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# MATH 3214 Fourier Analysis: Some useful results and formulae (2)

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*Please let me know of any errors, omissions or obscurities!*

*Riemann–Lebesgue lemma for Fourier series.* If $f : \mathbb{R} \to \mathbb{C}$ is continuous and $2\pi$-periodic, then $\hat{f}(n) \to 0$ as $n \to \infty$.

*Uniqueness theorem.* If $f$, $g$ are continuous and $2\pi$-periodic with $\hat{f}(n) = \hat{g}(n)$ for all $n \in \mathbb{Z}$, then $f = g$.

*Approximation (1).* For any $f : \mathbb{R} \to \mathbb{C}$ continuous and $2\pi$-periodic and $\epsilon > 0$ there is a trigonometric polynomial $p$ such that $|f(x) - p(x)| < \epsilon$ for all $x$. (The trigonometric polynomials are *dense* in the continuous periodic functions.)

*Approximation (2).* For any continuous $f : [a, b] \to \mathbb{C}$ and $\epsilon > 0$ there is a polynomial $q$ (in the usual sense) such that $|f(x)-q(x)| < \epsilon$ on $[a, b]$. (The *Weierstrass approximation theorem*: the polynomials are dense in the continuous functions on $[a, b]$.)

*Complex inner product notation.* For $f$, $g$, Riemann integrable on $[a, b]$ define $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$ and $\|f\|^2 = \langle f, f\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx$. Then the inner product is linear in the first argument, conjugate-symmetric, and conjugate-linear in the second argument. Write $e_n(x) = e^{inx}$, $n \in \mathbb{Z}$. Note that $\langle f, e_n\rangle = \hat{f}(n)$, and $s_k(f) = \sum_{n=-k}^{k} \langle f, e_n\rangle e_n$.

*Mean square convergence.* For $f : \mathbb{R} \to \mathbb{C}$ continuous we have $\|f - s_k(f)\|^2 \to 0$ as $k \to \infty$.

*Properties of the $(e_n)$.* (i) $\left\|\sum_{n=-k}^{k} c_n e_n\right\|^2 = \sum_{n=-k}^{n} |c_n|^2$.

(ii) $\left\|f - \sum_{n=-k}^{k} \langle f, e_n\rangle e_n\right\|^2 = \|f\|^2 - \sum_{n=-k}^{k} |\langle f, e_n\rangle|^2$.

*Bessel's inequality.* $\sum_{n=-k}^{k} |\hat{f}(n)|^2 \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx$, so if $f$ is square-integrable then $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 < \infty$.

*Best approximation.* For any scalars $(c_n)$, we have
$\left\|f - \sum_{n=-k}^{k} c_n e_n\right\|^2 = \left\|f - \sum_{n=-k}^{k} \langle f, e_n\rangle e_n\right\|^2 + \sum_{n=-k}^{k} |c_n - \langle f, e_n\rangle|^2$. Thus taking $c_n = \langle f, e_n\rangle$ minimizes the mean-square error.

*Parseval's identity.* For $f$ continuous, $2\pi$-periodic, $\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx$.

*Absolutely convergent Fourier series.* Parseval implies a stronger result, namely, every $f$ that is $C^1$ has an a.c.F.s.

*Gibbs phenomenon.* For the sawtooth function $f(x) = x$ on $(-\pi, \pi]$, $(s_k f)(\pi - \pi/k) \to 2\int_0^{\pi} \frac{\sin x}{x}\,dx \approx 1.18\pi$.

*Du Bois Reymond theorem.* There is a continuous $2\pi$-periodic function such that $s_k f$ does not converge at 0, indeed the sequence of values is unbounded. Note that $(s_k f)(0) = f * D_k(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_k(t)\,dt$, and $\frac{1}{2\pi}\int_{-\pi}^{\pi} |D_k(x)|\,dx \ge \frac{4}{\pi^2}\ln k$.

*Fourier transforms.* Assume always that $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$, and that $f$ is Riemann integrable over every finite interval, implying that $\int_{-\infty}^{\infty} f(x)\,dx$ converges. Define $\hat{f}(w) = \int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx$ for $w \in \mathbb{R}$.

*Important examples.* If $f(x) = e^{-x^2}$, then $\hat{f}(w) = \sqrt{\pi}e^{-w^2/4}$. If $f(x) = \chi_{[-1,1]}(x)$, then

$\hat{f}(w) = 2\,\mathrm{sinc}\,w = \begin{cases} \frac{2\sin w}{w} & \text{if } w \neq 0, \\ 2 & \text{if } w = 0. \end{cases}$

*Linearity.* $(af+bg)\hat{}(w) = a\hat{f}(w) + b\hat{g}(w)$ for $a$, $b \in \mathbb{C}$.

*Translations and dilations.* Let $(T_a f)(x) = f(x-a)$ and $(D_b f)(x) = f(x/b)$ for $a \in \mathbb{R}$ and $b > 0$. Then $(T_a f)\hat{}(w) = e^{-iaw}\hat{f}(w)$ and $(D_b f)\hat{}(w) = b\hat{f}(bw)$.
So for $f(x) = e^{-x^2/c}$ with $c > 0$ we get $\hat{f}(w) = \sqrt{\pi}\sqrt{c}e^{-cw^2/4}$.

*Properties of $\hat{f}$.* We have $|\hat{f}(w)| \leq \int_{-\infty}^{\infty} |f(x)|\,dx$. Also $\hat{f}$ is continuous.

*Riemann–Lebesgue lemma for transforms*: $\hat{f}(w) \to 0$ as $|w| \to \infty$.

*Transform of derivatives.* If $f$ is $C^1$ and $\int_{\mathbb{R}} |f'| < \infty$, then $(f')\hat{}(w) = iw\hat{f}(w)$. Thus there is a constant $C > 0$ such that $|\hat{f}(w)| \leq C/|w|$.

*Inverse Fourier transform.* $\check{g}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(w)e^{iwx}\,dw = \frac{\hat{g}(-x)}{2\pi}$.

*Convolution.* If $\int_{\mathbb{R}} |f| < \infty$ and $g$ is bounded, let $(f*g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt$.

*Kernels.* Let $K_1 : \mathbb{R} \to \mathbb{R}$ be continuous, nonnegative, with total integral 1 and s.t. $\exists C > 0$ with $K_1(x) \leq \frac{C}{1+x^2}$ for all $x$. Example $K_1(x) = \frac{1}{\sqrt{\pi}}e^{-x^2}$. Define $K_m(x) = mK_1(mx)$. Then $K_m * f \to f$ pointwise and $\hat{K}_m(w) \to 1$ for all $w$.

*Fourier's inversion theorem.* If $f : \mathbb{R} \to \mathbb{C}$ is continuous, $\int_{\mathbb{R}} |f| < \infty$ and $\int_{\mathbb{R}} |\hat{f}| < \infty$, then $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwx}dw$, i.e., $(\hat{f})\check{} = f$.

*Uniqueness theorem.* Take $f$, $g$ continuous, $\int_{\mathbb{R}} |f|$, $\int_{\mathbb{R}} |g| < \infty$. If $\hat{f} = \hat{g}$ then $f = g$.

*Transforms of convolutions.* If $g$ bounded, then $(f*g)\hat{} = \hat{f}\hat{g}$.

*Plancherel's theorem.* If $\int_{\mathbb{R}} |f| < \infty$ and $\int_{\mathbb{R}} |f|^2 < \infty$, then
$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(w)|^2\,dw$.

*Schwartz class $\mathcal{S}$.* This is the class of all infinitely differentiable functions $f : \mathbb{R} \to \mathbb{C}$ such that every derivative is rapidly decreasing: for all $n$, $k$, there is $C_{n,k} > 0$ such that $|f^{(n)}(x)| \leq \frac{C_{n,k}}{(1+|x|)^k}$ for all $x \in \mathbb{R}$. Example: $e^{-ax^2}$, $a > 0$.

Now if $\int_{\mathbb{R}} |x||f(x)|\,dx < \infty$, we have $\hat{f}$ differentiable, and
$(\hat{f})'(w) = -\int_{-\infty}^{\infty} ixf(x)e^{-ixw}\,dx$. This is used to show that the Fourier transform is a linear bijection from $\mathcal{S}$ to itself.

*Dirac delta function.* Define $\delta_a(f) = f(a)$ for $f \in \mathcal{S}$; $\delta = \delta_0$ is a *distribution*, a linear mapping from $\mathcal{S}$ to $\mathbb{C}$, and its $n$th derivative $\delta^{(n)}$ can be interpreted as $\delta^{(n)}(f) = (-1)^n f^{(n)}(0)$. In a precise sense, the Fourier transform of $\delta^{(n)}$ is $(iw)^n$; the transform of $\delta$ is identically 1.

*The Whittaker–Kotel'nikov–Shannon sampling theorem.* Suppose that $\hat{f}$ is $C^1$ and vanishes outside $[-T,T]$. Let $\delta$ satisfy $0 < \delta < \pi/T$. If $\phi : \mathbb{R} \to \mathbb{R}$ is $C^1$ with $\hat{\phi} = 1$ on $[-T,T]$, 0 outside $[-\pi/\delta, \pi/\delta]$, then $f(x) = \delta\sum_{n=-\infty}^{\infty} f(n\delta)\phi(x-n\delta)$.

*Fourier transforms in $\mathbb{R}^n$.* $\hat{f}(\mathbf{w}) = \int_{\mathbb{R}^n} f(\mathbf{x})e^{-i\mathbf{w}.\mathbf{x}}\,d\mathbf{x}$. *Inversion theorem:* $f(\mathbf{x}) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} \hat{f}(\mathbf{w})e^{i\mathbf{w}.\mathbf{x}}\,d\mathbf{w}$, if $f$ is continuous and $\int_{\mathbb{R}^n} |f|$ and $\int_{\mathbb{R}^n} |\hat{f}|$ are both finite. *Partial derivatives:* $\frac{\partial f}{\partial x_k}$ has transform $iw_k\hat{f}(\mathbf{w})$, and so the Laplacian $\nabla^2 f = \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_k^2}$ has transform $-\|\mathbf{w}\|^2\hat{f}(\mathbf{w})$; hence there are applications to solving PDEs.