Homological properties of modules over group algebras

H. G. Dales and M. E. Polyakov *

1 Introduction

Let G be a locally compact group, with left Haar measure m, and let $L^1(G)$ be the group algebra of G, so that $L^1(G)$ is the Banach algebra of all integrable functions on G, with the norm $\|\cdot\|_1$ specified by

$$\|f\|_1 = \int_G |f(t)| \, \mathrm{d}m(t) \quad (f \in L^1(G)), \tag{1.1}$$

and equipped with the convolution product $\star$, where

$$(f \star g)(t) = \int_G f(s)g(s^{-1}t) \, \mathrm{d}m(s) \quad (t \in G)$$

for each $f, g \in L^1(G)$. For the theory of this Banach algebra, see [8], [14], [17], and [2, §3.3], for example.

There are many standard left (and right) Banach $L^1(G)$-modules. Here we determine when these modules have certain well-known homological properties; we shall summarize some known results, and establish various new ones. In fact, we are seeking to characterize the locally compact groups G such that various modules are, respectively, projective, injective, and flat. Our conclusions are summarized in a table at the end of the paper.

*This paper was written whilst the second author held a Royal Society Fellowship at the University of Leeds. We acknowledge with thanks this financial support. We are also grateful to Professor A. Ya. Helemskii for some comments on a draft of the article. These comments were made during a visit of Professor Helemskii to Leeds; we are grateful to the London Mathematical Society for their support of this visit.

First, we recall the definitions and basic relationships of the standard homological properties that we shall consider; for full details and proofs, see [7].

Let A be an algebra. A *character* on A is a non-zero homomorphism from A onto $\mathbb{C}$. Each character on a Banach algebra is continuous.

Let A be an algebra, and let E be a left A-module. Then E is *faithful* if, for each $x \in E \setminus \{0\}$, there exists $a \in A$ with $a \cdot x \neq 0$. We set

$$A \cdot E = \{a \cdot x : a \in A,\, x \in E\}$$

and

$$AE = \text{lin}\, A \cdot E\,,$$

the linear span of $A \cdot E$.

Let A be a Banach algebra, and denote by A–**mod**, by **mod**–A, and by A–**mod**–A the categories of Banach left A-modules, of Banach right A-modules, and of Banach A-bimodules, respectively. Also, we denote by A–**unmod** the category of unital Banach left A-modules, etc. (These classes of modules are defined in [7, 0, §3.2] and [2, Definition 2.6.1].) For example, let A be a closed subalgebra of a Banach algebra B. Then it is immediate that $B \in A$–**mod**–A in an obvious way. Suppose that A is a Banach algebra and that $E \in A$–**mod**. Then E is *essential* if $\overline{AE} = E$. In the case where A has a bounded approximate identity, then in fact $\overline{AE} = A \cdot E$ [2, Corollary 2.9.26]; in particular, $A^{[2]} = A$, where $A^{[2]} = A \cdot A$. We use similar notation when $E \in$ **mod**–A and $E \in A$–**mod**–A; we say that $E \in A$–**mod**–A is *essential* if $\overline{AE} = \overline{EA} = E$.

The dual of a Banach space E is denoted by E'. Let $E \in A$–**mod**. Then $E' \in$ **mod**–A is the *dual module* of E, with the module operation specified by the formula

$$\langle x,\, \lambda \cdot a\rangle = \langle a \cdot x,\, \lambda\rangle \quad (a \in A,\, x \in E,\, \lambda \in E')\,.$$

Similarly, $E' \in A$–**mod** when $E \in$ **mod**–A; in this case, the module operation in E' is specified by

$$\langle x,\, a \cdot \lambda\rangle = \langle x \cdot a,\, \lambda\rangle \quad (a \in A,\, x \in E,\, \lambda \in E')\,.$$

In the case where $E \in A$–**mod**–A, also $E' \in A$–**mod**–A with respect to these two operations.

Let E and F be Banach spaces. Then $\mathcal{B}(E,F)$ is the Banach space of all bounded linear operators from E to F. We write $\mathcal{B}(E)$ for the Banach algebra $\mathcal{B}(E,E)$; the identity operator on E is denoted by I_E. The Banach space which is the projective tensor product of E and F is denoted by $E \widehat{\otimes} F$, with norm $\|\cdot\|_\pi$; for $z \in E \widehat{\otimes} F$, we have

$$\|z\|_\pi = \inf\left\{\sum_{j=1}^{\infty} \|x_j\|\,\|y_j\| : z = \sum_{j=1}^{\infty} x_j \otimes y_j,\ x_j \in E,\ y_j \in F\ (j \in \mathbb{N})\right\},$$

as in [2, §A.3]. The key property of $E \widehat{\otimes} F$ is that, for each continuous, bilinear map $B : E \times F \to G$, where G is a Banach space, there is a continuous linear map $T : E \widehat{\otimes} F \to G$ with $\|T\| = \|B\|$ and

$$T(x \otimes y) = B(x,y) \quad (x \in E,\, y \in F)\,.$$

The dual space $(E \widehat{\otimes} F)'$ of the Banach space $(E \widehat{\otimes} F, \|\cdot\|_\pi)$ is identified with $\mathcal{B}(E, F')$: if $\lambda \in (E \widehat{\otimes} F)'$, then $T_\lambda \in \mathcal{B}(E, F')$ is defined by

$$\langle y,\, T_\lambda(x)\rangle = \langle x \otimes y,\, \lambda\rangle \quad (x \in E,\, y \in F)\,,$$

and the map $\lambda \mapsto T_\lambda$ is an isometric linear isomorphism.

Let F be a closed subspace of a Banach space E. A *projection from E onto F* is an element $P \in \mathcal{B}(E)$ with $P^2 = P$ and $P(E) = F$; if there is such a projection, then F is *complemented* in E. Let E and F be Banach spaces, and let $T \in \mathcal{B}(E,F)$. Then T is *admissible* if $\ker T$, the kernel of T, is complemented in E and $\operatorname{im} T$, the image of T, is closed and complemented in F. Equivalently, T is admissible if there exists $S \in \mathcal{B}(F,E)$ with $T \circ S \circ T = T$.

Let A be a Banach algebra, and let $E, F \in A$–**mod**. Then ${}_A\mathcal{B}(E,F)$ is the closed linear subspace of $\mathcal{B}(E,F)$ consisting of the left A-module morphisms. We write ${}_A\mathcal{B}(E)$ for ${}_A\mathcal{B}(E,E)$. Similarly, we define $\mathcal{B}_A(E,F)$ for the space of right A-module morphisms when $E, F \in$ **mod**–A. Let $E, F \in A$–**mod**, and let $T \in {}_A\mathcal{B}(E,F)$. Then:

• T is a *retraction* if there exists $S \in {}_A\mathcal{B}(F,E)$ with $T \circ S = I_F$ (so that S is a *right inverse* to T), and in this case F is a *retract* of E;

• T is a *coretraction* if there exists $S \in {}_A\mathcal{B}(F,E)$ with $S \circ T = I_E$ (so that S is a *left inverse* to T).

Similar definitions apply when $E, F \in$ **mod**–A and when $E, F \in A$–**mod**–A.

Definition 1.1 *Let A be a Banach algebra, and let $P \in A$–**mod**. Then P is* projective *if, for each $E, F \in A$–**mod**, for each admissible epimorphism $T \in {}_A\mathcal{B}(E, F)$, and for each $S \in {}_A\mathcal{B}(P, F)$, there exists $R \in {}_A\mathcal{B}(P, E)$ with $T \circ R = S$. The map R* lifts *S.*

Similar definitions apply when $P \in$ **mod**–A and when $P \in A$–**mod**–A. Clearly the retract of a projective module is also projective. A Banach algebra is *biprojective* [7, IV, Definition 5.1] if it is projective in A–**mod**–A.

For an interesting variant of the above definition, taking into account the value of $\|R\|$, see [20, Definition 2.2].

Let A be an algebra, and denote by $A^\flat$ the algebra formed by adjoining an identity, called $e^\flat$, to A, as in [2, Definition 1.3.3]. (This algebra is denoted by A_+ in [7]; note that $A^\flat$ is different from A even when A already contains an identity.) In the case where A is a Banach algebra, $A^\flat$ is also a Banach algebra; if $E \in A$–**mod**, then also $E \in A^\flat-$**unmod**.

Let E be a Banach space, and set $P = A^\flat \widehat{\otimes} E$, with module operation specified by

$$a \cdot (b \otimes x) = ab \otimes x \quad (a \in A,\, b \in A^\flat,\, x \in E)\,,$$

so that $A^\flat \widehat{\otimes} E$ is the *free* Banach left A-module, as in [7, III.1.19] and [2, Example 2.6.7(i)]. Then P is a projective left A-module. We define $\pi \in {}_A\mathcal{B}(A^\flat \widehat{\otimes} E, E)$ to be such that

$$\pi(a \otimes x) = a \cdot x \quad (a \in A^\flat,\, x \in E)\,.$$

The restriction of π to $A \widehat{\otimes} E$ will also be denoted by π, and called the *canonical morphism*. We shall use the following facts from [7, IV.1.1, IV.1.2, and IV.4.5]; part (ii) was originally proved in [18].

Proposition 1.2 *Let A be a Banach algebra.*

(i) *Let $E \in A$–**mod**. Then the module E is projective if and only if the morphism $\pi \in {}_A\mathcal{B}(A^\flat \widehat{\otimes} E, E)$ is a retraction. In the case where E is essential, E is projective if and only if the canonical morphism $\pi \in {}_A\mathcal{B}(A \widehat{\otimes} E, E)$ is a retraction.*

(ii) *Suppose that $E \in A$–**mod** is projective and that at least one of the Banach spaces A and E has the approximation property. Then, for each $x \in E \setminus \{0\}$, there exists $T \in {}_A\mathcal{B}(E, A^\flat)$ such that $Tx \neq 0$. In the case where E is essential, we may suppose that $T \in {}_A\mathcal{B}(E, A)$.* □

Thus every $E \in A$–**mod** is a quotient of a projective module, namely of $A^\flat \widehat{\otimes} E$.

For example, let A be a unital Banach algebra with identity e_A. Then the map

$$\rho : a \mapsto a \otimes e_A, \;\; A \to A \widehat{\otimes} A\,,$$

belongs to ${}_A\mathcal{B}(A, A \widehat{\otimes} A)$ and is a right inverse to $\pi \in {}_A\mathcal{B}(A \widehat{\otimes} A, A)$, and so A is projective as a left A-module. Similarly, $A^\flat$ is always projective as a left A-module.

We prove the following result about projective modules in A–**mod**.

Proposition 1.3 *Let A be a Banach algebra such that A is a closed subalgebra of a Banach algebra B. Suppose that B is faithful in A–**mod**, that there exists $e \in B$ with $ae = ea = a$ $(a \in A)$, and that there exists $x_0 \in B$ such that $Ax_0 \subset A$ and $\{e+\overline{AB}, x_0+\overline{AB}\}$ is linearly independent in $B/\overline{AB}$. Then B is not projective in A–**mod**.*

Proof We identify $A^\flat$ with $\mathbb{C}e + A \subset B$, and set $F = \overline{AB}$, a closed submodule of B, with $A \subset F$.

Since $\{e+F, x_0+F\}$ is linearly independent in B/F, there exists $\lambda_0 \in B'$ such that

$$\langle e,\, \lambda_0\rangle = 1, \quad \langle x_0,\, \lambda_0\rangle = 0, \quad \lambda_0 \mid F = 0\,. \tag{1.2}$$

Assume towards a contradiction that B is projective in A–**mod**. Then there exists $\rho \in {}_A\mathcal{B}(B, A^\flat \widehat{\otimes} B)$ with $\pi \circ \rho = I_B$. For each $x \in B$, the element $\rho(x)$ has the form $e \otimes y + z$, where $y \in B$ and $z \in A \widehat{\otimes} B$. Note that $\pi(z) \in F$. Further, $x = y + \pi(z)$, and so $\rho(x) = e \otimes (x - \pi(z)) + z$. In particular, there exist $z_e, z_0 \in A \widehat{\otimes} B$ such that

$$\begin{aligned} \rho(e) &= e \otimes (e - \pi(z_e)) + z_e\,, \\ \rho(x_0) &= e \otimes (x_0 - \pi(z_0)) + z_0\,. \end{aligned}$$

For each $a \in A$, we have $ax_0 \in A$ and so $ax_0 \cdot \rho(e) = \rho(ax_0) = a \cdot \rho(x_0)$. Thus

$$ax_0 \otimes (e - \pi(z_e)) - a \otimes (x_0 - \pi(z_0)) = a \cdot (z_0 - x_0 \cdot z_e)\,. \tag{1.3}$$

There is a continuous linear map $T : A \widehat{\otimes} B \to A$ such that

$$T(a \otimes x) = \langle x,\, \lambda_0\rangle a \quad (a \in A,\, x \in B)\,,$$

and, indeed, $T \in {}_A\mathcal{B}(A \widehat{\otimes} B, A)$. We apply T to the elements in equation (1.3) to see that

$$ax_0 = a \cdot T(z_0 - x_0 \cdot z_e) \quad (a \in A)\,,$$

where we are using equations (1.2). Since B is faithful in A–**mod**, it follows that $x_0 = T(z_0 - x_0 \cdot z_e) \in A \subset F$, a contradiction.

Thus B is not projective in A–**mod**. □

Let A be a Banach algebra. Then there is a product $\Box$ on A'' such that $B := (A'', \Box)$ is a Banach algebra containing A as a closed subalgebra and such that $a \cdot \Phi = a \Box \Phi$ and $\Phi \cdot a = \Phi \Box a$ for each $a \in A$ and $\Phi \in A''$. (See [2, Theorem 2.6.15], for example; the product $\Box$ is called the *first Arens product* on A''.) Now suppose, further, that A has a bounded approximate identity, say (e_α). Then $Aa \neq \{0\}$ for each $a \in A$ with $a \neq 0$, and so B is faithful in A–**mod**. Let Φ_0 be a weak-$*$ accumulation point of (e_α) in A'', so that Φ_0 is a right identity for B and $\Phi_0 \Box a = a \Box \Phi_0 = a$ $(a \in A)$. Suppose that A is non-unital, and assume towards a contradiction that $\Phi_0 \in \overline{AB}$. Then $e_\alpha = e_\alpha \cdot \Phi_0 \to \Phi_0$ in $(A, \|\cdot\|)$, and so Φ_0 is the identity of A, a contradiction. Thus $\Phi_0 \notin \overline{AB}$.

Corollary 1.4 *Let A be a Banach algebra with a bounded approximate identity having a weak-$*$ accumulation point $\Phi_0 \in A''$. Suppose that there exists $\Phi_1 \in A''$ such that $A \cdot \Phi_1 \subset A$ and $\{\Phi_0 + \overline{AB}, \Phi_1 + \overline{AB}\}$ is linearly independent in $B/\overline{AB}$. Then A'' is not projective in A–**mod**.*

Proof This follows immediately from Proposition 1.3, given the above analysis. □

We remark that the conclusion of the above corollary may not hold if the hypothesis on the existence of Φ_1 be omitted. Indeed, let J be the James space, regarded as a Banach algebra. As explained in [2, Example 4.1.45], J is non-unital and has a bounded approximate identity. Further, $(J'', \Box)$ is identified with the unital algebra $J^\flat$, and so J'' is projective in J-**mod**.

Definition 1.5 *Let A be a Banach algebra, and let $J \in A$–**mod**. Then J is* injective *if, for each $E, F \in A$–**mod**, for each admissible monomorphism $T \in {}_A\mathcal{B}(E, F)$, and for each $S \in {}_A\mathcal{B}(E, J)$, there exists $R \in {}_A\mathcal{B}(F, J)$ with $R \circ T = S$.*

Similar definitions apply when $J \in \mathbf{mod}\text{–}A$ and when $J \in A\text{–}\mathbf{mod}\text{–}A$. For a variant of the above definition, again taking into account the value of $\|R\|$, see [20, Definition 3.2].

Let A be a Banach algebra, and let E be a Banach space. We write $J = \mathcal{B}(A^\flat, E)$, and define

$$(a \cdot T)(b) = T(ba), \quad (T \cdot a)(b) = T(ab) \quad (a \in A,\, b \in A^\flat,\, T \in J)\,.$$

Then $J \in A\text{–}\mathbf{mod}\text{–}A$ for these two module operations (*cf.* [7, 0, §4.2] and [2, Example 2.6.2(viii)]); the modules of this form are called the *cofree* modules [7, III.1.29]. Each cofree module is injective in $A-\mathbf{mod}$ [7, Proposition III.1.31]. Similarly, $\mathcal{B}(A, E)$ is a Banach A-bimodule for analogous products.

Let A be a Banach algebra, and suppose that $E \in A\text{–}\mathbf{mod}$. We define a *canonical embedding* $\Pi^\flat : E \to \mathcal{B}(A^\flat, E)$ by the formula

$$\Pi^\flat(x)(a) = a \cdot x \quad (a \in A^\flat,\, x \in E)\,.$$

Similarly, in the case where $E \in \mathbf{mod}\text{–}A$, we define $\Pi^\flat : E \to \mathcal{B}(A^\flat, E)$ by the formula

$$\Pi^\flat(x)(a) = x \cdot a \quad (a \in A^\flat,\, x \in E)\,.$$

The first map $\Pi^\flat : E \to \mathcal{B}(A^\flat, E)$ is a continuous left A-module embedding, and so every $E \in A\text{–}\mathbf{mod}$ is a closed submodule (complemented as a Banach space) of the injective left A-module $\mathcal{B}(A^\flat, E)$.

For a clear introduction to the theory of projective and injective modules in a more general situation, see [16]. We also note that projectivity and injectivity can be characterized in terms of the functor Ext. Indeed, by [7, III, Proposition 4.5], the module $P \in A\text{–}\mathbf{mod}$ is projective if and only if $\mathrm{Ext}^1_A(P, F) = \{0\}$ for each $F \in A\text{–}\mathbf{mod}$, and the module $J \in A\text{–}\mathbf{mod}$ is injective if and only if $\mathrm{Ext}^1_A(E, J) = \{0\}$ for each $E \in A\text{–}\mathbf{mod}$.

The following characterization of injective modules is given in [7, III.1.31].

Proposition 1.6 *Let A be a Banach algebra, and let $E \in A\text{–}\mathbf{mod}$. Then E is injective if and only if the map $\Pi^\flat : E \to \mathcal{B}(A^\flat, E)$ is a coretraction in $A\text{–}\mathbf{mod}$.* □

We shall use the following small variation of the above proposition; we now define $\Pi : E \to \mathcal{B}(A, E)$ by setting $\Pi(x) = \Pi^\flat(x) \mid A$ $(x \in E)$.

Proposition 1.7 *Let A be a Banach algebra, and let $E \in A$–**mod** be faithful. Then E is injective if and only if the map $\Pi : E \to \mathcal{B}(A, E)$ is a coretraction in A–**mod**.*

Proof Suppose first that E is injective. Then there is a left A-module morphism $\rho^\flat : \mathcal{B}(A^\flat, E) \to E$ such that $\rho^\flat \circ \Pi^\flat = I_E$.

First, take $S \in \mathcal{B}(A^\flat, E)$ such that $S \mid A = 0$. Then $a \cdot S = 0 \ (a \in A)$, and so $a \cdot \rho^\flat(S) = \rho^\flat(a \cdot S) = 0 \ (a \in A)$. Since E is faithful, it follows that $\rho^\flat(S) = 0$.

Now take $T \in \mathcal{B}(A, E)$, and extend T to $\widetilde{T} \in \mathcal{B}(A^\flat, E)$ by requiring that $\widetilde{T}(e^\flat) = 0$. Set $\rho(T) = \rho^\flat(\widetilde{T}) \ (T \in \mathcal{B}(A, E))$. Then $\rho : \mathcal{B}(A, E) \to E$ is a bounded linear operator, and

$$\rho(a \cdot T) - a \cdot \rho(T) = \rho^\flat(\widetilde{a \cdot T} - a \cdot \widetilde{T}) = 0$$

because $(\widetilde{a \cdot T} - a \cdot \widetilde{T}) \mid A = 0$. It follows that ρ is a left A-module morphism. Clearly, $(\rho \circ \Pi)(x) = x \ (x \in E)$, and so $\Pi : E \to \mathcal{B}(A, E)$ is a coretraction.

Conversely, suppose that $\Pi : E \to \mathcal{B}(A, E)$ is a coretraction. Then there exists $\rho \in {}_A\mathcal{B}(\mathcal{B}(A, E), E)$ with $\rho \circ \Pi = I_E$. Define

$$\rho^\flat(T) = \rho(T \mid A) \quad (T \in \mathcal{B}(A^\flat, E))\,.$$

Then $\rho^\flat \in {}_A\mathcal{B}(\mathcal{B}(A^\flat, E), E)$ and $\rho^\flat \circ \Pi^\flat = I_E$, and so $\Pi^\flat : E \to \mathcal{B}(A^\flat, E)$ is a coretraction. By Proposition 1.6, E is injective. □

Proposition 1.8 *Let A be a Banach algebra. Suppose that A is injective in A–**mod**. Then A has a right identity.*

Proof By Proposition 1.6, there exists an operator $\rho^\flat \in {}_A\mathcal{B}(\mathcal{B}(A^\flat, A), A)$ such that $\rho^\flat \circ \Pi^\flat = I_A$, where $\Pi^\flat \in {}_A\mathcal{B}(A, \mathcal{B}(A^\flat, A))$ is defined by

$$(\Pi^\flat a)(b) = ba \quad (a \in A,\, b \in A^\flat)\,.$$

Let $P \in \mathcal{B}(A^\flat, A)$ be the natural projection of $A^\flat$ onto A, so that $P \mid A = I_A$ and $a \cdot P = \Pi^\flat a \ (a \in A)$. Set $p = \rho^\flat(P)$. Then we have

$$a = (\rho^\flat \circ \Pi^\flat)(a) = \rho^\flat(a \cdot P) = ap \quad (a \in A)\,.$$

Thus p is a right identity for A. □

Corollary 1.9 *Let A be a Banach algebra which has an approximate identity. Suppose that A is injective in A–**mod**. Then A has an identity.*

Proof Let (e_α) be an approximate identity for A. By the proposition, A has a right identity, say p. We have $e_\alpha \to p$ in A, and so $e_\alpha a \to a = pa \ (a \in A)$. Thus p is an identity for A. □

Let A be a Banach algebra, and let $E \in A$–**mod**, so that $E' \in$ **mod**–A. Then the dual module of $A^\flat \widehat{\otimes} E$ is $\mathcal{B}(A^\flat, E')$ with the prescribed module operations (*cf.* [2, Theorem 2.6.4]), and the dual of $\pi \in \mathcal{B}(A^\flat \widehat{\otimes} E, E)$ is $\pi' = \Pi^\flat \in \mathcal{B}(E', \mathcal{B}(A^\flat, E'))$. It follows that the dual E' of a projective left A-module E is an injective right A-module. Similarly, the dual of a projective right A-module is an injective left A-module.

Definition 1.10 *Let A be a Banach algebra, and let $E \in A$–**mod**. Then E is* flat *if E' is injective in* **mod**–A.

An apparently different definition of a flat module is given in [7, VII.1.1] in terms of module tensor products (and this definition reflects the standard algebraic concept), but the two definitions are shown to be equivalent for Banach modules in [7, VII.1.14]; see also [20, §4]. Thus every projective module is flat.

Similar definitions apply in the cases where $E \in$ **mod**$-A$ and where $E \in A$–**mod**–A. A Banach algebra A is *biflat* [7, VII.2.5] if it is flat as an A-bimodule.

A very important class of Banach algebras is that of the *amenable* Banach algebras (see [2, Definition 2.8.57] and [7, Chapter VII]). There is a multitude of equivalent characterizations of amenability for Banach algebras. One is given as Definition VII.2.16 of [7]: a Banach algebra A is amenable if and only if $A^\flat$ is biflat.

We shall use the following basic fact about amenable Banach algebras [7, VII.2.29]; we shall essentially prove the converse result in the special case where $A = L^1(G)$ in §4.

Proposition 1.11 *Let A be an amenable Banach algebra, and let $E \in A$–**mod** or $E \in$ **mod**–A. Then E' is injective, equivalently E is flat.* □

We shall refer to some standard algebras; for details of these algebras, see [2], for example.

Let Ω be a locally compact space. Then: $C(\Omega)$ is the algebra of all continuous functions on Ω (with pointwise multiplication); $(C^b(\Omega), |\cdot|_\Omega)$ is the Banach algebra of all bounded, continuous functions on Ω with the uniform norm $|\cdot|_\Omega$; $C_0(\Omega)$ is the closed ideal in $C^b(\Omega)$ consisting of the functions which vanish at infinity. We denote by $\operatorname{supp} f$ the support of an element $f \in C(\Omega)$; this is the closure of the set $\{x \in \Omega : f(x) \neq 0\}$. We take $C_{00}(\Omega)$ to be the ideal in $C(\Omega)$ and $C^b(\Omega)$ consisting of the functions with compact support. We set: $c_0 = C_0(\mathbb{N})$, the space of null sequences; $c = C_0(\mathbb{N} \cup \{\infty\})$, the space of convergent sequences; and $\ell^\infty = C^b(\mathbb{N})$, the space of bounded sequences, identified with $C(\beta\mathbb{N})$.

Let μ be a positive, regular Borel measure on a locally compact space Ω. Then for $p \geq 1, (L^p(\Omega, \mu), \|\cdot\|_p)$ is the standard Banach space, and $(L^\infty(\Omega, \mu), \|\cdot\|_\infty)$ is the space of essentially bounded, μ-measurable functions on Ω (where, as usual, we identify functions which are equal save on a μ-locally null set). For $p \geq 1$, the dual of $L^p(\Omega, \mu)$ is $L^q(\Omega, \mu)$, where q is the conjugate index to p.

Further for $p \geq 1$, we denote by ℓ^p the Banach algebra of sequences $\alpha = (\alpha_n)$ such that

$$\|\alpha\|_p = \left(\sum_{n=1}^\infty |\alpha_n|^p \right)^{1/p} < \infty \,,$$

taken with the coordinatewise product.

Let $(E, \|\cdot\|_E)$ be a Banach space, and let μ be a positive measure on a set S. A function $f : S \to E$ is *μ-measurable* if there is a sequence (f_n) of simple functions from S into E such that

$$\lim_{n\to\infty} \|f_n(s) - f(s)\|_E = 0 \quad \text{for almost all } s \in S \,.$$

The function f is *Bochner integrable* if, further, $\int_S \|f(s)\|_E \,\mathrm{d}\mu(s) < \infty$. The space of Bochner integrable functions is denoted by $L^1(S, E, \mu)$; it is a Banach space with respect to the norm $\|\cdot\|_1$, where

$$\|f\|_1 = \int_S \|f(s)\|_E \,\mathrm{d}\mu(s) \quad (f \in L^1(S, E, \mu)) \,.$$

See [5, §II.2] for details of this space. In the case where μ is a σ-finite, positive, regular Borel measure on a locally compact space Ω, each function $f \in C(\Omega, E)$ with $\int_\Omega \|f(s)\|_E \,\mathrm{d}\mu(s) < \infty$ belongs to $L^1(\Omega, E, \mu)$.

The bilinear map taking the element $(f, x) \in L^1(S, \mu) \times E$ to the function in $L^1(S, E, \mu)$ whose value at $s \in S$ is $f(s)x$ is continuous with norm equal to 1, and so there is a natural continuous linear operator

$$J : L^1(S, \mu) \widehat{\otimes} E \to L^1(S, E, \mu)$$

with $\|J\| = 1$. By a theorem of Grothendieck (see [5, Example VIII.1.10] and [4, p. 29]), J is an isometric surjection, and so

$$L^1(S, \mu) \widehat{\otimes} E \cong L^1(S, E, \mu) .$$

In connection with Proposition 1.2(ii), it is useful to note that each Banach space $L^1(\Omega, \mu)$ and each $C_0(\Omega)$ has the approximation property.

We shall utilize the following famous 'Phillips Lemma'; for an easy proof, see [6, Theorem (0.1.16)].

Proposition 1.12 *The subspace c_0 of ℓ^∞ is not complemented.* □

In fact, Conway [1] has proved a generalization of Phillips Lemma: $C_0(\Omega)$ is not complemented in $C^b(\Omega)$ whenever Ω is a locally compact space such that $C^b(\Omega) \neq C(\Omega)$. For a generalization of Conway's result, see [19, §2].

We shall also use the following related result.

Proposition 1.13 *Let Ω be an infinite, locally compact space which is σ-compact. In the case where Ω is compact, suppose, further, that Ω contains a convergent sequence consisting of distinct points. Let μ be a positive, regular Borel measure on Ω. Then $C_0(\Omega)$ is not complemented in $L^\infty(\Omega, \mu)$.*

Proof In the case where Ω is compact, there is a sequence (x_n) in Ω consisting of distinct points such that (x_n) converges, say $\lim_n x_n = x_0$; we may suppose that $x_0 \neq x_n \ (n \in \mathbb{N})$. In the case where Ω is not compact, there is a sequence (x_n) in Ω such that $\lim_n x_n = \infty$ because Ω is σ-compact; in the latter case, we set $x_0 = \infty$ for convenience.

In both cases, there is a sequence (K_n) of compact subsets of Ω such that $x_n \in \operatorname{int} K_n \ (n \in \mathbb{N})$, such that $K_m \cap K_n = \emptyset$ whenever $m, n \in \mathbb{N}$ with $m \neq n$, and such that $x_0 \notin \bigcup_{n=1}^\infty K_n$. We have $0 < \mu(K_n) < \infty \ (n \in \mathbb{N})$. For each $n \in \mathbb{N}$, choose $\lambda_n \in C_{00}(\Omega)$ such that $\lambda_n(x_n) = |\lambda_n|_\Omega = 1$ and $\operatorname{supp} \lambda_n \subset K_n$.

The map

$$I : (\alpha_n) \mapsto \sum_{n=1}^{\infty} \alpha_n \lambda_n, \quad \ell^\infty \to L^\infty(\Omega, \mu)\,,$$

is an isometric embedding with $I(c_0) \subset C_0(\Omega)$, and the map

$$R : \lambda \mapsto (\lambda(x_n) - \lambda(x_0) : n \in \mathbb{N}), \quad C_0(\Omega) \to c_0\,,$$

is a continuous linear operator (where $\lambda(\infty) = 0$).

Assume towards a contradiction that there is a projection

$$P : L^\infty(\Omega, \mu) \to C_0(\Omega)\,,$$

and set $Q = R \circ P \circ I : \ell^\infty \to c_0$. Then $Q \in \mathcal{B}(\ell^\infty, c_0)$ and clearly $Q(\alpha) = \alpha \ (\alpha \in c_0)$, and so Q is a projection of ℓ^∞ onto c_0, a contradiction of Proposition 1.12. Thus $C_0(\Omega)$ is not complemented in $L^\infty(\Omega, \mu)$. □

The identity of a group G is denoted by e_G.

Proposition 1.14 *Let G be an infinite compact group. Then $C(G)$ is not complemented in $L^\infty(G)$.*

Proof Assume towards a contradiction that there is a projection of $L^\infty(G)$ onto $C(G)$.

By [8, Theorem (8.7)], there is a compact, normal subgroup N of G such that $H := G/N$ is metrizable; we can also arrange that H be infinite. For each $\lambda \in C(G)$, define

$$\widetilde{\lambda}(x) = \int_N \lambda(x\zeta)\,\mathrm{d}m(\zeta) \quad (x \in H)\,,$$

where $\mathrm{d}m(\zeta)$ denotes the Haar measure on N. Then $\widetilde{\lambda} \in C(H)$ by [17, Proposition 3.1.10].

We now take a sequence (x_n) in the metrizable group H with $x_n \to e_H$ as $n \to \infty$, and then define

$$R : \lambda \mapsto \left(\widetilde{\lambda}(x_n) - \widetilde{\lambda}(e_H) : n \in \mathbb{N}\right), \quad C(G) \to c_0\,.$$

Essentially as in the above proof, we obtain a projection from ℓ^∞ onto c_0, again a contradiction. □

In fact, a more general result is true: it is proved in [11, Theorem 2] that, for each infinite, locally compact group G, the space $C_0(G)$ is not complemented in $L^\infty(G)$.

We now give some (probably well-known) examples that distinguish easily between some of the above homological properties.

(i) In the case where $A = \mathbb{C}$, all modules in A–**mod** are both projective and injective.

Let $W = \ell^1(\mathbb{Z})$ with convolution multiplication (see below), so that W is the *Wiener algebra*. Then W is a unital, commutative Banach algebra, and so W is projective in W–**mod**. Also W is the dual of the closed submodule $c_0(\mathbb{Z})$ of $W' = \ell^\infty(\mathbb{Z})$, and W is an amenable Banach algebra, and so, by Proposition 1.11, W is injective in W–**mod**. This is also a special (easy) case of Corollary 4.8, below.

(ii) Let A be a Banach algebra, and let $E \in A$–**mod** be trivial, so that $E \neq \{0\}$ and $A \cdot E = \{0\}$.

Suppose that A has a right identity p, and define

$$\rho(x) = (e^\flat - p) \otimes x \quad (x \in E)\,,$$

so that $\rho \in {}_A\mathcal{B}(E, A^\flat \widehat{\otimes} E)$ because $\rho(a \cdot x) = a \cdot \rho(x) = 0 \ (a \in A,\, x \in E)$, and ρ is clearly a right inverse to π. Thus E is projective in A–**mod**.

Suppose that either A or E has the approximation property and that E is projective in A–**mod**. Take $x \in E \setminus \{0\}$. By Proposition 1.2(ii), there exists $T \in {}_A\mathcal{B}(E, A^\flat)$ with $Tx \neq 0$, say $Tx = ze^\flat - p$, where $z \in \mathbb{C}$ and $p \in A$. We have $za = ap \ (a \in A)$, and so either $z = 0$ and $Ap = \{0\}$ or $z \neq 0$ and p/z is a right identity for A.

For $T \in \mathcal{B}(A^\flat, E)$, define $\rho^\flat(T) = T(e^\flat) \in E$. Since E is trivial, we have $\rho^\flat(a \cdot T) = a \cdot \rho^\flat(T) = 0$ for each $a \in A$ and $T \in \mathcal{B}(A^\flat, E)$, and also $(\rho^\flat \circ \Pi^\flat)(x) = x \ (x \in E)$. Thus $\rho^\flat$ is a left inverse to $\Pi^\flat$, and hence E is injective. Similarly, E' is injective in **mod**–A, and so E is flat.

Thus we easily find examples where the trivial module is injective and flat, but not projective.

(iii) Let $A = c_0$, and set $F = A^\flat$, so that $F = c$. Clearly, $A \in A$–**mod** and A is essential and faithful; also $F \in A$–**mod**.

We *claim* that A is not injective. For let $T \in {}_A\mathcal{B}(A, F)$ be the natural embedding of c_0 in c; T is admissible because c_0 has codimension 1 in c. Assume towards a contradiction that A is injective, so that there exists

$R \in {}_A\mathcal{B}(F, A)$ with $R \circ T = I_A$, and set $\alpha = (\alpha_k) = R(1)$, where 1 denotes the sequence $(1, 1, \dots)$ in c. For $k \in \mathbb{N}$, set $\delta_k = (\delta_{j,k} : j \in \mathbb{N})$. Then

$$\delta_k = R(\delta_k) = R(\delta_k \cdot 1) = \delta_k \cdot \alpha = \alpha_k \delta_k \quad (k \in \mathbb{N}),$$

and so $\alpha_k = 1 \ (k \in \mathbb{N})$, a contradiction of the fact that $\alpha \in c_0$.

On the other hand, A is a projective module in A–**mod**. For define

$$\rho : \sum_{k=1}^{\infty} \alpha_k \delta_k \mapsto \sum_{k=1}^{\infty} \alpha_k \delta_k \otimes \delta_k, \quad c_0 \to c_0 \widehat{\otimes} c_0 .$$

Let $\alpha \in c_0$ with $|\alpha|_{\mathbb{N}} = 1$. Then $\rho(\alpha)$ is a 1-diagonal element in the sense of [7, II.2.43], and so, by [7, II.2.44], $\|\rho(\alpha)\|_\pi \leq 1$. (See also [2, Proposition A.3.68].) Thus $\rho \in \mathcal{B}(A, A \widehat{\otimes} A)$. Clearly ρ is a left A-module morphism, and so $\rho \in {}_A\mathcal{B}(A, A \widehat{\otimes} A)$. Also $\pi \circ \rho = I_A$. Thus ρ is a right inverse to π, and so A is projective by Proposition 1.2(i). For these remarks, see [7, IV, Example 5.10].

Thus A is projective, but not injective, in A–**mod**.

We shall also see, by combining Theorem 3.1 with Theorem 3.8, that, in the case where G is an infinite, compact group, $C_0(G)$ is a projective, but not injective, left $L^1(G)$-module. Also, by combining Theorem 2.4 with Corollary 4.8, we shall see that $\ell^1(G)$ is a projective, but not injective, left $\ell^1(G)$-module for each non-amenable group G.

(iv) Set $A = \ell^1$, with coordinatewise multiplication, and set $E = c_0$, so that $E \in A$–**mod** and E is essential and faithful. By exactly the argument in (i), E is not injective.

The dual of the module E is the algebra A itself. Since A does not have an identity, it follows from Proposition 1.8 that A is not injective, and so E is not flat. Indeed a stronger result is true: there is no continuous linear map $\rho : \mathcal{B}(A, A) \to A$ with $\rho \circ \Pi = I_A$. To see this, first note that $\Pi(\delta_j)(\delta_k) = \delta_{j,k}\delta_k \ (j, k \in \mathbb{N})$. Assume that there exists such a map ρ, and set $f_n = \sum_{j=1}^n \delta_j / j \in A$ and $T_n = \Pi(f_n)$ for each $n \in \mathbb{N}$. Set $\alpha_{j,k}^n = T_n(\delta_j)(\delta_k) \ (j, k, n \in \mathbb{N})$. Then $T_n = (\alpha_{j,k}^n)$ is the diagonal operator with $\alpha_{j,j}^n = 1/j$ for $j = 1, \dots, n$ and $\alpha_{j,j}^n = 0$ for $j > n$. Thus (T_n) converges to the diagonal operator $T = (\alpha_{j,k})$ in $\mathcal{B}(A, A)$, where $\alpha_{j,j} = 1/j$ for $j \in \mathbb{N}$, and hence $f_n \to \rho(T)$ in A, a contradiction because (f_n) is not a Cauchy sequence in A.

(v) Let $A = C(\mathbb{I})$, the Banach algebra of all continuous functions on $\mathbb{I} = [0, 1]$, and let E be the Banach space $(L^2(\mathbb{I}), \|\cdot\|_2)$. Then $E \in A$–**mod**–A, where the module product is pointwise multiplication. Clearly, the dual module E' is E itself (with the same operations).

The algebra A is amenable [2, Theorem 5.6.2(i)], and so, by Proposition 1.11, the module E is injective in A–**mod**.

Assume towards a contradiction that E is a projective module in A–**mod**. By Proposition 1.2(ii), there exists $T \in {}_A\mathcal{B}(E, A)$ such that $T(1) \neq 0$; here 1 denotes the element of E which is constantly equal to 1 on $\mathbb{I}$. Next set $f = T(1) \in A$. Then there exists $s \in (0, 1)$ with $f(s) \neq 0$. Let (f_n) be a sequence of functions in A such that $f_n(s) = 1 \ (n \in \mathbb{N})$ and $\|f_n\|_2 \to 0$ as $n \to \infty$. Then

$$f_n f = T(f_n \cdot 1) = T(f_n) \quad (n \in \mathbb{N}),$$

and so $|f_n f|_{\mathbb{I}} \leq \|T\| \, \|f_n\|_2 \to 0$ as $n \to \infty$. However,

$$|f_n f|_{\mathbb{I}} \geq |f_n(s)f(s)| = |f(s)| \quad (n \in \mathbb{N}),$$

a contradiction.

Thus $E = E'$ is injective and flat, but neither E nor E' is projective in A–**mod**.

We shall also see, by combining Theorem 2.6, Corolllary 2.5, and Corollary 4.7, that, in the case where G is an amenable, but not discrete, locally compact group, $M(G)$ is an injective and flat, but not projective, left $L^1(G)$-module.

(vi) Set $A = L^1(G)$ and $E = C_0(G) \in A$–**mod**, where G is a non-compact, locally compact group. Then we shall prove in Theorem 3.1 that E is not projective and in Theorem 3.8 that E is not injective. However, by Corollary 4.7, E is flat if and only if G is an amenable group. Thus, for example, $c_0(\mathbb{F}_2)$ is neither flat nor injective in $\ell^1(\mathbb{F}_2)$-**mod**; here, $\mathbb{F}_2$ is the free group on two generators, so that $\mathbb{F}_2$ is a non-amenable group.

(vii) Suppose that G is a non-amenable locally compact group. Then, by Theorem 2.4, $L^\infty(G)$ is injective, but, by Corollary 4.7, $L^\infty(G)$ is not flat in $L^1(G)$-**mod**.

We conclude that the only general relationship between the notions 'E is projective', 'E is injective', and 'E is flat' for $E \in A$–**mod** is the trivial one that E is flat whenever E is projective; counter-examples to all other possible inclusions are given by one or more of the above examples.

2 The modules $L^1(G)$ and $M(G)$

Let G be a group. For $S, T \subset G$, we set $ST = \{st : s \in S,\ t \in T\}$, $S^2 = SS$, and $S^{-1} = \{s^{-1} : s \in S\}$; S is *symmetric* if $S = S^{-1}$. The characteristic function of the set S is denoted by χ_S.

Let G be a locally compact group. We write $L^1(G)$ for $L^1(G, m)$, where m is left Haar measure on G. In the case where G is infinite and compact, we suppose that $m(G) = 1$; in the case where G is discrete, we suppose that $m(\{s\}) = 1$ for each $s \in G$.

The Banach algebra $(L^1(G), \star)$ has a bounded approximate identity contained in $C_{00}(G)$ [2, Theorem 3.3.23], and so we have $L^1(G)^{[2]} = L^1(G)$. The algebra $L^1(G)$ has an identity if and only if G is a discrete group; in this latter case, we denote $L^1(G)$ by $\ell^1(G)$.

We write $L^\infty(G)$ for $L^\infty(G, m) = L^1(G)'$, the dual module of $L^1(G)$. The left and right module operations of $L^1(G)$ on $L^\infty(G)$ are given by the formulae

$$\left.\begin{aligned} (f \cdot \lambda)(t) &= \int_G f(s)\lambda(ts)\,\mathrm{d}m(s)\,, \\ (\lambda \cdot f)(t) &= \int_G f(s)\lambda(st)\,\mathrm{d}m(s)\,, \end{aligned}\right\} \quad (t \in G)\,, \tag{2.1}$$

where $f \in L^1(G)$ and $\lambda \in L^\infty(G)$. The space $C_0(G)$ is a closed $L^1(G)$-submodule of the Banach left $L^1(G)$-module $L^\infty(G)$, and $C_0(G)$ is essential and faithful.

Let f be a function on G, and let $s \in G$. The *left-translate* $L_s f$ of f is defined by

$$(L_s f)(t) = f(s^{-1}t) \quad (t \in G)\,.$$

We note that

$$L_s(f \star g) = L_s f \star g \quad (f, g \in L^1(G))\,. \tag{2.2}$$

We denote by $M(G)$ the measure algebra of the locally compact group G, so that $M(G)$ is the algebra of all complex-valued, regular Borel measures on G, with norm $\|\cdot\|$ given by $\|\mu\| = |\mu|(G)$ $(\mu \in M(G))$, and the product $\mu \star \nu$ of $\mu, \nu \in M(G)$ specified by

$$(\mu \star \nu)(B) = \int_G \nu(s^{-1}B)\,\mathrm{d}\mu(s)$$

for each Borel subset B of G. For details, see [2, §3.3].

We regard $L^1(G)$ as a complemented, closed ideal in $M(G)$ and $\ell^1(G)$ as a complemented, closed subalgebra of $M(G)$, as in [2, Theorem 3.3.36]. For $s \in G$, we denote by δ_s the element of $M(G)$ which is the point mass at s; clearly, for each $f \in L^1(G)$, we have

$$(\delta_s \star f)(t) = f(s^{-1}t) = (L_s f)(t) \quad (t \in G)\,,$$

so that $\delta_s \star f = L_s f$ and $\|L_s f\|_1 = \|f\|_1$ for each $s \in G$. We shall usually write $s \star f$ for $\delta_s \star f$.

There is always one character on a measure algebra $M(G)$; this is the *augmentation character* φ_G, defined by

$$\varphi_G : \mu \mapsto \mu(G), \quad M(G) \to \mathbb{C}\,.$$

The restriction of φ_G to $L^1(G)$ has the form

$$\varphi_G : f \mapsto \int_G f(t)\,\mathrm{d}m(t), \quad L^1(G) \to \mathbb{C}\,. \tag{2.3}$$

Let 1 denote the element of $L^\infty(G)$ that is constantly equal to 1. Then it follows from equations (2.1) that

$$f \cdot 1 = 1 \cdot f = \varphi_G(f)1 \quad (f \in L^1(G))\,. \tag{2.4}$$

The space $\mathbb{C}$ is an $L^1(G)$-bimodule for the operation given by

$$f \cdot z = z \cdot f = \varphi_G(f)z \quad (z \in \mathbb{C},\ f \in L^1(G))\,;$$

as such it is denoted by $\mathbb{C}_{\varphi_G}$.

Let E be an essential Banach left $L^1(G)$-module. Then the module operation extends so that E is a unital Banach left $M(G)$-module; the module operation satisfies

$$\mu \cdot (f \cdot x) = (\mu \star f) \cdot x \quad (\mu \in M(G),\ f \in L^1(G),\ x \in E)\,.$$

Similarly, if E is a Banach right $L^1(G)$-module, then E is a unital Banach right $M(G)$-module. Further, E is an $M(G)$-bimodule when E is an $L^1(G)$-bimodule. For these remarks, see [2, Theorem 5.6.34].

For example, $C_0(G)$ is an essential Banach $L^1(G)$-bimodule, and so it is a unital Banach $M(G)$-bimodule; its dual space is $M(G)$, and the $M(G)$-module products are just the usual products in $M(G)$. As we said in

§2, $L^1(G)$ is a closed ideal in $M(G)$, and hence an $M(G)$-bimodule; thus $L^\infty(G) = L^1(G)'$ and $L^\infty(G)' = L^1(G)''$ are $M(G)$-bimodules.

Let E be Banach left $M(G)$-module. We often write $s \cdot x$ for $\delta_s \cdot x$ when $s \in G$ and $x \in E$; similarly, we write $x \cdot s$ in the case where E is a Banach right $M(G)$-module. The element $x \in E$ is *left-invariant* if $s \cdot x = x \ (s \in G)$. We note that

$$(\lambda \cdot s)(t) = \lambda(st), \quad (s \cdot \lambda)(t) = \lambda(ts) \quad (s, t \in G)$$

for each $\lambda \in L^\infty(G)$.

Let E be a Banach left $L^1(G)$-module. As a Banach space, we have $L^1(G) \widehat{\otimes} E \cong L^1(G, E)$. The module action is now given by

$$(f \cdot F)(t) = \int_G f(u)F(u^{-1}t)\,\mathrm{d}m(u) \quad (t \in G) \tag{2.5}$$

for $f \in L^1(G)$ and $F \in L^1(G, E)$. Further, in the case where $E \subset C^b(G)$, we have

$$\pi(F)(t) = \int_G F(s)(ts)\,\mathrm{d}m(s) \quad (t \in G) \tag{2.6}$$

for $F \in L^1(G, E)$.

Definition 2.1 *Let G be a locally compact group. An element Λ of $L^\infty(G)'$ is a* mean *on $L^\infty(G)$ if $\langle 1, \Lambda\rangle = \|\Lambda\| = 1$. The group G is* amenable *if there is a left-invariant mean on $L^\infty(G)$.*

Let G be a locally compact group. We set

$$P(G) = \{f \in L^1(G) : f \geq 0, \ \|f\|_1 = 1\}.$$

We shall use the following characterization of the amenability of G given in [15, Proposition (0.8)]; it is called *Reiter's condition.*

Proposition 2.2 *Let G be a locally compact group. Then G is amenable if and only if there is a net (h_α) in $P(G)$ such that $\lim_\alpha \|L_s h_\alpha - h_\alpha\|_1 = 0$ for each $s \in G$.* □

The following famous theorem of Johnson [9] (see also [2, Theorem 5.6.42] and [7, VII, Proposition 2.35]) determines when $L^1(G)$ is amenable; for a characterization of the groups G such that the measure algebra $M(G)$ is amenable, see [3].

Theorem 2.3 *Let G be a locally compact group. Then the following are equivalent:*

(a) *the Banach algebra $L^1(G)$ is amenable;*

(b) *the locally compact group G is amenable;*

(c) *the module $\mathbb{C}_{\varphi_G}$ is flat in $L^1(G)$–**mod**.* □

Set $E = L^1(G)$ itself, so that $E \in L^1(G)$–**mod**, and E is an essential left $L^1(G)$-module. We record the following standard theorem, given in [7, IV.2.17] and [2, Theorem 3.3.32], for example.

Theorem 2.4 *Let G be a locally compact group. Then $L^1(G)$ is a projective, and hence a flat, left $L^1(G)$-module, and $L^\infty(G)$ is an injective left $L^1(G)$-module.* □

Corollary 2.5 *Let G be a locally compact group such that G is either amenable or discrete. Then $M(G)$ is a flat left $L^1(G)$-module.*

Proof Suppose that G is amenable. Then $L^1(G)$ is an amenable Banach algebra, and so $M(G)$ is flat by Proposition 1.11.

Suppose that G is discrete. Then $M(G) = \ell^1(G)$, which is a flat left $\ell^1(G)$–module by Theorem 2.4. □

We *conjecture* that, in fact, $M(G)$ is a flat left $L^1(G)$-module for each locally compact group G.

We can also easily determine when $M(G)$ is projective.

Theorem 2.6 *Let G be a locally compact group. Then $M(G)$ is a projective left $L^1(G)$-module if and only if G is discrete.*

Proof Set $A = L^1(G)$ and $B = M(G) \in A$–**mod**, so that B is a unital Banach algebra containing A as a closed ideal.

Suppose that G is discrete, so that $A = B = \ell^1(G)$. Then A is a unital Banach algebra, and so A is projective in A–**mod**.

Conversely, suppose that G is not discrete. By [2, Corollary 3.3.24], B is faithful in A–**mod**, and certainly $\dim(B/A) \geq 2$, and so, by Proposition 1.3, B is not projective in A–**mod**. □

It follows easily from Corollary 1.9 that the Banach algebra $L^1(G)$ can only be injective in $L^1(G)$-**mod** in the case where G is discrete; we shall determine when $L^1(G)$ is injective in Theorem 4.9.

Theorem 2.7 *Let G be a locally compact group, and suppose that $L^1(G)''$ is projective in $L^1(G)$-**mod**. Then G is a discrete group that contains no infinite, amenable subgroup.*

Proof Set $A = L^1(G)$ and $B = (A'', \Box)$.

First assume towards a contradiction that G is not discrete, so that A is not unital; A has an approximate identity bounded by 1, so that B has a right identity, say Φ_0, with $\|\Phi_0\| = 1$. The algebra B is a Banach $M(G)$-bimodule, and the map $\mu \to \mu \cdot \Phi_0$, $M(G) \to B$, is an isometric embedding. Indeed, B is identified with the semi-direct product $M(G) \ltimes I$ for a certain closed ideal I in B. Clearly $\overline{AB} \subset A \ltimes I$. Set $\Phi_1 = s \cdot \Phi_0$ for some $s \in G \setminus \{e_G\}$. Then $f \cdot \Phi_1 = (f \star s) \cdot \Phi_0 = f \star s \in A$ $(f \in A)$, and $\{\Phi_0 + \overline{AB}, \Phi_1 + \overline{AB}\}$ is linearly independent in $B/\overline{AB}$. By Corollary 1.4, B is not projective in A–**mod**, a contradiction. Thus G is discrete.

Second, suppose that G is discrete. Assume towards a contradiction that G contains an infinite, amenable subgroup H, and let $R : A \to \ell^1(H)$ be the restriction map, so that $R(s \star f) = s \star R(f)$ $(s \in H, f \in A)$. There is a canonical isometric embedding $\iota : \ell^1(H) \to A$, and the map $\iota'' : \ell^1(H)'' \to A''$ is also an isometric embedding; clearly, we have $\iota(s \star f) = s \star \iota(f)$ $(s \in H, f \in \ell^1(H))$, and so

$$\iota''(s \cdot \Lambda) = s \cdot \iota(\Lambda) \quad (s \in H, \Lambda \in \ell^1(H)'').$$

Let $\Lambda_0 \in \ell^1(H)''$ be a left-invariant mean on $\ell^\infty(H)$. Then $\iota''(\Lambda_0)$ is a non-zero element of B. By Proposition 1.2(ii), there exists $T \in {}_A\mathcal{B}(B, A)$ such that $T(\iota''(\Lambda_0)) \neq 0$ in A. There exists $s_0 \in G$ such that $g_0 \neq 0$, where $g_0 = R(T(\iota''(\Lambda_0)) \cdot s_0)$. Consider the map

$$V : \Lambda \mapsto R(T(\iota''(\Lambda)) \cdot s_0), \quad \ell^1(H)'' \to \ell^1(H).$$

We have $V(s \cdot \Lambda) = s \cdot V(\Lambda)$ $(s \in H, \Lambda \in \ell^1(H)'')$, and so g_0 is a non-zero element of $\ell^1(H)$ such that $s \star g_0 = g_0$ $(s \in H)$. But there is no such element in $\ell^1(H)$ because H is infinite, the required contradiction.

The result follows. $\Box$

Let G be a group such that $\ell^1(G)''$ is projective. Then certainly each element of G has finite order. An example of an infinite group G_0, called *Ol'shanskii's group*, with the property that every proper subgroup is finite, is given in [12]. It is remarked at the end of the paper [13] that this group is not amenable (the proof uses a condition of Grigorchuk that is discussed in [15]). Thus certainly G_0 has no infinite, amenable subgroup. We do not know whether or not $\ell^1(G_0)''$ is projective in $\ell^1(G_0)$-**mod**.

3 The modules $C_0(G)$ and $L^\infty(G)$

Let G be a locally compact group. The main thrust of the present section is to determine when $C_0(G)$ is either projective or injective in $L^1(G)$-**mod**.

Theorem 3.1 *Let G be a locally compact group, and let E be a closed, left $L^1(G)$-submodule of $L^\infty(G)$ such that $C_{00}(G) \subset E \subset C^b(G)$. Then E is projective in $L^1(G)$-**mod** if and only if G is a compact space.*

Proof Set $A = L^1(G)$. We shall determine when the map $\pi \in {}_A\mathcal{B}(A^\flat \widehat{\otimes} E, E)$ is a retraction; the result will then follow from Proposition 1.2(i).

Suppose first that the group G is compact, so that $C(G) = E \subset A$, $m(G) = 1$, and the constant function 1 belongs to E. We have $A \widehat{\otimes} E \cong L^1(G, E)$.

For $\lambda \in E$, define $\rho(\lambda) : G \to E$ by

$$\rho(\lambda)(s)(t) = \lambda(ts^{-1}) \quad (s, t \in G)\,.$$

Then $\rho(\lambda) \in C(G, E)$ because $E = C(G)$, and so $\rho(\lambda) \in L^1(G, E)$. Clearly $\|\rho(\lambda)\| \le |\lambda|_G$. Thus $\rho \in \mathcal{B}(E, L^1(G, E))$.

Also, for $g \in A$ and $\lambda \in E$, we see, using equations (2.1) and (2.5), that

$$\begin{aligned} \rho(g \cdot \lambda)(s)(t) &= (g \cdot \lambda)(ts^{-1}) \\ &= \int_G g(u)\lambda(ts^{-1}u)\,\mathrm{d}m(u) \\ &= \int_G g(u)\rho(\lambda)(u^{-1}s)(t)\,\mathrm{d}m(u) \\ &= (g \cdot \rho(\lambda))(s)(t) \quad (s, t \in G)\,, \end{aligned}$$

and so $\rho(g \cdot \lambda) = g \cdot \rho(\lambda)$, whence $\rho \in {}_A\mathcal{B}(E, L^1(G, E))$. Finally, it follows from equation (2.6) that

$$((\pi \circ \rho)(\lambda))(t) = \int_G \rho(\lambda)(s)(ts)\,\mathrm{d}m(s) = \int_G \lambda(t)\,\mathrm{d}m(s) = \lambda(t) \quad (t \in G)$$

for $\lambda \in E$, and so $\pi \circ \rho = I_E$. Thus π is a retraction. (This conclusion also follows from the facts that $L^1(G)$ is biprojective whenever G is compact [7, IV, Theorem 5.13] and from [7, IV, Theorem 5.3], which shows that, in this case, each essential module in $L^1(G)$-**mod** is projective.)

For the converse, assume towards a contradiction that E is a projective left A-module, but that G is not compact.

Let V and W be compact, symmetric neighbourhoods of e_G such that $V^2 \subset W$, and take $f \in C_{00}(G)$ with $f(G) \subset \mathbb{I}$, with $f(e_G) = 1$, and with $\operatorname{supp} f \subset V$, such that $|f|_G = 1$, such that $\|f\|_1 \leq 1$, and such that $\operatorname{supp}(f \cdot f) \subset W$. Clearly $f \cdot f \neq 0$, and so, by Proposition 1.2(ii), there exists $T \in {}_A\mathcal{B}(E, A^\flat)$ with $T(f \cdot f) \neq 0$. By replacing T by the map $\lambda \mapsto (T\lambda) \star h,\ E \to A$, for suitable $h \in A$, we may suppose that $T(E) \subset A$. Set $\eta = \|f \star Tf\|_1 / 2 > 0$.

Fix $k \in \mathbb{N}$, and choose $g \in C_{00}(G)$ such that $\|Tf - g\|_1 < 1/k$ and $\|f \star (Tf - g)\|_1 < \eta$, so that $\|f \star g\|_1 > \eta$. Set $K = \operatorname{supp} g$. Since G is not compact, there exist $s_1, \dots, s_k \in G$ such that the sets $s_j(V \cup K)^2$ are pairwise disjoint for $j = 1, \dots, k$. Set $f_j = s_j \star f\ (j = 1, \dots, k)$. Since the sets $s_1V, \dots, s_kV$ are pairwise disjoint, we have

$$\left|\sum_{j=1}^k f_j\right|_G = 1 \quad \text{and} \quad \left\|\sum_{j=1}^k f_j \cdot g\right\|_1 = k\,\|f \cdot g\|_1 \,.$$

Now set $\lambda = \sum_{j=1}^k f_j \cdot f \in C_{00}(G)$, so that $|\lambda|_G = |f \cdot f|_G$, which is independent of k. However,

$$\begin{aligned} \|T\lambda\|_1 &= \left\|\sum_{j=1}^k f_j \star Tf\right\|_1 \geq \left\|\sum_{j=1}^k f_j \star g\right\|_1 - 1 \\ &= k\,\|f \star g\|_1 - 1 > k\eta - 1\,. \end{aligned}$$

Thus $k\eta \leq \|T\|\,|f \cdot f|_G + 1$. This holds for each $k \in \mathbb{N}$, the required contradiction. $\square$

The above theorem applies to the module $E = C_0(G)$. It also applies to the following closed, left $L^1(G)$-submodules of $C^b(G)$: $E = C^b(G)$; $E = LUC(G)$, the *bounded, left uniformly continuous functions* (so that $\lambda \in LUC(G)$ if and only if $\lambda \in C^b(G)$ and the map $t \mapsto L_t\lambda,\ G \to C^b(G)$, is continuous); $E = AP(G)$, the *almost periodic functions* on G (so that

$\lambda \in AP(G)$ if and only if $\lambda \in C^b(G)$ and the set $\{L_t\lambda : t \in G\}$ is relatively compact in $(L^\infty(G), \|\cdot\|_\infty)$; $E = WAP(G)$, the *weakly almost periodic functions* on G (so that $\lambda \in WAP(G)$ if and only if $\lambda \in C^b(G)$ and the set $\{L_t\lambda : t \in G\}$ is relatively compact in $L^\infty(G)$ with respect to the weak topology).

The above theorem does not deal with the case where $E = L^\infty(G)$. In fact, this module is rarely projective, as Theorem 3.3, below, shows.

Lemma 3.2 *Let G be a compact group. Suppose that $L^\infty(G)$ is a projective $L^1(G)$-module. Then there is a projection of $L^\infty(G)$ onto $C(G)$.*

Proof Set $A = L^1(G)$ and $E = L^\infty(G)$. Since E is projective, it follows from Proposition 1.2(i) that there exists $\rho \in {}_A\mathcal{B}(E, A^\flat \widehat{\otimes} E)$ such that $\pi \circ \rho = I_E$. Thus there exists $T \in \mathcal{B}(E)$ such that

$$\rho(\lambda) - e_A \otimes T\lambda \in A \widehat{\otimes} E \quad (\lambda \in E)\,.$$

It follows that $\lambda - T\lambda \in \pi(A \widehat{\otimes} E) = AE$ $(\lambda \in E)$. By [2, Proposition 3.3.13], $AE \subset C(G)$. Set $P\lambda = \lambda - T\lambda$ $(\lambda \in E)$, so that $P \in \mathcal{B}(E, C(G))$. Now suppose that $\lambda \in C(G)$, and take (e_α) to be a bounded approximate identity in A. Then $e_\alpha \cdot \lambda \to \lambda$ in $C(G)$ because $A \cdot C(G) = C(G)$ by [2, Proposition 3.3.23]. Hence

$$\rho(\lambda) = \lim_\alpha \rho(e_\alpha \cdot \lambda) = \lim_\alpha e_\alpha \cdot \rho(\lambda) \in A \widehat{\otimes} E\,,$$

and so $T\lambda = 0$ and $P\lambda = \lambda$. Thus $P : L^\infty(G) \to C(G)$ is a projection. □

Theorem 3.3 *Let G be a locally compact group. Then $L^\infty(G)$ is a projective $L^1(G)$-module if and only if G is finite.*

Proof Certainly $L^\infty(G)$ is a projective $L^1(G)$-module when G is finite.

For the converse, first suppose that G is not compact. Then the second part of the proof of Theorem 3.1 produces a contradiction. Thus G is compact. Now Lemma 3.2 shows that there is a projection of $L^\infty(G)$ onto $C(G)$, again a contradiction of Proposition 1.14 in the case where G is infinite. Hence G is finite. □

We now consider when the left $L^1(G)$-module $C_0(G)$ is injective.

In the case where G is a finite, and hence amenable, group, $L^1(G)$ is amenable and $C_0(G)$ is a dual module, and so, by Proposition 1.11, $C_0(G)$

is injective. We now show that $C_0(G)$ is not injective whenever the group G is infinite; as a first step, we shall show that it suffices to establish this in the special case where G is σ-compact.

Let G be an infinite locally compact group, and let $\{s_i : i \in \mathbb{N}\}$ be an infinite subset of G. The group G contains a compact, symmetric neighbourhood K of e_G. Define a sequence (K_n) of compact subsets of G inductively by setting $K_1 = K \cup \{s_1, s_1^{-1}\}$ and

$$K_{n+1} = (K_n \cup \{s_n, s_n^{-1}\})K_n \quad (n \in \mathbb{N}) .$$

Finally, set $H = \bigcup\{K_n : n \in \mathbb{N}\}$. Then H is an infinite, σ-compact set, and H is an open and closed subgroup of G. Further, we can write

$$G = \bigcup\{sH : s \in S\} = \bigcup\{Hs^{-1} : s \in S\}$$

for a suitable subset S of G. (The above is essentially [8, Theorem (5.7)].) The restriction of m to the family of Borel subsets of H is a left Haar measure on H, and so we can identify $L^1(G)$ with $\ell^1(S,\, L^1(sH))$ as a Banach space. Similarly, we can identify $C_0(G)$ with $c_0(S,\, C_0(sH))$.

Lemma 3.4 *Assume that the Banach left $L^1(G)$-module $C_0(G)$ is injective. Then the Banach left $L^1(H)$-module $C_0(H)$ is also injective.*

Proof We write $\mathcal{B}_G$ and $\mathcal{B}_H$ for $\mathcal{B}(L^1(G), C_0(G))$ and $\mathcal{B}(L^1(H), C_0(H))$, respectively; we write I_G and I_H for the identity maps on $C_0(G)$ and $C_0(H)$, respectively; and we write $\Pi_G : C_0(G) \to \mathcal{B}_G$ and $\Pi_H : C_0(H) \to \mathcal{B}_H$ for the appropriate canonical embeddings.

Take $T \in \mathcal{B}_H$. For each $s \in S$, there is a corresponding linear map

$$T_s \in \mathcal{B}(L^1(s^{-1}H), C_0(Hs)) ,$$

so that $T_s(f) = s^{-1} \cdot T(s \star f)$ $(f \in L^1(s^{-1}H))$, and there is a map

$$\widetilde{T} = \bigoplus\{T_s : s \in S\} \in \mathcal{B}_G$$

which is defined 'coordinatewise'. Obviously, the map

$$Q : T \mapsto \widetilde{T}, \quad \mathcal{B}_H \to \mathcal{B}_G ,$$

is an isometric linear operator. Further, Q is a morphism of left $L^1(H)$-modules. To see this, take an element $f \in L^1(G)$, say $f = \sum_{s \in S} f_s$, where $f_s \in L^1(s^{-1}H)$ $(s \in S)$, take $g \in L^1(H)$, and take $T \in \mathcal{B}_H$. Then

$$\begin{aligned} Q(g \cdot T)(f) &= \sum_{s \in S} s^{-1} \cdot ((g \cdot T)(s \star f_s)) = \sum_{s \in S} s^{-1} \cdot T((s \star f_s) \star g) \\ &= \sum_{s \in S} s^{-1} \cdot T(s \star (f \star g)_s) = (g \cdot Q(T))(f)\,, \end{aligned}$$

as required.

Let $I : C_0(H) \to C_0(G)$ be the natural embedding, and let $R : C_0(G) \to C_0(H)$ be the restriction map, so that $R \circ I = I_H$. Take $\lambda \in C_0(G)$, say $\lambda = \sum_{s \in S} \lambda_s$, where $\lambda_s \in C_0(Hs)$, so that $R(\lambda) = \lambda_e$, and take $g \in L^1(H)$. Then $R(g \cdot \lambda) = (g \cdot \lambda)_e = g \cdot \lambda_e = g \cdot R(\lambda)$, and so R is a morphism of left $L^1(H)$-modules.

We *claim* that $Q \circ \Pi_H = \Pi_G \circ I : C_0(H) \to \mathcal{B}_G$. Indeed, take $\lambda \in C_0(H)$ and $f \in L^1(G)$, say $f = \sum_{s \in S} f_s$, where $f_s \in L^1(s^{-1}H)$. Then

$$\begin{aligned} (Q \circ \Pi_H(\lambda))(f) &= \sum_{s \in S} (s^{-1} \cdot \Pi_H(\lambda))(s \star f_s) = \sum_{s \in S} s^{-1} \cdot ((s \star f_s) \cdot \lambda) \\ &= \sum_{s \in S} f_s \cdot \lambda = \sum_{s \in S} (f \cdot \lambda)_s = (\Pi_G \circ I(\lambda))(f)\,, \end{aligned}$$

and so the claim follows.

By Proposition 1.7, there is a continuous morphism $\rho_G : \mathcal{B}_G \to C_0(G)$ with $\rho_G \circ \Pi_G = I_G$. Define

$$\rho_H = R \circ \rho_G \circ Q : \mathcal{B}_H \to C_0(H)\,.$$

Then ρ_H is a continuous linear operator which is a morphism of left $L^1(H)$-modules, and, further,

$$\rho_H \circ \Pi_H = R \circ \rho_G \circ Q \circ \Pi_H = R \circ \rho_G \circ \Pi_G \circ I = I_H\,,$$

so that Π_H is a coretraction. By Proposition 1.7, $C_0(H)$ is injective. □

We also require a technical lemma.

Let G be a locally compact group, and let $E \in L^1(G)$–**mod**. Set $\mathcal{B} = \mathcal{B}(L^1(G), E)$, and take $T \in \mathcal{B}$. Then T *has support in* B for a Borel subset B of G, written supp $T \subset B$, if $Tf = 0$ in E whenever $f \in L^1(G)$ with

$f \mid B = 0$; T has *compact support* if there is a compact subset K of G such that supp $T \subset K$.

Now suppose that (T_n) is a bounded sequence in $\mathcal{B}$ such that supp $T_n \subset B_n$ $(n \in \mathbb{N})$ for a pairwise disjoint family $\{B_n : n \in \mathbb{N}\}$ of Borel subsets of G. In this case, define

$$Tf = \sum_{n=1}^{\infty} T_n f \quad (f \in L^1(G)) .$$

Then $Tf \in E$, and $T \in \mathcal{B}$ with $\|T\| = \sup_{n\in\mathbb{N}} \|T_n\|$; we set $T = \sum_{n=1}^{\infty} T_n$.

Let (t_n) be a sequence in G. Then we say that

$$\operatorname*{Lim}_{n\to\infty} t_n = \infty$$

if, for each compact subset K of G, we have $t_n \notin K$ eventually.

Lemma 3.5 *Let G be a locally compact group which is σ-compact and non-compact. Let $\rho : \mathcal{B}(L^1(G), C_0(G)) \to C_0(G)$ be a continuous linear operator which is a left $L^1(G)$-morphism. Then $\rho(T) = 0$ whenever $T \in \mathcal{B}(L^1(G), C_0(G))$ and T has compact support.*

Proof Write $\mathcal{B} = \mathcal{B}(L^1(G), C_0(G))$, and suppose that $T \in \mathcal{B}$ is such that supp $T \subset K$, where K is a compact subset of G.

Set $\lambda_0 = \rho(T) \in C_0(G)$, and assume towards a contradiction that $\lambda_0 \neq 0$. Then there exists $f \in L^1(G)$ such that $f \cdot \lambda_0 \neq 0$ and supp $f \subset L$ for a certain compact subset L of G. Next set $g = f \cdot \lambda_0$; by modifying f, we may suppose that $g(e_G) = 1$.

We shall construct by induction a sequence (t_n) in G, and then set

$$x_{m,n} = ((t_n \star f) \cdot \lambda_0)(t_m^{-1}) = g(t_m^{-1} t_n) \quad (m, n \in \mathbb{N}) ,$$

so that $x_{m,m} = 1$ $(m \in \mathbb{N})$. We choose the sequence (t_n) to satisfy the following conditions:

(i) $|x_{m,n}| \leq 1/2^{|m-n|+2}$ for $m, n \in \mathbb{N}$ with $m \neq n$;

(ii) the subsets $KL^{-1}t_n^{-1}$ are pairwise disjoint in G;

(iii) $\operatorname{Lim}_{n\to\infty} t_n = \infty$.

Such a construction is possible because K and L are compact, G is σ-compact, and g vanishes at infinity.

For each $n \in \mathbb{N}$, set $T_n = (t_n \star f) \cdot T$, so that $\|T_n\| = \|f \cdot T\| \ \ (n \in \mathbb{N})$, and hence $(\alpha_n T_n)$ is a bounded sequence in $\mathcal{B}$ whenever $(\alpha_n) \in \ell^\infty$. Further, $\operatorname{supp} \alpha_n T_n \subset KL^{-1}t_n^{-1} \ \ (n \in \mathbb{N})$ for each such sequence (α_n), and so it follows from (ii), above, that we can define $\sum_{n=1}^\infty \alpha_n T_n$ in $\mathcal{B}$.

Define

$$Q_1 : (\alpha_n) \mapsto \sum_{n=1}^\infty \alpha_n T_n, \quad \ell^\infty \to \mathcal{B}\,,$$

so that $Q_1 \in \mathcal{B}(\ell^\infty, \mathcal{B})$, and define

$$Q_2 : \lambda \mapsto (\lambda(t_n^{-1}) : n \in \mathbb{N}), \quad C_0(G) \to c_0\,,$$

so that $Q_2 \in \mathcal{B}(C_0(G), c_0)$; we note that $Q_2\lambda \in c_0 \ \ (\lambda \in C_0(G))$ because $\operatorname{Lim}_{n\to\infty} t_n^{-1} = \infty$ by (iii), above. Finally, consider the operator

$$Q_3 = Q_2 \circ \rho \circ Q_1 : \ \ell^\infty \to c_0\,.$$

We *claim* that $Q := Q_3 \mid c_0 : c_0 \to c_0$ is invertible in $\mathcal{B}(c_0)$. In fact,

$$Q(\alpha)_m = \sum_{n=1}^\infty \alpha_n x_{m,n} \quad (m \in \mathbb{N})$$

for each $\alpha \in c_0$, and so Q has the form $I_{c_0} - S$, where $S \in \mathcal{B}(c_0)$ satisfies the condition

$$\begin{aligned} \|S\| &= \sup_{m\in\mathbb{N}} \sum \{|x_{m,n}| : n \in \mathbb{N} \setminus \{m\}\} \\ &\leq \sum \{2^{-|m-n|-2} : n \in \mathbb{Z} \setminus \{m\}\} \\ &\leq \sum \{2^{-n-2} : n \in \mathbb{Z} \setminus \{0\}\} = 1/2\,, \end{aligned}$$

and so Q is invertible, giving the claim.

It follows that there is a projection on ℓ^∞ with range c_0, a contradiction of Proposition 1.12. Hence $\rho(T) = 0$, as required. □

Proposition 3.6 *Let G be a locally compact group which is σ-compact and non-compact. Then the left $L^1(G)$-module $C_0(G)$ is not injective.*

Proof We write $A = L^1(G)$, $E = C_0(G)$, and $\mathcal{B} = \mathcal{B}(A, E)$. The canonical embedding is $\Pi : E \to \mathcal{B}$, so that

$$\Pi(\lambda)(f)(t) = \int_G f(s)\lambda(ts)\,\mathrm{d}m(s) \quad (t \in G) \tag{3.1}$$

for $\lambda \in E$ and $f \in A$.

Assume towards a contradiction that E is injective, so that, by Proposition 1.7, there exists $\rho \in {}_A\mathcal{B}(\mathcal{B}, E)$ with $\rho \circ \Pi = I_E$.

Since G is σ-compact and non-compact, there is a sequence (K_n) of compact subsets of G such that $K_n \subsetneq \operatorname{int} K_{n+1}$ $(n \in \mathbb{N})$ and

$$G = \bigcup \{K_n : n \in \mathbb{N}\}.$$

For each $n \in \mathbb{N}$, set $L_n = K_n \setminus K_{n-1}$ (with $L_1 = K_1$), so that L_n is a non-empty Borel set in G, and take $k_n \in C_{00}(G)$ such that $k_n(G) \subset \mathbb{I}$, $k_n \mid K_n = 1$, and $\operatorname{supp} k_n \subset K_{n+1}$. Then define $Q_n : L^\infty(G) \to \mathcal{B}$ by

$$Q_n(\lambda)(f)(t) = \int_{L_n} f(s)\lambda(ts)k_n(ts)\,\mathrm{d}m(s) \quad (t \in G)$$

for $\lambda \in L^\infty(G)$ and $f \in A$. As a function of t on G, $Q_n(\lambda)(f)$ is continuous by [2, Proposition 3.3.13], being a convolution of a function in A with a function in $L^\infty(G)$. Also $Q_n(\lambda) \in \mathcal{B}$ with $\|Q_n(\lambda)\| \leq \|\lambda\|_\infty$, and so we see that $Q_n \in \mathcal{B}(L^\infty(G), \mathcal{B})$ with $\|Q_n\| \leq 1$. Finally, $\operatorname{supp} Q_n(\lambda) \subset L_n$.

For each $\lambda \in L^\infty(G)$, the sequence $(Q_n(\lambda))$ is bounded in $\mathcal{B}$ and is such that $\operatorname{supp} Q_n(\lambda) \subset L_n$ $(n \in \mathbb{N})$; since $\{L_n : n \in \mathbb{N}\}$ is a pairwise disjoint family of Borel subsets of G, we can define

$$Q(\lambda) = \sum_{n=1}^{\infty} Q_n(\lambda) \in \mathcal{B}.$$

Clearly, Q is a linear operator with $\|Q(\lambda)\| = \sup_n \|Q_n(\lambda)\| \leq \|\lambda\|_\infty$, and so we see that $Q \in \mathcal{B}(L^\infty(G), \mathcal{B})$.

Consider the composition $\rho \circ Q \in \mathcal{B}(L^\infty(G), E)$. We shall prove that

$$(\rho \circ Q) \mid E = I_E.$$

In fact, it suffices to show that $(\rho \circ Q)(\lambda) = \lambda$ $(\lambda \in C_{00}(G))$. Thus, take $\lambda \in C_{00}(G)$, and choose $n_0 \in \mathbb{N}$ such that $\operatorname{supp} \lambda \subset K_{n_0}$. Define $T = Q\lambda - \Pi\lambda$. We *claim* that $\operatorname{supp} T \subset K_{n_0}$. Indeed, take $f \in A$ with $f \mid K_{n_0} = 0$. Then

$$Tf = \lim_{n\to\infty} \sum_{m=n_0}^{n} \int_{L_m} f(s)\lambda(ts)(k_m(ts) - 1)\,\mathrm{d}m(s).$$

Consider the expression $r := \lambda(ts)(k_m(ts) - 1)$ for $m \geq n_0$. If $ts \in G \setminus K_{n_0}$, then $\lambda(ts) = 0$. If $ts \in K_{n_0}$, then $k_m(ts) = 1$. Thus $r = 0$ in each case, and so $Tf = 0$. This establishes the claim. By Lemma 3.5, $\rho(T) = 0$, and so $\rho(Q\lambda) = \rho(\Pi\lambda) = \lambda$, as required. Thus $C_0(G)$ is not injective.

We have shown that $\rho \circ Q$ is a projection of $L^\infty(G)$ onto $C_0(G)$, a contradiction of Proposition 1.13. □

Proposition 3.7 *Let G be an infinite and compact group. Then the left $L^1(G)$-module $C(G)$ is not injective.*

Proof This is more elementary version of Proposition 3.6: we now define $Q = \Pi$, where Π is specified in equation (3.1), now for $\lambda \in L^\infty(G)$ and $f \in A$, and then follow the final part of the above proof. We again obtain a projection of $L^\infty(G)$ onto $C(G)$, and this is a contradiction of Proposition 1.14. □

Theorem 3.8 *Let G be a locally compact group. Then the left $L^1(G)$-module $C_0(G)$ is injective if and only if G is finite.*

Proof We have remarked that $C_0(G)$ is injective whenever G is finite.

Assume towards a contradiction that G is infinite and that $C_0(G)$ is injective. By Lemma 3.4, there is an open and closed subgroup H of G such that H is infinite and σ-compact and $C_0(H)$ is an injective $L^1(H)$-module. In the cases where H is not compact and compact, respectively, this is a contradiction of Propositions 3.6 and 3.7, respectively, and so $C_0(G)$ is not injective in both cases. □

4 Amenability and injectivity

Again let G be a locally compact group. Let us consider a left $L^1(G)$-module E which is the dual of a right $L^1(G)$-module. In the case where G is amenable, it follows from Proposition 1.11 that E is injective. In this section, we shall seek a form of converse to this result. In particular, we shall show that the left $L^1(G)$-modules $M(G)$ and $L^1(G)''$ are each injective if and only if the group G is amenable.

Definition 4.1 *Let G be a locally compact group, and let E be a Banach left $L^1(G)$-module. An element $\lambda \in E'$ is an* augmentation-invariant *functional if*

$$\langle f \cdot x, \lambda \rangle = \varphi_G(f) \langle x, \lambda \rangle \quad (f \in L^1(G), \, x \in E) \, .$$

The module E is augmentation-invariant *if there is a non-zero, augmentation-invariant functional in E'.*

Example 4.2 Let G be a locally compact group.

(i) Set $E = L^\infty(G)$. An element $\Lambda \in E'$ is augmentation-invariant if and only if $\Lambda \cdot f = \varphi_G(f)\Lambda \;\; (f \in L^1(G))$, and this is just the classical definition of a topologically right-invariant mean, as given in [15], for example. Thus the module E is augmentation-invariant if and only if G is an amenable group [15, Corollary (1.10)].

(ii) Set $E = M(G)$, a faithful $L^1(G)$-module. Then $\lambda = \varphi_G$ is a non-zero augmentation-invariant functional in E' because

$$\varphi_G(f \star \mu) = \varphi_G(f)\varphi_G(\mu) \quad (f \in A, \, \mu \in M(G)) \, ,$$

and so $M(G)$ is augmentation-invariant.

(iii) Set $E = L^\infty(G)'$, again a faithful $L^1(G)$-module. Let λ be the constant function 1 on G, regarded as an element of $L^\infty(G)$, and hence as an element of $E' = L^\infty(G)''$. To see that λ is an augmentation-invariant functional in E', we must check that

$$\langle 1 \cdot f, \Phi \rangle = \varphi_G(f) \langle 1, \Phi \rangle \quad (f \in A, \, \Phi \in E) \, .$$

But this follows from equation (2.4), and so the module $L^\infty(G)'$ is augmentation-invariant.

(iv) In a similar way, each of the modules $C^b(G)'$, $LUC(G)'$, $AP(G)'$, and $WAP(G)'$ is augmentation-invariant. □

Again let G be a locally compact group, and set $A = L^1(G)$. We fix $E \in A$–**mod** and $s \in G$. We have defined the left-translate $L_s f = s \star f$ of $f \in A$. We recall the definitions of the *right-translate* $T \cdot s$ of $T \in \mathcal{B}(A, E)$ and the *left-translate* $s \cdot \Lambda$ of $\Lambda \in \mathcal{B}(A, E)'$. Indeed,

$$(T \cdot s)(f) = T(L_s f) \quad (f \in A) \, , \quad \langle T, \, s \cdot \Lambda \rangle = \langle T \cdot s, \, \Lambda \rangle \quad (T \in \mathcal{B}(A, E)) \, .$$

We note that, for $f, g \in A$, we have

$$
\begin{aligned}
((T \cdot s) \cdot f)(g) &= (T \cdot s)(f \star g) = T(L_s(f \star g)) \\
&= T(L_s f \star g) = (T \cdot L_s f)(g)\,, \qquad (4.1)
\end{aligned}
$$

where we are using equation (2.2), and so $(T \cdot s) \cdot f = T \cdot L_s f \ \ (f \in A)$.

We also define $f^{\triangleleft}$ for $f \in A$ by the formula

$$
f^{\triangleleft}(s) = f(s^{-1})\Delta_G(s^{-1}) \quad (s \in G)\,,
$$

where Δ_G is the modular function on G (so that $f^{\triangleleft} = \overline{f^*}$, where $f \mapsto f^*$ is the standard involution on A, as defined in [2, Definition 3.3.16]). Clearly the map $f \mapsto f^{\triangleleft}$ is a linear isometry on A, and we have $f^{\triangleleft\triangleleft} = f \ \ (f \in A)$ and $(f \star g)^{\triangleleft} = g^{\triangleleft} \star f^{\triangleleft} \ \ (f, g \in A)$. Further, $\varphi_G(f^{\triangleleft}) = \varphi_G(f) \ \ (f \in A)$.

For $T \in \mathcal{B}(A, E)$, we define $T^{\triangleleft} \in \mathcal{B}(A, E)$ by setting

$$
T^{\triangleleft}(f) = T(f^{\triangleleft}) \quad (f \in A)\,,
$$

so that the map $T \mapsto T^{\triangleleft}$ is a linear isometry on $\mathcal{B}(A, E)$ and

$$
(T \cdot f)^{\triangleleft} = f^{\triangleleft} \cdot T^{\triangleleft} \quad (f \in A)\,.
$$

Now suppose that the module $E \in A$–**mod** is augmentation-invariant, take a non-zero, augmentation-invariant functional $\lambda_0 \in E'$, and then take $x_0 \in E$ with $\langle x_0,\, \lambda_0 \rangle = 1$. Set

$$
T_0 = \Pi(x_0) \in \mathcal{B}(A, E)\,.
$$

Assume that E is faithful and injective. By Proposition 1.7, there exists $\rho \in {}_A\mathcal{B}(\mathcal{B}(A, E), E)$ with $\rho \circ \Pi = I_E$; in particular, $\rho(T_0) = x_0$. We shall consider the element $T_0^{\triangleleft} \in \mathcal{B}(A, E)$. After adjustment of λ_0 and x_0 by suitable non-zero constants, we may suppose that

$$
\|T_0\| = \|T_0^{\triangleleft}\| = 1\,. \qquad (4.2)
$$

Lemma 4.3 *There exists a left-invariant element $\Lambda_0 \in \mathcal{B}(A, E)'$ such that $\langle T_0^{\triangleleft}, \Lambda_0 \rangle = 1$.*

Proof First consider the element $\lambda_0 \circ \rho \in \mathcal{B}(A, E)'$. For each $f \in A$ and $T \in \mathcal{B}(A, E)$, we see that

$$
(\lambda_0 \circ \rho)(f^{\triangleleft} \cdot T^{\triangleleft}) = \varphi_G(f)(\lambda_0 \circ \rho)(T^{\triangleleft}) \qquad (4.3)
$$

because $\langle f^{\lhd} \cdot \rho(T^{\lhd}),\, \lambda_0\rangle = \varphi_G(f^{\lhd})\langle \rho(T^{\lhd}), \lambda_0\rangle$.

Now define Λ_0 by setting

$$\langle T,\, \Lambda_0\rangle = \langle T^{\lhd},\, \lambda_0 \circ \rho\rangle \quad (T \in \mathcal{B}(A, E))\,.$$

Then $\Lambda_0 \in \mathcal{B}(A, E)'$ and $\langle T_0^{\lhd},\, \Lambda_0\rangle = \langle T_0,\, \lambda_0 \circ \rho\rangle = \langle x_0,\, \lambda_0\rangle = 1$.

Take $f \in A$ with $\varphi_G(f) = 1$. Then, for each $T \in \mathcal{B}(A, E)$, we have

$$\langle T \cdot f,\, \Lambda_0\rangle = \langle f^{\lhd} \cdot T^{\lhd},\, \lambda_0 \circ \rho\rangle = \langle T^{\lhd},\, \lambda_0 \circ \rho\rangle = \langle T,\, \Lambda_0\rangle$$

by (4.3). This implies that, for each $s \in G$, we have

$$\begin{aligned} \langle T,\, L_s(\Lambda_0)\rangle &= \langle T \cdot s,\, \Lambda_0\rangle = \langle (T \cdot s) \cdot f,\, \Lambda_0\rangle \\ &= \langle T \cdot L_s f,\, \Lambda_0\rangle = \langle g \cdot T^{\lhd},\, \lambda_0 \circ \rho\rangle\,, \end{aligned}$$

by (4.1), where $g = (L_s f)^{\lhd}$, and so

$$\langle T,\, L_s(\Lambda_0)\rangle = \langle g \cdot \rho(T^{\lhd}),\, \lambda_0\rangle = \langle \rho(T^{\lhd}),\, \lambda_0\rangle$$

because $\varphi_G(g) = 1$ and λ_0 is augmentation-invariant.

Finally, we see that

$$\langle T,\, L_s(\Lambda_0)\rangle = \langle T^{\lhd},\, \lambda_0 \circ \rho\rangle = \langle T,\, \Lambda_0\rangle \quad (s \in G)$$

for each $T \in \mathcal{B}(A, E)$, and so $L_s(\Lambda_0) = \Lambda_0$ $(s \in G)$, as required. □

The further assumption for the next lemma is that E is a dual module, say $E = F'$, where $F \in$ **mod**–A. As in §1, $\mathcal{B}(A, E) \cong (A \widehat{\otimes} F)'$ as a Banach space. Set $X = A \widehat{\otimes} F$ and $\sigma = \sigma(X, X')$, the weak topology on X. For $s \in G$, we define

$$L_s(f \otimes y) = L_s f \otimes y \quad (f \in A,\, y \in F)\,,$$

so that $\langle T,\, L_s x\rangle = \langle T \cdot s,\, x\rangle$ $(x \in X,\, T \in \mathcal{B}(A, E))$, as before.

Lemma 4.4 *There is a net* (v_α) *in* X *such that* $\langle T_0^{\lhd}, v_\alpha\rangle = 1$ *for each* α *and such that* $\lim_\alpha \|L_s v_\alpha - v_\alpha\|_\pi = 0$ *for each* $s \in G$.

Proof First, a net (u_α) is indexed by the family of all pairs

$$\alpha = (\{s_1, \ldots, s_k\},\, \{T_1, \ldots, T_\ell\})\;,$$

where $\{s_1, \dots, s_k\}$ is a finite subset of G such that $s_1 = e_G$ and $\{T_1, \dots, T_\ell\}$ is a finite subset of $\mathcal{B}(A, E)$, with the ordering specified by the inclusion of the two components of the pair. For each such α, choose $u_\alpha \in X$ such that $\langle T_0^{\triangleleft}, u_\alpha \rangle = 1$ and $\langle T_j \cdot s_i, u_\alpha \rangle = \langle T_j \cdot s_i, \Lambda_0 \rangle$ $(i = 1, \dots, k,\ j = 1, \dots, \ell)$, where $\Lambda_0 \in X''$ was specified in Lemma 4.3.

For each $s \in G$ and $T \in \mathcal{B}(A, E)$, we have

$$\langle T,\, L_s u_\alpha \rangle = \langle T \cdot s,\, u_\alpha \rangle = \langle T \cdot s,\, \Lambda_0 \rangle = \langle T,\, L_s(\Lambda_0) \rangle = \langle T,\, \Lambda_0 \rangle = \langle T,\, u_\alpha \rangle$$

for each sufficiently large α, and so $\lim_\alpha (L_s u_\alpha - u_\alpha) = 0$ in (X, σ).

Let $\{s_1, \dots, s_k\}$ be a finite subset of G. Consider the Banach space $Y = \bigoplus_{i=1}^k X_i$, where $X_i = X$ $(i = 1, \dots, k)$ and we are taking the ℓ^1-sum, and also consider the linear operator

$$W : x \mapsto (L_{s_1} x - x, \dots, L_{s_k} x - x), \quad X \to Y\,.$$

The set $C = \{x \in X : \langle T_0^{\triangleleft},\, x \rangle = 1\}$ is convex in X, and so $W(C)$ is also convex in Y. We have shown that 0 belongs to the $\sigma(Y, Y')$-closure of $W(C)$ in Y. But, by Mazur's theorem (see [2, Theorem A.3.29(ii)]), it follows that 0 belongs to the $\|\cdot\|$-closure of $W(C)$ in Y. The existence of the required net (v_α) follows. $\square$

In the following lemma, we maintain the above hypotheses that G is a locally compact group and that $E = F'$ is a faithful, augmentation-invariant, and injective module in $L^1(G)$-**mod**.

Lemma 4.5 *There is a net (h_α) in $P(G)$ such that* $\lim_\alpha \|L_s h_\alpha - h_\alpha\|_1 = 0$ *for each $s \in G$.*

Proof The space $X = A \widehat{\otimes} F = L^1(G) \widehat{\otimes} F$ is now identified with the space $L^1(G, F)$. Let (v_α) be the net in $L^1(G, F)$ specified in Lemma 4.4, and set

$$k_\alpha(t) = \|v_\alpha(t)\|_F \quad (t \in G)$$

for each α, so that (k_α) is a net in A. For each α, we have from (4.2) that

$$\langle k_\alpha,\, 1 \rangle = \int_G k_\alpha(t)\, \mathrm{d}m(t) = \|v_\alpha\|_F \geq |\langle T_0^{\triangleleft}, v_\alpha \rangle| = 1\,.$$

Set $h_\alpha = k_\alpha / \langle k_\alpha, 1 \rangle$. Then $\langle h_\alpha, 1 \rangle = 1$ and $h_\alpha \geq 0$, and so $h_\alpha \in P(G)$.

Now take $s \in G$. We have $(L_s v_\alpha)(t) = v_\alpha(s^{-1}t)$ $(t \in G)$, and so $(L_s h_\alpha)(t) = h_\alpha(s^{-1}t)$ $(t \in G)$. It follows that

$$\begin{aligned}\|L_s h_\alpha - h_\alpha\|_1 &\leq \|L_s k_\alpha - k_\alpha\|_1 \leq \int_G \|(L_s v_\alpha - v_\alpha)(t)\|_F \,\mathrm{d}m(t) \\ &= \|L_s v_\alpha - v_\alpha\|_\pi ,\end{aligned}$$

and so $\lim_\alpha \|L_s h_\alpha - h_\alpha\|_1 = 0$, as required. □

Theorem 4.6 *Let G be a locally compact group, and let E be a Banach left $L^1(G)$-module such that E is the dual of a Banach right $L^1(G)$-module. Suppose that E is faithful and augmentation-invariant. Then E is injective if and only if G is amenable.*

Proof Suppose that G is amenable. By Theorem 2.3, $L^1(G)$ is an amenable Banach algebra, and so, by Proposition 1.11, the dual module E is injective.

Conversely, suppose that E is injective. Then the above analysis applies, and so there is a net (h_α) in $P(G)$ as specified in Lemma 4.5. By Proposition 2.2, the group G is amenable. □

An analogous result holds if we exchange 'right' and 'left' in the above theorem.

Corollary 4.7 *Let G be a locally compact group. Then the following conditions are equivalent:*

(a) *the group G is amenable;*

(b) *$M(G)$ is injective as a Banach left $L^1(G)$-module;*

(c) *$L^\infty(G)$ is flat as a Banach left $L^1(G)$-module;*

(d) *$C_0(G)$ is flat as a Banach left $L^1(G)$-module.*

Proof We have remarked that $M(G)$ and $L^\infty(G)'$ satisfy the conditions on the module E in the theorem. Thus equivalences (a)⇔(b)⇔(c) follow immediately from that result. Since $M(G) = C_0(G)'$ is injective as a right $L^1(G)$-module if and only if G is amenable, we also have (a)⇔(d). □

Corollary 4.8 *Let G be a group. Then $\ell^1(G)$ is injective as a Banach $\ell^1(G)$-module if and only if G is amenable.* □

Theorem 4.9 *Let G be a locally compact group. Then $L^1(G)$ is injective as a Banach left $L^1(G)$-module if and only if G is discrete and amenable.*

Proof The algebra $L^1(G)$ has a bounded approximate identity, and so it follows from Corollary 1.9 that $L^1(G)$ is unital, and hence that G is discrete, in the case where $L^1(G)$ is injective. The result now follows from Corollary 4.8. $\square$

Corollary 4.10 *There a locally compact group G such that $L^1(G)$ is not injective, but $L^1(G)''$ is injective, in $L^1(G)$-**mod**.*

Proof Take G to be a locally compact group which is amenable, but not discrete. $\square$

5 The modules $L^p(G)$

There is a further class of modules that we investigate. Let G be a locally compact group, and take $p \geq 1$, and let $E = L^p(G)$ be the Banach space $L^p(G, m)$. Then $L^p(G)$ is a faithful, essential Banach $L^1(G)$-bimodule for the products $(f, g) \mapsto f \star_p g$ and $(f, g) \mapsto g \star_p f$ from $L^1(G) \times E$ into E, where $f \star_p g = f \star g$ and $g \star_p f$ is specified by

$$(g \star_p f)(t) = \int_G g(ts^{-1})f(s)\Delta(s^{-1})^{1/p}\,\mathrm{d}m(s) \quad (t \in G)\,.$$

See [2, Theorem 3.3.19] for further details.

Take $p > 1$, and let q be the conjugate index to p. Then the dual of the bimodule $(L^p(G), \star_p)$ is $(L^q(G), \cdot_q)$ for a certain module operation $\cdot_q$, and the dual of $(L^q(G), \cdot_q)$ is $(L^p(G), \star_p)$ (see [2, p. 381]).

The following theorem confirms a conjecture of Selivanov [19, p. 166].

Theorem 5.1 *Let G be a locally compact group, and take p with $1 < p < \infty$. Then $L^p(G)$ is projective in $L^1(G)$-**mod** if and only if G is a compact space.*

Proof This is almost the same as the proof of Theorem 3.1, and we shall omit the details. Set $A = L^1(G)$ and $E = L^p(G)$.

Suppose first that G is compact. For $g \in E$, now define $\rho(g) \in L^1(G, E)$ by

$$\rho(\lambda)(s)(t) = \lambda(st) \quad (s, t \in G)\,.$$

Then it is easily checked that ρ is a right inverse to $\pi \in {}_A\mathcal{B}(A^\flat \widehat{\otimes} E, E)$, and so E is projective by Proposition 1.2(i).

For the converse, again assume towards a contradiction that E is projective, but that G is not compact.

Choose $f \in C_{00}(G)$, $T \in {}_A\mathcal{B}(E, A^\flat)$, $k \in \mathbb{N}$, and $g \in C_{00}(G)$ as in the proof of Theorem 3.1, so that $\|f \cdot f\|_p \neq 0$, and again set $\eta = \|f \star Tf\|_1 / 2 > 0$ and $f_j = s_j \star f \ (j = 1, \dots, k)$ for suitable $s_1, \dots, s_k \in G$. Now we have

$$\left\| \sum_{j=1}^{k} f_j \cdot f \right\|_p = k^{1/p} \|f \cdot f\|_p \,,$$

and we conclude that $k\eta \leq k^{1/p} \|T\| \|f \cdot f\|_p + 1$. This holds for each $k \in \mathbb{N}$, again a contradiction. □

Again, a very small variation of the above proof shows that $(L^q(G), \cdot_q)$ is injective in $L^1(G)$-**mod** if and only if G is compact.

We now fix p with $1 < p < \infty$, and seek to determine when the module $L^p(G)$ is injective. Since $L^p(G)$ is a dual module, it follows from Theorem 2.3 and Proposition 1.11 that this is the case whenever G is an amenable locally compact group. We *conjecture* that the converse is also true, but we are not able to prove this. Indeed, we are only able to make reasonable progress in the case where G is discrete, and so we suppose this henceforth.

Throughout, we set $A = \ell^1(G)$ and $E = \ell^p(G)$, so that E is a left A-module for the operation $\star$.

We begin with a lemma. For $n \in \mathbb{N}$, we set

$$D_n = \{(d_1, \dots, d_n) : d_j = \pm 1 \ (j = 1, \dots, n)\} \,.$$

Lemma 5.2 *Fix $n \in \mathbb{N}$, $\lambda_1, \dots, \lambda_n \in E$, and $s_1, \dots, s_n \in G$, and set*

$$C = \max \left\{ \left\| \sum_{i=1}^{n} d_i \lambda_i \right\|_p : d = (d_1, \dots, d_n) \in D_n \right\} .$$

Then

$$\left\| \sum_{i=1}^{n} \lambda_i(s_i) \delta_{s_i} \right\|_p \leq C \,.$$

Proof The result is trivial in the case where $n = 1$, and so we may suppose that $n \geq 2$.

Take $d \in D_n$, and set $x_{j,d} = \sum_{i=1}^n d_i \lambda_i(s_j)$ $(j = 1, \dots, n)$. By hypothesis, we have $\sum_{j=1}^n |x_{j,d}|^p \leq C^p$. Since there are 2^n elements in D_n, we have

$$\sum_{j=1}^n \sum_{d \in D_n} |x_{j,d}|^p \leq 2^n C^p \,.$$

We can write the term $\sum_{d \in D_n} |x_{j,d}|^p$ as

$$2 \sum_{d \in D_{n-1}} \left| \lambda_n(s_n) + \sum_{i=1}^{n-1} d_i \lambda_i(s_n) \right|^p \,.$$

The set D_{n-1} can be partitioned into two disjoint sets S and T such that $T = \{-d : d \in S\}$, and $|S| = |T| = 2^{n-2}$. Since the function $t \mapsto |t|^p$ is convex on $\mathbb{R}$, we have $2\,|a|^p \leq |a+b|^p + |a-b|^p$ $(a, b \in \mathbb{R})$, and so

$$\sum_{d \in D_{n-1}} \left| \lambda_n(s_n) + \sum_{i=1}^{n-1} d_i \lambda_i(s_n) \right|^p \geq 2^{n-2} \cdot 2\,|\lambda_n(s_n)|^p \,.$$

A similar estimate holds for $|\lambda_j(s_j)|^p$ for $j = 1, \dots, n-1$, and so we see that

$$2^n \sum_{j=1}^n |\lambda_j(s_j)|^p \leq \sum_{j=1}^n \sum_{d \in D_n} |x_{j,d}|^p \leq 2^n C^p \,.$$

Hence we have $\sum_{j=1}^n |\lambda_j(s_j)|^p \leq C^p$, which is a reformulation of the claimed inequality. $\square$

We now consider the space $\mathcal{B} = \mathcal{B}(A, E)$. We recall that

$$\|U\| = \sup\{\|U(\delta_t)\|_p : t \in G\}$$

for each $U \in \mathcal{B}$. We shall identify $\mathcal{B}$ with a space of functions contained in $\mathbb{C}^{G \times G}$.

Let $U : G \times G \to \mathbb{C}$ be a function. For $s, t \in G$, define

$$U_t : s \mapsto U(t, s), \quad U^s : t \mapsto U(t, s), \quad G \to \mathbb{C} \,.$$

For $U \in \mathcal{B}$, define

$$U(t, s) = U(\delta_t)(s) \quad (s, t \in G) \,.$$

We obtain a space, also called $\mathcal{B}$, of functions on $G\times G$; the space is characterized by the property that $U_t \in E$ $(t\in G)$ and $\|U\| = \sup_{t\in G}\|U_t\|_p < \infty$. Note that the function

$$(t,s)\mapsto U(t,rs),\quad G\times G\to\mathbb{C}\,,$$

belongs to $\mathcal{B}$ whenever $U\in\mathcal{B}$ and $r\in G$, and that the norm is unchanged.

For $U\in\mathbb{C}^{G\times G}$ and $r\in G$, define

$$U^{(r)}(t,s) = \delta_{r,s}U(t,s)\quad (s,t\in G)\,,$$

so that $U^{(r)}$ is equal to the product of U and the characteristic function of the 'horizontal slice' $G\times\{r\}$ in $G\times G$. For each $s,t\in G$, we have

$$U(t,s) = \sum_{r\in G} U^{(r)}(t,s)\,.$$

Let $U\in\mathcal{B}$ and $r\in G$. Then $U^{(r)}\in\mathcal{B}$ with $\left\|U^{(r)}\right\|\leq\|U\|$. Since $|U^r(t)| = \left\|U_t^{(r)}\right\|\leq\|U_t\|$ $(t\in G)$, we have $U^r\in\ell^\infty(G)$ with $\|U^r\|_\infty\leq\|U\|$. Further, the map $U\mapsto U^r$, $\mathcal{B}\to\ell^\infty(G)$, is a linear surjection.

Let $U\in\mathbb{C}^{G\times G}$. For each $r\in G$, we define

$$(r\,\cdot\,U)(t,s) = U(tr,s)\quad (r,s,t\in G)\,. \tag{5.1}$$

In the case where $U\in\mathcal{B}$, the above map is precisely the action of A on $\mathcal{B}$. Also define

$$Q(U)(t,s) = U(t,t^{-1}s)\quad (s,t\in G)\,,$$

so that Q is a linear map on $\mathbb{C}^{G\times G}$. Suppose that $U\in\mathcal{B}$. Then, for each $t\in G$, we have $Q(U)_t\in E$ with $\|Q(U)_t\|_p = \|U_t\|_p$, and so $Q(U)\in\mathcal{B}$ with $\|Q(U)\| = \|U\|$. This shows that Q is a linear isometry on $\mathcal{B}$.

Take $\lambda\in\mathbb{C}^G$. Then we define

$$\widetilde{\lambda}(t,s) = \lambda(s)\quad (s,t\in G)\,.$$

In the case where $\lambda\in E$, we have $\widetilde{\lambda}\in\mathcal{B}$ with $\left\|\widetilde{\lambda}\right\| = \|\lambda\|_p$, and the map $\lambda\mapsto\widetilde{\lambda}$, $E\to\mathcal{B}$, is a linear isometry.

Let $\lambda\in E$. We recall that $(\Pi\lambda)(\delta_t) = t\star\lambda$ $(t\in G)$, and so we see that $Q(\widetilde{\lambda})(t,s) = \lambda(t^{-1}s) = (\Pi\lambda)(t,s)$ $(s,t\in G)$. Thus

$$Q(\widetilde{\lambda}) = \Pi\lambda\quad (\lambda\in E)\,. \tag{5.2}$$

We shall also require a further modification of the space $\mathcal{B}$.

For each $U \in \mathbb{C}^{G\times G}$, define

$$S(U)(t,s) = U(ts^{-1},s), \quad S^{-1}(U)(t,s) = U(ts,s) \quad (s,t\in G)\,.$$

Then S and S^{-1} are linear maps on $\mathbb{C}^{G\times G}$, and S^{-1} really is the inverse of S. We note that

$$(Q\,\circ\,S^{-1})(U)(t,s) = U(s,t^{-1}s) \quad (s,t\in G) \tag{5.3}$$

because the composition of the maps $(t,s)\mapsto(t,t^{-1}s)$ and $(u,v)\mapsto(uv,v)$ is the map $(t,s)\mapsto(s,t^{-1}s)$ on $G\times G$. We define

$$S(\mathcal{B}) = \{S(U) : U\in\mathcal{B}\}\,.$$

Then $S(\mathcal{B})$ is a linear subspace of $\mathbb{C}^{G\times G}$, linearly isomorphic to $\mathcal{B}$.

For $U\in\mathbb{C}^{G\times G}$ and $r\in G$, define

$$(L_rU)(t,s) = U(t,r^{-1}s) \quad (s,t\in G)\,,$$

so that each L_r is a linear map on the space $\mathbb{C}^{G\times G}$ and an isometry on $\mathcal{B}$. We *claim* that

$$(Q\,\circ\,S^{-1}\,\circ\,L_r)(U) = r\,\cdot\,((Q\,\circ\,S^{-1})(U)) \quad (r\in G) \tag{5.4}$$

for each $U\in\mathbb{C}^{G\times G}$. This holds because the two maps $(t,s)\mapsto(s,t^{-1}s)\mapsto(s,r^{-1}t^{-1}s)$ and $(t,s)\mapsto(tr,s)\mapsto(s,(tr)^{-1}s)$ on $G\times G$ are equal; we have used equation (5.3) here.

Note that

$$(S^{-1}(U))^{(r)} = S^{-1}(U^{(r)}) \quad \text{and} \quad L_u(U^{(r)}) = (L_uU)^{(ur)} \tag{5.5}$$

and that

$$(S^{-1}(U))^{r} = r\,\cdot\,U^{r} \quad \text{and} \quad U^{r} = (L_uU)^{ur} \tag{5.6}$$

for each $u\in G$.

Let $r,s,t\in G$ and $\lambda\in E$. Then clearly $(Q(\widetilde{\lambda}^{(r)}))(t,s)$ is equal to $\lambda(r)$ when $s=tr$ and to 0 otherwise, and so, by (5.2), we have

$$Q(\widetilde{\lambda}^{(r)}) = \lambda(r)\,\Pi(\delta_r) \quad (r\in G,\lambda\in E)\,. \tag{5.7}$$

We now consider when the module $E = \ell^p(G)$ is injective. Since E is a faithful module, this is the case if and only if there exists $\rho \in {}_A\mathcal{B}(\mathcal{B}, E)$ with $\rho \circ \Pi = I_E$.

Let ρ be such a map, and define

$$\overline{\rho} = \rho \circ Q : \mathcal{B} \to E\,.$$

Certainly $\overline{\rho} \in \mathcal{B}(\mathcal{B}, E)$ and $\|\overline{\rho}\| = \|\rho\|$. For $\lambda \in E$, we have

$$\overline{\rho}(\widetilde{\lambda}) = \rho(Q(\widetilde{\lambda})) = (\rho \circ \Pi)(\lambda) = \lambda\,,$$

where we are using (5.2). Also $\overline{\rho} \circ S^{-1}$ is a map from $S(\mathcal{B})$ to E such that

$$(\overline{\rho} \circ S^{-1})(L_r U) = \rho(r \cdot ((Q \circ S^{-1})(U))) = r \star (\overline{\rho} \circ S^{-1})(U) \quad (r \in G) \tag{5.8}$$

for each $U \in S(\mathcal{B})$; here we are using (5.4).

Again take $U \in \mathcal{B}$. The next step is to define

$$\rho_1(U)(s) = \overline{\rho}(U^{(s)})(s) \quad (s \in G)\,.$$

We note that $\rho_1(U^{(s)})(t) = 0$ $(t \in G \setminus \{s\})$. We *claim* that $\rho_1(U) \in E$. To see this, take $s_1, \dots, s_n$ to be distinct elements of G, take $(d_1, \dots, d_n) \in D_n$, and set $V = \sum_{i=1}^n d_i U^{(s_i)}$, so that $\|V\| \leq \|U\|$. Then clearly $\overline{\rho}(V) \in E$ with

$$\left\|\overline{\rho}(V)\right\|_p \leq \|\rho\|\,\|U\|\ .$$

By applying Lemma 5.2 with $\lambda_i = \overline{\rho}(U^{(s_i)})$ for $i = 1, \dots, n$, we see that

$$\left\|\sum_{i=1}^n \overline{\rho}(U^{(s_i)})(s_i)\delta_{s_i}\right\|_p \leq \|\rho\|\,\|U\|\ .$$

This shows that

$$\left\|\sum_{i=1}^n \rho_1(U)(s_i)\delta_{s_i}\right\|_p \leq \|\rho\|\,\|U\|\ .$$

But this is true for each choice of $s_1, \dots, s_n \in G$, and so $\rho_1(U) \in E$ with $\|\rho_1(U)\| \leq \|\rho\|\,\|U\|$, as claimed. It follows that $\rho_1 \in \mathcal{B}(\mathcal{B}, E)$ and that $\|\rho_1\| \leq \|\rho\|$.

We also note that

$$\rho_1(U)(s) = \rho_1(U^{(s)})(s) \quad (s \in G,\, U \in \mathcal{B})\,. \tag{5.9}$$

For each $r \in G$ and $\lambda \in E$, we have $\rho_1(\widetilde{\lambda})(r) = \rho(Q(\widetilde{\lambda}^{(r)}))(r) = \lambda(r)$ by (5.7), and so

$$\rho_1(\widetilde{\lambda}) = \lambda \quad (\lambda \in E)\,. \tag{5.10}$$

The final map that we define is

$$\rho_2 = \rho_1 \circ S^{-1} : S(\mathcal{B}) \to E\,.$$

We now combine equations (5.4), (5.5), and (5.8) in the following calculation. Let $U \in S(\mathcal{B})$ and $r \in G$. Then, for each $s \in G$, we have

$$\begin{aligned}\rho_2(L_rU)(s) &= \overline{\rho}((S^{-1}(L_rU))^{(s)})(s) = \overline{\rho}(S^{-1}((L_rU)^{(s)}))(s)\\ &= (\overline{\rho} \circ S^{-1})(L_r(U^{(r^{-1}s)}))(s) \qquad \text{by (5.5)}\\ &= (r \star (\overline{\rho} \circ S^{-1})(U^{(r^{-1}s)}))(s)\\ &= ((\overline{\rho} \circ S^{-1})(U^{(r^{-1}s)}))(r^{-1}s) = (r \star \rho_2(U))(s)\,.\end{aligned}$$

Hence we see that

$$\rho_2(L_rU) = r \star \rho_2(U) \quad (r \in G,\, U \in S(\mathcal{B}))\,. \tag{5.11}$$

We can now reformulate the condition that E be injective.

Theorem 5.3 *Let G be a group, and take p with $1 < p < \infty$. Suppose that the Banach left $\ell^1(G)$-module $\ell^p(G)$ is injective. Then there are a bounded linear operator $P : \ell^\infty(G, \ell^p(G)) \to \ell^p(G)$ and $\Lambda \in \ell^\infty(G)'$ such that the following conditions are satisfied:*

(a) $P(U)(r) = \langle r^{-1} \cdot U^r,\, \Lambda\rangle \quad (r \in G,\, U \in \ell^\infty(G, \ell^p(G)))$;

(b) $\langle 1, \Lambda\rangle = 1$.

Proof Set $A = \ell^1(G)$, $E = \ell^p(G)$, and $\mathcal{B} = \ell^\infty(G, \ell^p(G))$, with the above identifications.

Suppose that E is injective, and take P to be the map $\rho_1 \in \mathcal{B}(\mathcal{B}, E)$ of the above analysis. Let $U \in \mathcal{B}$. For each $s \in G$, the element $P(U^{(s)}) \in E$ has the form $\alpha_s\delta_s$ for some $\alpha_s \in \mathbb{C}$ with $|\alpha_s| \leq \|P\|\,\|U\|$. The map $U \mapsto \alpha_s$ is clearly linear. We set $\langle U^s, \Lambda_s\rangle = \alpha_s$.

We note that $P(U)(s) = \langle U^s, \Lambda_s\rangle \ \ (s \in G)$; this follows from equation (5.9).

Let $\rho_2 = P \circ S^{-1} : S(\mathcal{B}) \to E$, as above. Then it follows from (5.6) that

$$\rho_2(U)(s) = \langle (S^{-1}(U))^s, \Lambda_s\rangle = \langle s \cdot U^s,\, \Lambda_s\rangle \quad (s \in G)\,.$$

For each $r, s \in G$ and $V \in S(\mathcal{B})$, we have

$$(r \star \rho_2(V))(s) = \rho_2(V)(r^{-1}s) = \langle r^{-1}s \cdot V^{r^{-1}s}, \Lambda_{r^{-1}s}\rangle$$

and also, by (5.6), we have

$$(r \star \rho_2(V))(s) = \langle s \cdot (L_rV)^s, \Lambda_s\rangle = \langle s \cdot V^{r^{-1}s}, \Lambda_s\rangle\,.$$

Set $\lambda = r^{-1}s \cdot V^{r^{-1}s}$. Then we have shown that

$$\langle \lambda, \Lambda_{r^{-1}s}\rangle = (r \star \rho_2(V))(s) = \langle s \cdot V^{r^{-1}s}, \Lambda_s\rangle = \langle r \cdot \lambda, \Lambda_s\rangle\,.$$

We define $\Lambda \in \ell^\infty(G)'$ to be Λ_e, where e is the identity of G. For each $U \in \mathcal{B}$, we have $P(U)(r) = \langle U^r, \Lambda_r\rangle = \langle r^{-1} \cdot U^r, \Lambda\rangle$ $(r \in G)$. This establishes clause (a).

Take $U = \widetilde{\delta_e}$, so that U is the constant function 1 on $G \times G$, and U^e is identified with the sequence which is constantly 1 in $\ell^\infty(G)$. Then

$$\langle 1, \Lambda\rangle = P(U)(e) = \delta_e(e) = 1\,.$$

This establishes clause (b). □

Let $\mathbb{F}_2$ denote the free group on two generators (denoted by a and b). This group is the standard example of a group which is not amenable. We now show that Theorem 5.3 is already sufficient to show that the module $\ell^p(\mathbb{F}_2)$ is not injective in $\ell^1(\mathbb{F}_2)$-**mod** whenever $1 < p < \infty$.

The group $\mathbb{F}_2$ consists of the set of words in the two letters a and b and their inverse a^{-1} and b^{-1}, together with the empty word, which is denoted by e. We take each word to be reduced; the product of two words is formed by juxtaposition and reduction to the corresponding reduced form. For $k \in \mathbb{Z} \setminus \{0\}$, we denote by A_k the set of words that end in a^k, and we set

$$B = \bigcup\{A_k : k \in \mathbb{Z} \setminus \{0\}\}, \quad A_0 = \mathbb{F}_2 \setminus B\,.$$

Thus A_0 is the set of words which end in b^n for some $n \in \mathbb{Z} \setminus \{0\}$, and $\{A_k : k \in \mathbb{Z}\}$ is a partition of the group $\mathbb{F}_2$ into pairwise disjoint subsets. Clearly $A_ja^k = A_{j+k}$ $(j, k \in \mathbb{Z})$ and $Bb \subset A_0$. We also set

$$B_k := Bba^k \subset A_0a^k = A_k \quad (k \in \mathbb{N})\,.$$

The subsets $B_1, B_2, B_3, \ldots$ of $\mathbb{F}_2$ are pairwise disjoint.

Proposition 5.4 *Take p with $1 < p < \infty$. Then the Banach left $\ell^1(\mathbb{F}_2)$-module $\ell^p(\mathbb{F}_2)$ is not injective.*

Proof Assume towards a contradiction that the module $\ell^p(\mathbb{F}_2)$ is injective. By Proposition 5.3, there are an operator $P : \ell^\infty(\mathbb{F}_2, \ell^p(\mathbb{F}_2)) \to \ell^p(\mathbb{F}_2)$ and $\Lambda \in \ell^\infty(\mathbb{F}_2)'$ such that clauses (a) and (b) of that proposition are satisfied (with $G = \mathbb{F}_2$).

Since $\{A_0, B\}$ is a partition of $\mathbb{F}_2$ and since $\langle 1, \Lambda\rangle = 1$, we see that either $\langle \chi_{A_0}, \Lambda\rangle \neq 0$ or $\langle \chi_B, \lambda\rangle \neq 0$.

We suppose first that $c \neq 0$, where $c = \langle \chi_B, \lambda\rangle$.

Fix $n \in \mathbb{N}$, and set $S = \{ba^k : k = 1, \dots, n\}$. For $s, t \in G$, define $U(t, s) = \chi_{Bs}(t)$. For each $t \in G$, we have $t \in Bs$ for at most one value of $s \in S$, and so $\|U_t\|_p \leq 1$; for some values of t, we have $t \in Bs$ for some $s \in S$, and so $\|U_t\|_p = 1$ in these cases. Thus $U \in \mathcal{B}$ and $\|U\| = 1$. Clearly $U^{(s)}(t, s) = \chi_{Bs}(t)\ \ (s, t \in G)$, and so it follows from clause (a) of Theorem 5.3 that

$$P(U)(s) = \langle s^{-1} \star \chi_{Bs}, \Lambda\rangle = \langle \chi_B, \Lambda\rangle = c \quad (s \in S)\,.$$

Thus $\|P(U)\|_p \geq c\,|S|^{1/p} = cn^{1/p}$. This is true for each $n \in \mathbb{N}$, a contradiction, and so $\ell^p(\mathbb{F}_2)$ is not injective.

A similar contradiction arises in the case where $\langle \chi_{A_0}, \Lambda\rangle \neq 0$. $\square$

We now introduce a rather strange condition that is related to the amenability of a group G. Recall that G is amenable if and only if it satisfies *Følner's condition*: for each $\varepsilon > 0$ and each finite subset F of G, there is a non-empty, finite subset S of G such that

$$|Sx \Delta Sy| < \varepsilon\,|S| \quad (x, y \in F)\,.$$

Here Δ denotes the symmetric difference of two sets. (See [15, Chapter 4], for example.)

Let S be a set, and let $n \in \mathbb{N}$. We denote by $\mathcal{P}_n(S)$ the family of all subsets of S with cardinality n.

Definition 5.5 *Let G be a group. Then G is* pseudo-amenable *if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for each $n \geq n_0$ and each $F \in \mathcal{P}_n(G)$, there exists a non-empty, finite subset S of G such that $|SF| < \varepsilon n\,|S|$.*

It is easy to see that an amenable group is pseudo-amenable. Indeed, let G be an amenable group, and take $\varepsilon > 0$. Choose $n_0 \in \mathbb{N}$ such that $\varepsilon n_0 > 2$. Take $n \geq n_0$ and $F := \{t_1, \dots, t_n\} \in \mathcal{P}_n(G)$. By the Følner condition, there is a non-empty, finite subset S of G such that $2\,|Sx\Delta Sy| < \varepsilon\,|S|$ for each $x, y \in F$. Then

$$SF = \bigcup_{i=1}^{n} St_i \subset St_1 \cup (St_1 \Delta St_2) \cup \cdots \cup (St_{n-1} \Delta St_n)\,,$$

and so $|SF| \leq |S| + (n-1)\varepsilon\,|S|\,/2 < \varepsilon n\,|S|$. This shows that G is pseudo-amenable.

Lemma 5.6 *Each subgroup of a pseudo-amenable group is pseudo-amenable.*

Proof Let G be a pseudo-amenable group, and let H be a subgroup of G. We shall show that H is pseudo-amenable; we may suppose that H is infinite.

We write G as the disjoint union of sets $s_\alpha H$ for a family $\{s_\alpha\}$ in G.

Take $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that, for each $n \geq n_0$ and each $F \in \mathcal{P}_n(H)$, there exists a non-empty, finite subset S of G such that $|SF| < \varepsilon n\,|S|$. Set $S_\alpha = S \cap s_\alpha H$, so that $S_\alpha \neq \emptyset$ for only finitely many values of α; for each such α, choose $t_\alpha \in H$ so that the sets $T_\alpha := t_\alpha s_\alpha^{-1} S_\alpha$ are pairwise disjoint subsets of H. Take T to be the union of the sets T_α, so that $T \subset H$ and $|T| = |S|$. Then

$$|TF| \leq \sum_\alpha |T_\alpha F| = \sum_\alpha |S_\alpha F| = |SF| < \varepsilon n\,|S| = \varepsilon n\,|T|\ ,$$

and so H is pseudo-amenable. □

Lemma 5.7 *Let G be a pseudo-amenable group. Then, for each $\varepsilon > 0$ and each $k \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that, for each $n \geq n_0$ and each $F_1, \dots, F_k \in \mathcal{P}_n(G)$, there exists a non-empty, finite subset S of G such that $|SF_j| < \varepsilon n\,|S|\ \ (j = 1, \dots, k)$.*

Proof Let $\varepsilon > 0$ and $k \in \mathbb{N}$. Then there exists $n_0 \in \mathbb{N}$ such that, for each $m \geq n_0$ and each $F \in \mathcal{P}_m(G)$, there exists a non-empty, finite subset S of G such that $|SF| < (\varepsilon/k)m\,|S|$. Now take $F_1, \dots, F_k \in \mathcal{P}_n(G)$, and set $F = F_1 \cup \cdots \cup F_k$, so that $F \in \mathcal{P}_m(G)$ for some $m \in \mathbb{N}$ with $n \leq m \leq kn$. Then, for $j = 1, \dots, k$, we have $|SF_j| \leq |SF| \leq (\varepsilon/k)kn\,|S| = \varepsilon n\,|S|$, as required. □

Lemma 5.8 *The group $\mathbb{F}_2$ is not pseudo-amenable.*

Proof Set $\varepsilon = 1/3$. For $n \in \mathbb{N}$, consider the two sets $F_1 = \{ba, \dots, ba^n\}$ and $F_2 = \{aba, \dots, aba^n\}$, so that $F_1, F_2 \in \mathcal{P}_n(\mathbb{F}_2)$.

Assume towards a contradiction that there is a non-empty, finite subset S of $\mathbb{F}_2$ with

$$|SF_i| < \varepsilon n\, |S| = mn/3 \quad (i = 1, 2)\,,$$

where $|S| = m$. Define $S_j = S \cap A_j$ $(j \in \mathbb{N})$, where A_j was defined above. There exists $j \in \{0, -1\}$ such that $|S_j| \leq m/2$; we have $|S \setminus S_j| \geq m/2$. Set $T = Sa^{-j}b$ and $T_0 = T \cap A_0$, and consider the translation $\tau : s \mapsto sa^{-j}b$ on $\mathbb{F}_2$. Clearly $\tau(S \setminus S_j) \subset T_0$, and so $|T_0| \geq m/2$. We have $SF_{1-j} = \bigcup_{i=1}^n Ta^i$, and so $\bigcup_{i=1}^n T_0a^i \subset SF_{1-j}$. However, $T_0a^u \cap T_0a^v = \emptyset$ whenever $u, v \in \mathbb{Z}$ with $u \neq v$, and $|T_0a^u| = |T_0|$ for $u = 1, \dots, n$, and so $|SF_{1-j}| \geq mn/2$, a contradiction of the fact that $|SF_{1-j}| < mn/3$.

Thus, for each $n \in \mathbb{N}$, we have sets $F_1, F_2 \in \mathcal{P}_n(\mathbb{F}_2)$ such that the inequality $|SF_i| < \varepsilon n\, |S|$ fails for at least one of $i = 1$ and $i = 2$. By Lemma 5.7, the group $\mathbb{F}_2$ is not pseudo-amenable. □

The following result follows from Lemmas 5.6 and 5.8.

Theorem 5.9 *Let G be a group containing the group $\mathbb{F}_2$ as a subgroup. Then G is not pseudo-amenable.* □

We do not know whether or not every pseudo-amenable group is already amenable. In particular, we do not know whether or not Ol'shanskii's group G_0, mentioned in §2, is pseudo-amenable.

We shall now prove that G is pseudo-amenable whenever $\ell^p(G)$ is injective in $\ell^p(G)$-**mod**. First we require some further notation.

Let G be a group, and let S be a subset of G. We set

$$\mathcal{B}_S = \ell^\infty(G, \ell^p(S)) = \mathcal{B}(A, \ell^p(S))\,,$$

and we regard $\mathcal{B}_S$ as a closed subspace of $\mathcal{B}$. Recall that q is the conjugate index to p. We set

$$X_S = \ell^1(G, \ell^q(S))\,;$$

for $x \in X_S$, we write $x(t, s)$ for $x(\delta_t)(s)$ when $t \in G$ and $s \in S$, so that

$$\|x\| = \sum_{t \in G} \left(\sum_{s \in S} |x(t, s)|^q \right)^{1/q}\,.$$

The dual space X'_S of X_S is clearly isometrically isomorphic to $\mathcal{B}_S$; the action of $U \in \mathcal{B}_S$ on $x \in X_S$ is given by the formula:

$$\langle x, U\rangle = \sum_{t\in G}\sum_{s\in S} x(t,s)U(t,s)\,.$$

We shall also regard X_S as a closed linear subspace of $X''_S = \mathcal{B}'_S$. The weak-$*$ topology on $\mathcal{B}_S$ is denoted by σ.

We denote by $\ell^1_{00}(G)$ and $P_{00}(G)$ the sets of elements in $\ell^1(G)$ and $P(G)$, respectively, which have finite support.

The following two proofs were suggested by classical proofs of the conditions of Reiter and Følner for the amenability of a group; see [15, (0.7) and (4.27)], for example.

Proposition 5.10 *Let G be a group, and take p with $1 < p < \infty$. Suppose that the Banach left $\ell^1(G)$-module $\ell^p(G)$ is injective. Then there is a constant $C > 0$ such that, for each $m \in \mathbb{N}$ and each $F \in \mathcal{P}_m(G)$, there exists $x_0 \in X_F$ and $g_0 \in P_{00}(G)$ such that the following conditions are satisfied:*

(a) $x_0(t,s) = g_0(ts)\ \ (t \in G,\ s \in F)$;

(b) $\langle g_0,\, 1\rangle = 1$;

(c) $\|x_0\| \le Cm^{1/q}$.

Proof We fix P and Λ as in Theorem 5.3, and set $C = 2\,\|P\|$. As before we set $A = \ell^1(G)$ and $E = \ell^p(G)$.

Now take $m \in \mathbb{N}$ and $F \in \mathcal{P}_m(G)$.

Set $\tau(f) = \sum_{s\in F} f(s)\ \ (f \in E)$, and set $\mu = (\tau \circ P) \mid \mathcal{B}_F$. Then we see that $\mu \in \mathcal{B}'_F$ with $\|\mu\| \le m^{1/q}\,\|P\|$ and that

$$\langle U,\, \mu\rangle = \sum_{s\in F}\langle s^{-1}\cdot U^s,\, \Lambda\rangle \quad (U \in \mathcal{B}_F)\,.$$

Consider the convex set

$$K = \{g \in \ell^1_{00}(G) : \langle g,\, 1\rangle = 1,\ \ \|g\|_1 \le 2\,\|\Lambda\|\}$$

in A. There is a net (g_α) in K such that $\lim_\alpha g_\alpha = \Lambda$ in the weak-$*$ topology on $A'' = \ell^\infty(G)'$. Also there is a net (x_β) in $(X_F)_{[\|\mu\|]}$ such that $\lim_\beta x_\beta = \mu$ in $(\mathcal{B}'_F, \sigma)$. In fact, we can suppose that (x_β) is indexed by the same directed set as (g_α), and we write (x_α) for the former net.

For each $\Phi \in A''$, we set

$$\langle U, W(\Phi)\rangle = \sum_{s\in F}\langle s^{-1} \cdot U^s, \Phi\rangle \quad (U \in \mathcal{B}_F)\,.$$

Then $W(\Phi) \in \mathcal{B}'_F$ with $\|W(\Phi)\| \leq m\,\|\Phi\|$ for each $\Phi \in A''$, and hence $W \in \mathcal{B}(A'', \mathcal{B}'_F)$ with $\|W\| \leq m$. For each $g \in A$ and $U \in \mathcal{B}_F$, we have

$$\langle U, W(g)\rangle = \sum_{s\in F}\langle s^{-1} \cdot U^s, g\rangle = \sum_{s\in F, r\in G} U(rs^{-1}, s)g(r) = \sum_{s\in F, t\in G} U(t, s)g(ts)\,,$$

so that $W(g)(t, s) = g(ts)\ (t \in G,\, s \in F)$.

It is clear that $\lim_\alpha W(g_\alpha) = W(\Lambda) = \mu$, where we are taking the limit in $(\mathcal{B}'_F, \sigma)$. Thus

$$\lim_\alpha (W(g_\alpha) - x_\alpha) = 0$$

in $(\mathcal{B}'_F, \sigma)$. This shows that 0 belongs to the σ-closure of the convex set $W(K) - (X_F)_{[\|\mu\|]}$. By Mazur's theorem, 0 belongs to the $\|\cdot\|$-closure of the same convex set.

Take $h \in K$ and $y \in X_F$ with $\|y\| \leq \|\mu\|$ and $\|W(h) - y\| \leq \|\mu\|$, so that $\|W(h)\| \leq 2\,\|\mu\|$. We can write $h = h_1 - h_2 + \mathrm{i}(h_3 - h_4)$, where $h_1, \dots, h_4 \in \ell^1_{00}(G)$ and also $\|h_j\|_1 \leq \|h\|_1$ and $h_j \geq 0$ for $j = 1, \dots, 4$. We see that $\langle h_1, 1\rangle \geq 1$. Set

$$g_0 = h_1/\langle h_1, 1\rangle \in P_{00}(G)\,,$$

and define $x_0 = W(g_0) \in X_F$, so that $x_0(t, s) = g_0(ts)\ (t \in G,\, s \in F)$, giving clause (a). Clearly $\langle g_0, 1\rangle = 1$, and so clause (b) holds. Finally,

$$\|x_0\| \leq \|W(h_1)\| \leq \|W(h)\| \leq 2\,\|\mu\| \leq Cm^{1/q}\,,$$

giving clause (c). □

Proposition 5.11 *Let G be a group, and take p with $1 < p < \infty$. Suppose that the Banach left $\ell^1(G)$-module $\ell^p(G)$ is injective. Then there is a constant $C > 0$ such that, for each $m \in \mathbb{N}$ and each $F \in \mathcal{P}_m(G)$, there exists a non-empty, finite subset S of G with $|SF| < C\,|S|\,m^{1/q}$.*

Proof Let C be as in Proposition 5.10. Take $m \in \mathbb{N}$ and $F \in \mathcal{P}_m(G)$, so that $F^{-1} \in \mathcal{P}_m(G)$, and let $x_0 \in X_{F^{-1}}$ and $g_0 \in P_{00}(G)$ be as in Proposition 5.10.

We can write

$$g_0 = \sum_{j=1}^{n} \alpha_j \chi_{S_j} / |S_j| \ ,$$

where $S_1 \subset S_2 \subset \cdots \subset S_n \subset G$, where $\alpha_1, \ldots, \alpha_n > 0$, and $\sum_{j=1}^{n} \alpha_j = 1$. For convenience, set $S_{n+1} = G$.

Let $t \in G$. Choose j_0 to be the least element of the set $\{1, \ldots, n+1\}$ such that $t \in S_{j_0}F$. This implies that, for each $j \in \{j_0, \ldots, n+1\}$, there exists $s_j \in F^{-1}$ with $\chi_{S_j}(ts_j) = 1$, and hence that

$$\max_{s \in F^{-1}} x_0(t, s) = \max_{s \in F^{-1}} g_0(ts) = \sum_{j=j_0}^{n} \alpha_j / |S_j| \ .$$

Hence

$$\sum_{j=1}^{n} \alpha_j \beta_j = \sum_{t \in G} \max_{s \in F^{-1}} x_0(t, s) \le \|x_0\| \le C m^{1/q} \ ,$$

where $\beta_j = |S_j F| \,/\, |S_j| \quad (j = 1, \ldots, n)$. Since $\sum_{j=1}^{n} \alpha_j \beta_j$ is a convex combination of the numbers $\beta_1, \ldots, \beta_n$, there exists $j \in \{1, \ldots, n\}$ such that $\beta_j \le C m^{1/q}$. Set $S = S_j$. Then $|SF| < C\,|S|\, m^{1/q}$, as required. □

Theorem 5.12 *Let G be a group, and take p with $1 < p < \infty$. Suppose that the Banach left $\ell^1(G)$-module $\ell^p(G)$ is injective. Then the group G is pseudo-amenable.*

Proof Let C be the constant prescribed in Proposition 5.11. Take $\varepsilon > 0$, and choose $n_0 \in \mathbb{N}$ such that $C n^{1/q} < \varepsilon n \quad (n \ge n_0)$. Now take $n \ge n_0$ and $F \in \mathcal{P}_n(G)$. By Proposition 5.11, there is a non-empty, finite subset S of G with $|SF| < C\,|S|\, n^{1/q}$, and so $|SF| < \varepsilon n\, |S|$, proving that G is pseudo-amenable. □

We *conjecture* that, under the conditions of the above theorem, G must be an amenable group. If this were true, $\ell^p(G)$ would be injective in $\ell^1(G)$-**mod** if and only if G is amenable.

6 Summary

We summarize our results in the following table, which gives necessary and sufficient conditions for the specified Banach left $L^1(G)$-module in the first column to have the specified homological property in the top row. In the table, we take p with $1 < p < \infty$. The indication (1) means that the result holds whenever G is amenable and whenever G is discrete; we conjecture that it holds for all G. The indication (2) means that the result holds in the case where G is amenable; we *conjecture* that the converse is true. (In the discrete case, G must be pseudo-amenable.) The indication (3) means that the result implies that G is discrete and contains no infinite, amenable subgroup; we conjecture that in fact it implies that G is finite. The numbers in the boxes are those of the theorems in which the result is established.

	projective	injective	flat
$L^1(G)$	all G 2.4	G discrete and amenable 4.9	all G 2.4
$C_0(G)$	G compact 3.1	G finite 3.8	G amenable 4.7
$L^\infty(G)$	G finite 3.3	all G 2.4	G amenable 4.7
$M(G)$	G discrete 2.6	G amenable 4.7	(1) 2.5
$L^1(G)''$	(3) 2.7	G amenable 4.7	(2) 1.11
$L^p(G)$	G compact 5.1	(2) 1.11	(2) 1.11

References

[1] J. B. Conway, Projections and retractions, *Proc. American Math. Soc.*, 17 (1966), 843–847.

[2] H. G. Dales, *Banach algebras and automatic continuity*, London Mathematical Society Monographs, Volume 24, Clarendon Press, Oxford, 2000.

[3] H. G. Dales, F. Ghahramani, and A. Ya. Helemskii, The amenability of measure algebras, *J. London Math. Soc.* (2), 63 (2001), 215–226.

[4] A. Defant and K. Floret, *Tensor norms and operator ideals*, North-Holland, Amsterdam, 1993.

[5] J. Diestel and J. J. Uhl, Jr., *Vector measures*, Mathematical Surveys and Monographs, Vol. 15, American Math. Soc., Providence, Rhode Island, 1977.

[6] A. Ya. Helemskii, *Banach and locally convex algebras*, Clarendon Press, Oxford, 1993.

[7] A. Ya. Helemskii, *The homology of Banach and topological algebras*, Kluwer Academic Publishers, Dordrecht, 1986.

[8] E. Hewitt and K. A. Ross, *Abstract harmonic analysis, Volume 1*, (2nd edn.), Springer-Verlag, Berlin, 1979.

[9] B. E. Johnson, Cohomology in Banach algebras, *Memoirs American Math. Soc.*, 127 (1972).

[10] B. E. Johnson, Approximate diagonals and cohomology of certain annihilator Banach algebras, *American J. Maths.*, 94 (1972), 685–698.

[11] A. T.-M. Lau and V. Losert, Complementation of certain subspaces of $L_\infty(G)$ of a locally compact group, *Pacific J. Mathematics*, 141 (1990), 295–310.

[12] A. Yu. Ol'shanskii, An infinite group with subgroups of prime orders, *Izvestiya Akademic Nauk SSSR, Ser. Mat.*, 44 (1980), 309–321.

[13] A. Yu. Ol'shanskii, On the problem of the existence of an invariant mean on a group, *Uspekhi Mat. Nauk*, 35 (4), (1980), 199–200 = *Russian Math. Surveys*, 35 (4), (1980), 180–181.

[14] T. W. Palmer, *Banach algebras and the general theory of $*$-algebras, Volumes I and II*, Cambridge University Press, 1994 and 2001.

[15] A. L. T. Paterson, *Amenability*, Mathematical Surveys and Monographs, Volume. 29, American Math. Soc., Providence, Rhode Island, 1988.

[16] A. Yu. Pirkovskii, Injective topological modules, additivity formulas for homological dimensions, and related topics, in *Topological homology*, (ed. A. Ya. Helemskii), Nova Science, Huntington, New York, 2000, 93–144.

[17] H. Reiter and J. Stegeman, *Classical harmonic analysis and locally compact groups*, London Mathematical Society Monographs, Volume 22, Clarendon Press, Oxford, 2000.

[18] Yu. V. Selivanov, Biprojective Banach algebras, *Akad. Nauk SSR, Ser. Mat.,* 43 (1979), 1159–1174 = *Math. USSR, Izvestiya*, 15 (1980), 387–399.

[19] Yu. V. Selivanov, Coretraction problems and homological properties of Banach algebras, in *Topological homology*, (ed. A. Ya. Helemskii), Nova Science, Huntington, New York, 2000, 145–200.

[20] M. C. White, Injective modules for uniform algebras, *Proc. London Math. Soc.* (3), 73 (1996), 155–184.

Department of Pure Mathematics,
University of Leeds,
Leeds, LS2 9JT,
United Kingdom