee h, H^2).$$

Moreover $\|h_0 - \psi\|_{L^2(\mathbb{T}\setminus K)} = L$, unless f is already the trace on K of an H^2 function h such that $\|h - \psi\|_{L^2(\mathbb{T}\setminus K)}$ is less than L. One can characterize h_0 by

$$h_0 = (1 + \lambda)\psi - \lambda(I + \lambda T)^{-1} P_{H^2}(f \vee (1 + \lambda)\psi), \tag{8}$$

where T is the Toeplitz operator defined in (4), and $\lambda > -1$ a constant such that $\|h_0 - \psi\|_{L^2(\mathbb{T}\setminus K)} = L$.

Proof: the key observation is that, writing $k_0 = h_0 - \psi$, we have

$$\mathrm{Re}\,(P_{\overline{H_0^2}}(f \vee (k_0 + \psi)), u)_{L^2(\mathbb{T}\setminus K)} = 0$$

for all u in the tangent space to the sphere

$$S_L = \{k \in L^2(\mathbb{T} \setminus K) : \|k\|_{L^2(\mathbb{T}\setminus K)} = L\}.$$

As in [11], this leads us to conclude that there exists $\mu \in \mathbb{R}$ such that

$$P_{\overline{H_0^2}}(f \vee (k_0 + \psi)) = \mu k_0 \qquad \text{a.e. on } \mathbb{T} \setminus K.$$

Equivalently, writing $\rho = 1 - \mu$, we have

$$P_{H^2}(f \vee (k_0 + \psi) = \psi + \rho k_0 \qquad \text{a.e. on } \mathbb{T} \setminus K,$$

that is, $\psi + \rho k_0$ is the restriction of an H^2 function g with

$$P_{H^2}(f \chi_K + \rho^{-1} g \chi_{\mathbb{T}\setminus K} + (1 - \rho^{-1}) \psi \chi_{\mathbb{T}\setminus K}) = g,$$

or $(I - \rho^{-1} T) g = P_{H^2}(f \vee (1 - \rho^{-1}) \psi)$. Finally we write $\lambda = -1/\rho > -1$, and solve for $h_0 = \psi + (g - \psi)/\rho$. ■

In view of (5) and (8), we also get that

$$h_0 = (1 + \lambda)\, \psi - \lambda\, g_0(f, \psi, L/|\lambda|)\,, \tag{9}$$

for the solution $g_0(f, \psi, L/|\lambda|)$ to problem 3.3 associated to f and such that $\|g_0 - \psi\|_{L^2(\mathbb{T}\setminus K)} = L/|\lambda|$. This is to be compared to (7).

For $p = 2, \infty$, it follows from equations (7), (9) that $h_0 \in H^p_{|\mathbb{T}\setminus K}$ as soon as $\psi \in H^p_{|\mathbb{T}\setminus K}$, which is somewhat unexpected; this can be put in regard to the density / non-density properties of theorems 5.2 and 5.4. We do not know if this still holds for other values of p and what kind of relation should then remain, if any, between BEP and BCP.

7 Conclusion

We used here solutions to bounded extremal and completion problems 3.3 and 6.1 in order to solve for band-limited discrete interpolation issues in H^p.

When $p = \infty$, it is possible to formulate and solve more general extremal problems than problems 3.3 and 6.1, where we replace the set H^∞ by the set $H^\infty + R_N$ of functions that are meromorphic in the disc with at most N poles and are bounded in $\{z : r < |z| < 1\}$ for some $r < 1$, and where the constraints M and L are no longer constant. We refer to [9] for details.

When $p = 2$, the asymptotic behaviour of the constraints M and L as the parameter λ approaches -1 has been recently investigated in [5] where sharp estimates are established.

Some applications of these problems to the identification of linear time-invariant systems can be found in [4, 10, 23]. A further application to the study of Dirichlet–Neumann problems is given in [7].

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