MATH 2021 Complex Analysis: Some useful results and formulae

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Please let me know of any errors, omissions or obscurities!

Basics: $z = x + iy = r(\cos\theta + i\sin\theta)$, Re $z = x$, Im $z = y$, $|z| = r$, $\arg z = \theta$, $\bar{z} = x - iy$.

De Moivre: $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$.

Triangle Inequality: $|z_1 + z_2| \le |z_1| + |z_2|$.

Convergence of complex sequences: $c_n \to c$ when $|c_n - c| \to 0$, which happens iff $a_n \to a$ and $b_n \to b$, where $c_n = a_n + ib_n$ and $c = a + ib$.

Convergence of complex series: $\sum_{k=1}^{\infty} z_k$ converges if the sequence (c_n) of partial sums $c_n = \sum_{k=1}^{n} z_k$ converges. Suppose $z_k = x_k + iy_k$. Then $\sum_{k=1}^{\infty} z_k$ converges iff $\sum_{k=1}^{\infty} x_k$ and $\sum_{k=1}^{\infty} y_k$ converge.

Absolute convergence: $\sum_{k=1}^{\infty} |z_k| < \infty$, implies convergence of $\sum_{k=1}^{\infty} z_k$.

Exponential and trig. functions: $\exp(x \pm iy) = e^x(\cos y \pm i\sin y)$, $\log z = \log|z| + i\arg z$, $\cos z = (1/2)(e^{iz} + e^{-iz})$, $\sin z = (1/2i)(e^{iz} - e^{-iz})$, $\cos iz = \cosh z$, $\sin iz = i\sinh z$.

Open sets: U is open if for each $z \in U$ there is a disc Δ with centre z contained in U.

Continuity: f is continuous at c iff given $\epsilon > 0$ there is a $\delta > 0$ such that $|f(z) - f(c)| < \epsilon$ whenever $|z - c| < \delta$ (that is, $f(z) \to f(c)$ as $z \to c$).

Analyticity: f is analytic (complex differentiable) at c if $\dfrac{f(c+h) - f(c)}{h}$ tends to a limit (written $f'(c)$) as $h \to 0$.

Cauchy-Riemann equations: If $f = u + iv$ is analytic, then $f' = u_x + iv_x = v_y - iu_y$ and so $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$.

Harmonic functions: $u_{xx} + u_{yy} = 0$ (Laplace's equation). Real and imaginary parts of analytic functions are harmonic (there is a partial converse).

Power series: $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has a radius of convergence $R \ge 0$ (may be ∞) such that the series converges absolutely for $|z| < R$ and diverges for $|z| > R$. Usually determined by ratio test. Then $f'(w) = \sum_{n=1}^{\infty} n a_n w^{n-1}$ for $|w| < R$, and f is analytic (indeed infinitely differentiable) with $a_k = f^{(k)}(0)/k!$ for all k.

Triangle inequality for complex Riemann integrals: $\left|\int_a^b g(t)\,dt\right| \le \int_a^b |g(t)|\,dt$.

Contours: $\gamma : [a, b] \to \mathbb{C}$ piecewise continuously differentiable. It is *closed* if $\gamma(a) = \gamma(b)$.

Contour integrals: $\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt$, where $\gamma : [a, b] \to \mathbb{C}$.

Fundamental theorem of the calculus for contour integrals: $\int_\gamma F' = F(\gamma(b)) - F(\gamma(a))$.

Length of a contour: $L(\gamma) = \int_a^b |\gamma'(t)|\,dt$.

Estimation theorem: $\left|\int_\gamma f(z)\,dz\right| \le M \times L(\gamma)$ if $|f(z)| \le M$ on γ.

Winding number lemma: $\int_\gamma \frac{dz}{z-w} = 2\pi i n(\gamma, w)$. Here $n(\gamma, w)$ is the winding number of γ about w (anticlockwise turns counted positive, clockwise negative).

Cauchy's theorem: $U \subseteq \mathbb{C}$ open, $f : U \to \mathbb{C}$ analytic, γ a closed contour whose interior (points about which it has nonzero winding number) also lies in U. Then $\int_\gamma f = 0$.

Cauchy's integral formula: Same hypotheses, $w \in U \setminus \gamma$, then $\int_\gamma \frac{f(z)\,dz}{z-w} = 2\pi i n(\gamma, w) f(w)$.

Liouville's theorem: A bounded entire function (i.e., analytic on all of $\mathbb{C}$) is constant.

Maximum modulus theorem: If $|f(z)| \le M$ for z on γ then $|f(w)| \le M$ for w inside γ.

Fundamental theorem of algebra: A polynomial p of degree n has n complex roots, counting multiplicity (if not, p is a bounded entire function).

Taylor's theorem: Let f be analytic in U, take $w \in U$, and let $R = \operatorname{dist}(w, \mathbb{C}\backslash U)$. Then $f(w+h) = \sum_0^\infty a_n h^n$ for $|h| < R$, where $a_n = \frac{1}{2\pi i}\int_{|z-w|=r} \frac{f(z)\,dz}{(z-w)^{n+1}}$ for any $0 < r < R$.

Cauchy's integral formula (extended form): $f^{(n)}(w) = \frac{n!}{2\pi i}\int_{|z-w|=r} \frac{f(z)\,dz}{(z-w)^{n+1}}$ for all $n \ge 0$.

Product rule for power series: If $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$ are power series with r.c. at least R then so is $f(z)g(z) = \sum c_n z^n$ where $c_n = a_0 b_n + a_1 b_{n-1} \ldots a_n b_0$.

Pole of order $N \ge 1$: $f(p+h) = \sum_{n=-N}^{\infty} a_n h^n$ in some $0 < |h| < \delta$, with $a_{-N} \ne 0$.

Removable singularity: The same with $N = 0$ (so we can make it analytic at p, e.g. $(\sin z)/z$ at $p = 0$).

Essential singularity: $f : U \setminus \{p\}$ is analytic, but has neither a pole nor a removable singularity at p.

Residue of f at p: $\operatorname{Res}(f, p) = a_{-1}$ above.

Residue theorem: $\int_\gamma f = 2\pi i \sum_{j=1}^{m} \operatorname{Res}(f, p_j) n(\gamma, p_j)$, if f is analytic on $U \setminus \{p_1, \ldots, p_m\}$.

Formulae for residues:

(i) If $f(p) \ne 0$ and $g(p) = 0$ but $g'(p) \ne 0$, then $\operatorname{Res}(f, p) = f(p)/g'(p)$.

(ii) If f has a pole order N at p then $\operatorname{Res}(f, p) = \frac{1}{(N-1)!} \lim_{z \to p} \frac{d^{N-1}}{dz^{N-1}} \{(z-p)^N f(z)\}$.

Evaluation of definite integrals: Use a closed semicircular contour, radius R for $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,dx$, with p and q polynomials. Also the same if $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \sin ax\,dx$ or $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos ax\,dx$ for $a > 0$, in which case work with $\frac{p(z)e^{iaz}}{q(z)}$. Let $R \to \infty$ and show the contribution of the semicircular arc goes to 0.

Jordan's inequality: If $|f(z)| \le M_R$ on the semicircle $\gamma_R(t) = Re^{it}$ $(0 \le t \le \pi)$ and $a > 0$ then

$$\left| \int_{\gamma_R} f(z) e^{iaz}\,dz \right| \le \pi M_R / a.$$

Poles on axis: For a simple pole at p, with $\gamma_r(t) = p + re^{it}$ $(0 \le t \le \pi)$,

$$\lim_{r \to 0} \int_{\gamma_r} f = \pi i \operatorname{Res}(f, p).$$

Can use a contour looking like a cut-in-half CD to calculate things like $\int_0^\infty \frac{\sin x}{x}\,dx = \pi/2$.

Other contours: Use the unit circle as the contour for $\int_0^{2\pi} g(\cos\theta, \sin\theta)\,d\theta$, since g can be expressed in terms of $e^{i\theta}$.