MATH 4020 Banach Spaces and Algebras

Some useful results and formulae II

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Please let me know of any errors, omissions or obscurities!

Baire's theorem. Let X be a complete metric space and (U_n) a sequence of dense open sets in X. Then $\bigcap_{n=1}^\infty U_n$ is also dense. Recall: U is *dense*, if $\overline{U} = X$, i.e., U meets every nontrivial open set, i.e., $X \setminus U$ contains no nontrivial open set, i.e., $X \setminus U$ has empty interior.

Corollary of Baire. If $X = \bigcup_{n=1}^\infty F_n$, where the F_n are closed, then one of them has nonempty interior. Terminology: Y is a *nowhere dense* subset of X, when $\operatorname{int} \overline{Y} = \emptyset$. It is *meagre (first category)* when it is a countable union of nowhere dense sets, and *non-meagre (second category)* otherwise. So a non-empty complete metric space is non-meagre in itself.

Uniform boundedness. Let $\mathcal{S} \subset B(X,Y)$, where X is a Banach space. If $\{\|Tx\| : T \in \mathcal{S}\}$ is bounded for all $x \in X$, then $\sup\{\|T\| : T \in \mathcal{S}\} < \infty$. (Pointwise bounded implies uniformly bounded.) So if $\mathcal{S} \subset X$ and $\{f(x) : x \in \mathcal{S}\}$ is bounded for all $f \in X^*$, then $\mathcal{S}$ is bounded (weakly bounded implies norm bounded).

Open mapping theorem. Let X, Y be Banach spaces and $T : X \to Y$ a bounded linear surjection. Then T takes open sets to open sets. *Corollary (Banach's isomorphism theorem).* If T is in fact a bijection, then T^{-1} is also bounded, so T is an isomorphism.

Closed graph theorem. For $T : X \to Y$ linear, with X, Y Banach spaces, let $G(T) = \{(x, Tx) : x \in X\}$, a subspace of $X \oplus Y = \{(x,y) : x \in X, y \in Y\}$, with norm $\max\{\|x\|, \|y\|\}$. Then T is bounded if and only if $G(T)$ is closed. So to check whether T is continuous, it is enough to check that whenever we have $x_n \to x$ and $Tx_n \to y$ then it follows that $y = Tx$.

Banach Algebras. These are Banach spaces A with a multiplication defined, such that $\|ab\| \le \|a\|\,\|b\|$, and algebraic properties $a(bc) = (ab)c$, $a(b+c) = ab + ac$, $(a+b)c = ac + bc$, $\lambda(ab) = (\lambda a)b = a(\lambda b)$, for a, b, $c \in A$ and $\lambda \in \mathbb{F}$. *Commutative* if $ab = ba$ always, and *unital* if there is a unit (identity) e (sometimes written I or 1) such that $ea = ae = a$ for all a, and we assume also $\|e\| = 1$. *Examples:* $B(X)$, $C(K)$, L_∞, c_0, c, ℓ_∞, H_∞ (bounded analytic functions on the unit disc $\mathbb{D}$ with supremum norm), $A(\mathbb{D})$ (disc algebra), which is $C(\overline{\mathbb{D}}) \cap H_\infty$. *Wiener algebra W* of Fourier series $\sum_{n=-\infty}^\infty a_n e^{in\theta}$ with norm $\sum_{n=-\infty}^\infty |a_n| < \infty$. We get the positive part

W_+ by restricting to functions $\sum_{n=0}^{\infty} a_n e^{in\theta}$, and this is isometric to ℓ_1. Hence ℓ_1 is also a Banach algebra, where the multiplication is convolution $(a_n) * (b_n) = c_n$ and $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, as in power-series multiplication.

Spectrum. Let A be a *complex* unital Banach algebra, and $x \in A$. Then $\sigma(x) = \{\lambda \in \mathbb{C} : (x - \lambda e)$ has no inverse in $A\}$ (an *inverse* to y is any z such that $yz = zy = e$). *Spectral radius.* $r(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}$. *Properties:* $\sigma(x)$ is a non-empty compact set, contained in the closed disc about 0 of radius $\|x\|$. Ideas of proof: if $\|x\| < 1$, then $(1-x)^{-1} = 1+x+x^2+\ldots$; if y and z are invertible, then $(yz)^{-1} = z^{-1}y^{-1}$; for $\phi \in A^*$, $F(\lambda) = \phi((x - \lambda e)^{-1})$ is analytic (for $\lambda \notin \sigma(x)$) and goes to 0 at ∞. Use Liouville's theorem (an analytic function bounded on $\mathbb{C}$ is necessarily constant).

Gelfand–Mazur theorem. If A is a complex unital Banach algebra and also a field (i.e., all nonzero elements have inverses), then A is isometric to $\mathbb{C}$ (one-dimensional).

Subalgebras of a Banach algebra. These are linear subspaces B such that b_1, $b_2 \in B$ implies $b_1 b_2 \in B$. So if B is closed it is also a Banach algebra. Examples: $c_0 \subset c \subset \ell_\infty$, $A(\mathbb{D}) \subset H_\infty$, $W_+ \subset W$, (finite rank operators) $\subset B(X)$.

Stone–Weierstrass theorem. Real case. Let K be compact Hausdorff, A a subalgebra of real $C(K)$. Suppose that A separates points (i.e., whenever $x \neq y \in K$ we can find $f \in A$ with $f(x) \neq f(y)$); then its closure is either $C(K)$ or $\{f \in C(K) : f(k_0) = 0\}$ for some $k_0 \in K$. So, if in addition we know the constants are in A, then A is dense in $C(K)$. *Proof* uses two lemmas: (1) If A is a closed subalgebra, then $\max(f, g)$ and $\min(f, g)$ are in A whenever f and g are; (2) For any subset A closed under taking max and min, its norm closure is the set of functions f such that, for any $\epsilon > 0$ and x, $y \in K$, there is a function $f_{xy} \in A$ with $|f_{xy}(x) - f(x)| < \epsilon$ and $|f_{xy}(y) - f(y)| < \epsilon$ (two-point approximation).

Complex case. We need more (consider $\bar{z}$, not a limit of polynomials in $C(\overline{\mathbb{D}})$). The extra condition is that whenever $f \in A$ we also have $\overline{f} \in A$.

Ideals and Gelfand theory. For A a complex commutative unital Banach algebra, $I \subseteq A$ is an *ideal* if it's a subalgebra and $ax \in I$ whenever $a \in A$ and $x \in I$. A *maximal ideal* is a proper ideal (not A), not contained in any bigger proper ideal. They are closed. A *character* on A is a linear map $\delta : A \to \mathbb{C}$ with $\delta(e) = 1$ and $\delta(xy) = \delta(x)\delta(y)$. So the *character space* Δ_A is a subset of A^*. Every maximal ideal is the kernel of some character, and vice-versa. An element $x \in A$ is invertible if and only if $\delta(x) \neq 0$ for all $\delta \in \Delta_A$. Hence $\lambda \in \sigma(x)$ if and only if $\delta(x) = \lambda$ for some $\delta \in \Delta_A$. For $C(K)$, characters are evaluation at points of K; for W, W_+ and $A(\mathbb{D})$, they are evaluations at points of the unit circle/disc/disc respectively.