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Improper integrals. A definite integral is *proper* if it is of the form $\int_a^b f(x)dx$, where $a, b \in \mathbb{R}$ and $f(x)$ is *bounded* for $x \in [a, b]$, that is, there are $L, M \in \mathbb{R}$ with $L \leq f(x) \leq M$ for all $a \leq x \leq b$. Otherwise it is *improper.*
Integrals to infinity. Define $\int_a^\infty f(x)dx = \lim_{X\to\infty} \int_a^X f(x)dx$. Similarly $\int_{-\infty}^a f(x)dx = \lim_{X\to\infty} \int_{-X}^a f(x)dx$ and $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$.
Limits. A function $f(x)$ *tends to a limit* ℓ *as* x *tends to* ∞ if for all $\epsilon > 0$ there is some N such that $|f(x) - \ell| < \epsilon$ for all $x > N$. Write $f(x) \to \ell$ as $x \to \infty$, or $\lim_{x\to\infty} f(x) = \ell$. Also $f(x)$ *tends to* ℓ *as* x *tends to* $a \in \mathbb{R}$ if for all $\epsilon > 0$ there is some $\delta > 0$ such that $|f(x) - \ell| < \epsilon$ for all x with $0 < |x - a| < \delta$. Write $f(x) \to \ell$ as $x \to a$, or $\lim_{x\to a} f(x) = \ell$.
Arithmetic of limits. If $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} g(x) = m$, then $\lim_{x\to a} f(x) \pm g(x) = \ell \pm m$; $\lim_{x\to a} f(x).g(x) = \ell.m$; and $\lim_{x\to a} f(x)/g(x) = \ell/m$ if $m \neq 0$.
Squeeze rule. Suppose $f(x) \leq g(x) \leq h(x)$ for all x. If $\lim_{x\to a} f(x) = \ell = \lim_{x\to a} h(x)$ then also $\lim_{x\to a} g(x) = \ell$.
Continuous functions. A function $f(x)$ is *continuous at* $x = a$ if for all $\epsilon > 0$ there is some $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ for all x with $|x - a| < \delta$. The function $f(x)$ is continuous if it is continuous everywhere it is defined. Note that $f(x)$ is continuous at $x = a$ if and only if $f(x) \to f(a)$ as $x \to a$.
Examples. The constant function $f(x) = c$ is continuous; so is the identity function $f(x) = x$; if $f(x)$ and $g(x)$ are continuous at a, then so are $f(x) \pm g(x)$, $f(x)g(x)$ and $f(x)/g(x)$ (in case $g(a) \neq 0$). Hence polynomials are continuous. The rational function $p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials, is continuous where defined, which is at the points a with $q(a) \neq 0$. Trigonometric functions, the exponential function and the logarithm $\log(x)$ for $0 < x < \infty$ are continuous. The function $f(x) = |x|$ is continuous. But $f(x) = x/|x|$ is not continuous at at 0, however you define $f(0)$. Similarly $f(x) = 1/(x-a)$ is not defined at $x = a$, and can't be made into a continuous function at a.
Composition of functions. If $f(x)$ is continuous at a and $g(y)$ is continuous at $f(a)$ then $g(f(x))$ is continuous at a.
Intermediate Value Theorem. Suppose $f(x)$ is continuous on a closed interval $[a, b]$, and $f(a)$ and $f(b)$ have opposite signs. Then $\exists$ c in the open interval (a, b) with $f(c) = 0$. **Corollary**. A polynomial of odd degree has at least one real root.
Properties of continuous functions. A continuous function $f(x)$ on a closed interval $[a, b]$ is bounded, and attains its maximum and minimum values. That is, there are $c, d \in [a, b]$ with $f(c) = \max_{x\in[a,b]} f(x)$, and $f(d) = \min_{x\in[a,b]} f(x)$.
Differentiability. A function $f(x)$ is *differentiable at a point* $x_0 \in \mathbb{R}$ if $\lim_{h\to 0}(f(x_0 + h) - f(x_0))/h$ exists. The value of the limit is denoted $f'(x_0)$ or $\frac{df}{dx}|_{x_0}$

or $\frac{df}{dx}(x_0)$, and is the *derivative* of f at x_0. f is differentiable if it is differentiable everywhere it is defined. If $f(x)$ is differentiable at x_0, then it is continuous at x_0.
Examples. $f(x) = c$ is differentiable. So is $f(x) = x$. If $f(x)$ and $g(x)$ are differentiable at x_0, then so are $f(x) \pm g(x)$, $f(x)g(x)$ and $f(x)/g(x)$ (in case $g(x_0) \neq 0$). The usual formulae for derivatives apply. Hence polynomials are differentiable and rational functions $p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials, are differentiable where defined, i.e., where $q(x) \neq 0$. Trigonometric functions, the exponential function and the logarithm $\log(x)$ for $0 < x < \infty$ are differentiable.
Chain rule. If $f(x)$ is differentiable at x_0 and $g(y)$ is differentiable at $f(x_0)$ then the composition $g(f(x))$ is differentiable at x_0, with $(gf)'(x_0) = g'(f(x_0))f'(x_0)$.
Non-differentiable functions. $f(x) = 1/(x - a)$ is not defined at $x = a$, so not differentiable there; $f(x) = |x|$ is not differentiable at 0.
Turning points. If $f'(x_0) > 0$ then f is increasing at x_0, that is, there is $\delta > 0$ such that for $0 < |x - x_0| < \delta$, one has $f(x) > f(x_0)$ if $x > x_0$ and $f(x) < f(x_0)$ if $x < x_0$. Dually, if $f'(x_0) < 0$ then f is decreasing at x_0. If f is differentiable at x_0 and f has a local maximum or minimum at x_0, then $f'(x_0) = 0$.
Rolle's Theorem. If $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) and if $f(a) = f(b)$, then there is some $c \in (a, b)$ with $f'(c) = 0$.
Mean Value Theorem. If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there is some $c \in (a, b)$ such that $(f(b) - f(a))/(b - a) = f'(c)$.
Complex functions. Write $z = x + iy$, $f(z) = u(x, y) + iv(x, y)$, with
$u(x, y) = \Re e\, f(x + iy)$ and $v(x, y) = \Im m\, f(x + iy)$.
Open sets. The *disc* with centre $z_0 \in \mathbb{C}$ and radius $\epsilon > 0$ is $D(z_0, \epsilon) = \{z \in \mathbb{C} : |z - z_0| < \epsilon\}$. A subset U of $\mathbb{C}$ is *open* if for every $z_0 \in U$, some disc with centre z_0 is contained in U, i.e., for all $z_0 \in U$ there is an $\epsilon > 0$ such that $D(z_0, \epsilon) \subseteq U$.
Limits and continuity. If $w \in \mathbb{C}$, we say that $f(z) \to w$ *as* $z \to z_0$ if given $\epsilon > 0$ there is a $\delta > 0$ such that $|f(z) - w| < \epsilon$ whenever $0 < |z - z_0| < \delta$. f is *continuous at* z_0 if, given $\epsilon > 0$ there is some $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$. Equivalently, if $f(z) \to f(z_0)$ as $z \to z_0$.
Differentiability. f is *(complex)-differentiable at* z_0, with *derivative* $f'(z_0)$ if
$(f(z_0 + w) - f(z_0))/w \to f'(z_0)$ as $w \to 0$.
Examples of differentiable functions. $f(z) = c$, $f(z) = z$; if $f(z)$ and $g(z)$ are differentiable at z_0, then so are $f(z) \pm g(z)$, $f(z)g(z)$ and $f(z)/g(z)$ (if $g(z_0) \neq 0$). The usual formulae for derivatives hold. So polynomials are differentiable, and rational functions where defined. Also e^z, $\sin z$, $\cos z$, etc. *Chain rule:* if $f(z)$ is differentiable at z_0 and $g(w)$ is differentiable at $f(z_0)$ then the composition $g(f(x))$ is differentiable at z_0.
Analytic functions. Let U be an open subset of $\mathbb{C}$. A function $f(z)$ is *analytic* (or *holomorphic* or *regular*) on U if defined on U and differentiable at every point of U.

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Cauchy–Riemann equations. If a function $f(z) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = x_0 + iy_0$, then the partial derivatives of u and v exist at (x_0, y_0) and satisfy $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Note that $f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$. If we know the partial derivatives exist, are continuous, and satisfy Cauchy–Riemann, then indeed f must be analytic.

Logarithms. We define $\log z = \log(|z|) + i\mathrm{Arg}(z)$ for $z \neq 0$. If $z = re^{i\theta}$ with $-\pi < \theta \leq \pi$ then $\log z = \log r + i\theta$. Thus $e^{\log z} = e^{\log r}e^{i\theta} = re^{i\theta} = z$. The logarithm function is not continuous at points on the negative real axis. $\log z$ is analytic on $\mathbb{C} \setminus \{\text{negative real axis}\}$ with derivative $1/z$.

Harmonic functions. A real-valued function of two variables $\phi(x, y)$ is *harmonic* if it satisfies *Laplace's equation* $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$. Suppose that $f(z) = u(x, y) + iv(x, y)$ is analytic on an open subset U of $\mathbb{C}$; then u and v are harmonic functions. Note that v is called the *harmonic conjugate* of u.

Paths. A *path* is a continuous mapping $p : [a, b] \to \mathbb{C}$. One can write $p(t) = u(t)+iv(t)$, where $u, v : [a, b] \to \mathbb{R}$. Then p is continuous if and only if u and v are continuous. A path is *closed* if $p(a) = p(b)$. A path is *smooth* if it is differentiable and its derivative $p'(t)$ is continuous for all $t \in (a, b)$. If $p(t) = u(t) + iv(t)$, then p is differentiable means that u and v are differentiable, and $p'(t) = u'(t) + iv'(t)$.

Connected sets. A subset U of $\mathbb{C}$ is *connected* (strictly speaking, *path-connected*) if for any $z_1, z_2 \in U$ there is a path $p : [a, b] \to U$ such that $p(a) = z_1$ and $p(b) = z_2$. That is, there is a path lying entirely in U starting at z_1 and ending at z_2. A connected open set is often called a *domain.*

Path integrals. If f is a continuous function of a complex variable, and $p : [a, b] \to \mathbb{C}$ is a smooth path, then the path integral $\int_p f(z)dz$ is defined to be $\int_a^b f(p(t))p'(t)dt$. (Algebraically this looks like the rule of substitution, $z = p(t)$.)

Fundamental example. $f(z) = 1/z$, $p : [0, 2\pi] \to \mathbb{C}$, $p(t) = Re^{it}$, $R > 0$, a circle.
So $p'(t) = Rie^{it}$. Then $\int_p f(z)dz = \int_0^{2\pi} \frac{1}{Re^{it}}iRe^{it}dt = \int_0^{2\pi} idt = 2\pi i$.

Piecewise smooth paths. A path $p : [a, b] \to \mathbb{C}$ is *piecewise smooth* if $p'(t)$ exists and is continuous, except possibly at a finite number of points in (a, b). We extend the notion of a path-integral to this case, by cutting up the range of integration at points where p isn't smooth.

Fundamental Theorem of Path Integrals. Suppose $f(z)$ is a continuous function on an open set U in $\mathbb{C}$, and $p : [a, b] \to U$ is a piecewise-smooth path. Suppose there is an analytic function $g(z)$ on U with $g'(z) = f(z)$ for all $z \in U$. Then
$\int_p f(z)dz = [g(z)]_{p(a)}^{p(b)} = g(p(b)) - g(p(a))$. So if the path is closed (i.e., $p(a) = p(b)$), then $\int_p f(z)dz = 0$. Example: $f(z) = z^n$ for $n \neq -1$, and $g(z) = z^{n+1}/(n + 1)$.

Length of a path. If $p : [a, b] \to \mathbb{C}$ is a piecewise-smooth path, then the length of p is $L(p) = \int_a^b |p'(t)|dt$.

Estimate Lemma. Suppose $f(z)$ is a continuous function on an open set U, and $p : [a, b] \to U$ is a piecewise-smooth path. Then $|\int_p f(z)dz| \leq L(p).M$, where $M = \max_{t\in[a,b]} |f(p(t))|$. This uses the result that if $h : [a, b] \to \mathbb{C}$ is a continuous function, then $|\int_a^b h(t)dt| \leq \int_a^b |h(t)|dt$.

Combining and reversing paths. If $p : [a, b] \to \mathbb{C}$ and $q : [c, d] \to \mathbb{C}$ are piecewise-smooth paths with $p(b) = q(c)$, one can combine them to give a path "$p+q$". Explicitly one can take the path $(p + q) : [a, b + d - c] \to \mathbb{C}$, where $(p+q)(t) = \begin{cases} p(t) & \text{if } a \leq t \leq b, \\ q(t + c - b) & \text{if } b \leq t \leq d + b - c. \end{cases}$ So $\int_{p+q} f(z)dz = \int_p f(z)dz + \int_q f(z)dz$. The *reverse of* p is $r : [-b, -a] \to \mathbb{C}$ with $r(t) = p(-t)$ and $\int_r f(z)dz = -\int_p f(z)dz$.

Simple paths. A path $p : [a, b] \to \mathbb{C}$ is *simple* if it doesn't cross itself, that is if $p(s) \neq p(t)$ for all $s < t$, except possibly that $p(a) = p(b)$. *The Jordan Curve Theorem:* A simple closed path divides the plane into two parts, an "inside" and an "outside".

Cauchy's Theorem. Let $p : [a, b] \to U$ be a piecewise-smooth simple closed path, and suppose that $f(z)$ is analytic on an open set U that contains p and all points inside p. Then $\int_p f(z)dz = 0$.

Star-shaped domains. Suppose that U is an open subset which is *star-shaped*, meaning that there is a point $z_0 \in U$ such that the straight line joining z_0 to any other point in U is contained in U. Then for any analytic function $f(z)$ on U there is an analytic function $g(z)$ on U with $g'(z) = f(z)$. *Corollary:* Cauchy's Theorem holds for any analytic function $f(z)$ defined on a star-shaped open set U and any closed piecewise-smooth path $p : [a, b] \to U$.

Cauchy's Integral Formula. Let $p : [a, b] \to \mathbb{C}$ be a piecewise-smooth simple closed path, and let $f(z)$ be analytic on an open subset containing p and all points "inside" p. If z_0 is "inside" p, and p goes anti-clockwise around z_0, then $f(z_0) = \frac{1}{2\pi i}\int_p \frac{f(z)}{z-z_0}\,dz$. If z_0 is not "inside" p then the RHS is zero by Cauchy's Theorem. *Generalized Integral Formula:* $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_p \frac{f(z)}{(z-z_0)^{n+1}}\,dz$. Thus if f is analytic in an open set U, then all its derivatives $f^{(n)}(z)$ exist. So the derivative of an analytic function is analytic.

Power series. A power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ with z_0, $a_n \in \mathbb{C}$ converges at $z \in \mathbb{C}$ to $\ell \in \mathbb{C}$ if for all $\epsilon > 0$ there is some N such that $|\sum_{n=0}^{m} a_n(z - z_0)^n - \ell| < \epsilon$ for all $m \geq N$. There is an R, either $R \geq 0$ or $R = \infty$, the *radius of convergence*, such that the power series converges whenever $|z - z_0| < R$, and diverges whenever $|z - z_0| > R$ (in case $R \neq \infty$). If the radius of convergence is R, we define a function $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ on the disc $D(z_0, R)$ (or $\mathbb{C}$ if $R = \infty$). For z in this disc, the sum is *absolutely convergent*, i.e., $\sum_{n=0}^{\infty} |a_n(z - z_0)^n|$ converges. One can differentiate term by term. If $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ then $f'(z) = \sum_{n=0}^{\infty} na_n(z - z_0)^{n-1}$, and this power series has radius of convergence at least R. Thus $f(z)$ is analytic on $D(z_0, R)$.

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Taylor series. If $f(z)$ is analytic on the disc $D(z_0, R)$, then $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ for $z \in D(z_0, R)$, where $a_n = \frac{1}{2\pi i}\int_p \frac{f(z)}{(z-z_0)^{n+1}}$ with p a circle, centre z_0, radius $r < R$. Note $a_n = f^{(n)}(z_0)/n!$. Its radius of convergence is the radius of the largest disc $D(z_0, R')$ such that $f(z)$ can be defined as an analytic function on $D(z_0, R')$.

Cauchy's Inequality. If $f(z)$ is analytic for $|z-z_0| < R$, and if $r < R$ then $|f^{(n)}(z_0)| \leq Mn!/r^n$ for all $n \geq 0$, with M the maximum value of $|f(z)|$ on the circle with centre z_0 and radius r. In particular, $|f(z_0)| \leq M$.

Liouville's Theorem. A function $f(z)$ that is analytic on the whole of $\mathbb{C}$, and bounded (i.e., there is some M with $|f(z)| \leq M$ for all z) must be constant.

Fundamental Theorem of Algebra. Every nonconstant polynomial with complex coefficients has a root in $\mathbb{C}$.

Zeroes. Let $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ again. Then z_0 is a zero of f if $f(z_0) = 0$, or equivalently $a_0 = 0$; z_0 is a zero of (finite) order m if $a_0 = a_1 = \ldots = a_{m-1} = 0$ and $a_m \neq 0$; z_0 is a zero of infinite order if all a_n are zero, so $f(z) = 0$ for all z with $|z - z_0| < R$.

Isolated zeroes. A zero z_0 of finite order is isolated. That is, there is some $r > 0$ such that $f(z) \neq 0$ for all z with $0 < |z - z_0| < r$. If $f(z)$ is an analytic function on a connected open set U with a zero of infinite order, then $f(z) = 0$ for all $z \in U$.

Identity theorem: Let U be a connected open set, let $z_0 \in U$ and let S be a subset of U. Suppose that any disc with centre z_0 meets S at some point $\neq z_0$. If f and g are analytic functions on U with $f(z) = g(z)$ for $z = z_0$ and all $z \in S$, then $f(z) = g(z)$ for all $z \in U$.

Annuli. For any $0 \leq R_1 < R_2 \leq \infty$ the set $\{z \in \mathbb{C} : R_1 < |z-z_0| < R_2\}$ is an annulus. The cases $R_1 = 0$ and/or $R_2 = \infty$ give degenerate annuli.

Laurent series. These are expansions of the form $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ with coefficients $a_n \in \mathbb{C}$ for all $n \in \mathbb{Z}$. This will converge on some annulus. Suppose f is analytic on the annulus $R_1 < |z-z_0| < R_2$ where $0 \leq R_1 < R_2 \leq \infty$. Then $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ for z in this annulus, where $a_n = \frac{1}{2\pi i}\int_p \frac{f(z)}{(z-z_0)^{n+1}}\,dz$ with p an anti-clockwise circle about z_0 with radius r satisfying $R_1 < r < R_2$.

Singularities. A *singularity* of $f(z)$ is a point z_0 where it is not defined, or not differentiable. A singularity z_0 is *isolated* if $f(z)$ is analytic for $0 < |z - z_0| < R$, for some $R > 0$. By Laurent expansion, $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ for $0 < |z - z_0| < R$. The *principal part* is $\sum_{n=-\infty}^{-1} a_n(z - z_0)^n$.

Removable singularities. z_0 is a *removable singularity* if the principal part is zero, so $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$. If one defines $f(z_0) = a_0$, the resulting function is analytic on the whole disc $|z - z_0| < R$. In general, z_0 is a removable singularity if and only if $f(z)$ tends to a limit as $z \to z_0$.

Poles. z_0 is a *pole* if the principal part is finite. Thus there is some $d > 0$ with $a_{-d} \neq 0$ and $f(z) = \sum_{n=-d}^{\infty} a_n(z-z_0)^n$. In this case one says that $f(z)$ has a pole of order d at $z = z_0$; $d = 1/2/3$ are *simple/double/triple poles*. If one defines $g(z) = (z-z_0)^d f(z)$ then g has a removable singularity at z_0. Conversely, if $g(z)$ is analytic in an open subset containing z_0 and $g(z_0) \neq 0$, then the function $g(z)/(z-z_0)^d$ has a pole of order d at $z = z_0$. In general, z_0 is a pole of order d if and only if $(z-z_0)^d f(z)$ tends to a nonzero limit as $z \to z_0$.

Essential singularities. The point z_0 is an *essential singularity* if the principal part is infinite. Thus $a_n \neq 0$ for infinitely many $n < 0$. The function has wild behaviour near z_0.

Residues. If $f(z)$ has an isolated singularity at z_0 then the *residue* of f at $z = z_0$ is the coefficient of $(z-z_0)^{-1}$ in the Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ (valid for $0 < |z-z_0| < R$). Thus the residue is a_{-1}. Notation: $\operatorname{res}(f, z_0)$ or $\operatorname{res}(f(z), z = z_0)$.

Calculating residues. If $f(z)$ has a simple pole at z_0, then $\operatorname{res}(f, z_0) = \lim_{z\to z_0}(z-z_0)f(z)$. More generally, if $f(z)$ has a pole of order n at z_0, then $\operatorname{res}(f, z_0) = \lim_{z\to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} ((z-z_0)^n f(z))$. If $f(z) = p(z)/(z-z_0)$ with $p(z)$ analytic at z_0 and $p(z_0) \neq 0$ then f has a simple pole at z_0 with residue $p(z_0)$.

Cauchy's Residue Theorem Let f be analytic on an open subset U of $\mathbb{C}$, except for a finite number of isolated singularities. Suppose that p is a piecewise-smooth simple closed path in U which goes anti-clockwise around its "inside". Suppose U contains the "inside" of p, and that none of the singularities of f are on p. Let $z_1, \ldots, z_n$ be the singularities of f "inside" p. Then $\int_p f(z)\,dz = 2\pi i \sum_{j=1}^{n} \operatorname{res}(f, z_j)$.

Trigonometric integrals. For an integral of the form $\int_0^{2\pi} R(\cos t, \sin t)\,dt$, where R is a rational function, turn it into a path integral for the circle $p : [0, 2\pi] \to \mathbb{C}$, $p(t) = e^{it}$, and then use the residue theorem. If $z = p(t)$, then $\cos t = (z + z^{-1})/2$ and $\sin t = (z - z^{-1})/2i$. We have $dz = p'(t)dt = ie^{it}dt$, so set $dt = dz/(iz)$.

Some improper integrals. The doubly infinite integral $\int_{-\infty}^{\infty} f(x)dx$ can be computed as $\lim_{R\to\infty} \int_{-R}^{R} f(x)dx$. Suppose that f extends to an analytic function on the upper half plane, except for a finite number of singular points. Choose R sufficiently large that the upper semi-circle, radius R, contains them all. Let $p : [-R, R] \to \mathbb{C}$, $p(t) = t$ and $q : [0, \pi] \to \mathbb{C}$, $q(t) = Re^{it}$. Then $\int_{-R}^{R} f(x)dx = \int_p f(z)dz = 2\pi i \sum$ Residues in upper half plane $- \int_q f(z)dz$. If $\int_q f(z)dz \to 0$ as $R \to \infty$ then $\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum$ Residues in upper half plane.

Example: to find $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2} dx$ work out $\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)^2}\, dx$ and take the real part.

J. R. Partington, 9.2.07, based on notes by W. Crawley-Boevey