STABILITY OF MAPPINGS ON MULTI-NORMED SPACES

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Dedicated to the memory of M. E. Polyakov

Abstract. In this paper, we define multi-normed spaces, and investigate some properties of multi-bounded mappings on multi-normed spaces. Moreover, we prove a generalized Hyers–Ulam–Rassias stability theorem associated to the Cauchy additive equation for mappings from linear spaces into multi-normed spaces.

1. Introduction

In 1940, Ulam [22] first raised the stability problem of functional equations: 'For which metric groups G is it true that an ε-automorphism of G is necessarily near to an automorphism?' In the next year, Hyers [9] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces, and in 1978, Th. M. Rassias [19] formulated and proved the following theorem, which subsumes Hyers' theorem in the case where $p = 0$.

Let E and F be real normed spaces with F complete, let $f : E \to F$ be a mapping such that, for each fixed $x \in E$, the mapping $t \mapsto f(tx)$ is continuous on $\mathbb{R}$, and let $\varepsilon \geq 0$ and $p \in [0, 1)$ be such that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p) \quad (x, y \in E)\,. \tag{1.1}$$

Then there exists a unique linear mapping $T : E \to F$ such that

$$\|f(x) - Tx\| \leq \varepsilon\|x\|^p/(1 - 2^{p-1}) \quad (x \in E)\,.$$

In 1990, Th. M. Rassias [20] asked whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [7], following the same approach as in [19], gave an affirmative solution to this question when $p > 1$, but it was proved by Gajda [7] and Rassias and Šemrl [21] that one

2000 *Mathematics Subject Classification.* Primary 39B82; Secondary 39B52, 46B99, 47A99.

Key words and phrases. generalized Hyers–Ulam–Rassias stability, multi-normed space, multi-null sequence, multi-bounded mapping, fixed point, the Cauchy additive equation.

cannot prove an analogous theorem when $p = 1$. The result of Rassias, which is also true for $p < 0$ (where we adopt the convention that $\|0\|^p = \infty$), has influenced the development of what is now called the *Hyers–Ulam–Rassias stability phenomenon*. In 1994, a generalization of Rassias' theorem, the so-called generalized Hyers–Ulam–Rassias stability theorem, was obtained by Găvruta [8]: he replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ in (1.1) by a general control function $\varphi(x, y)$.

During the last decades several stability problems of functional equations have been investigated by a number of mathematicians [3, 5, 10, 15, 20]. These results have many applications in information theory, physics, economic theory and social and behavioural sciences [1, 2]. We briefly mention some applications within the theory of Banach algebras. In each case a key step is the use of an analogue of our Theorem 3.5.

About 20 years ago, papers appeared dealing with perturbation theory: they asked when is it true that a mathematical object which satisfies a certain property approximately is necessarily close to something which satisfies the property exactly. Many of these result are described in [12]. For example, let A be a Banach algebra, and let $T : A \to \mathbb{C}$ be a linear functional such that $|T(ab) - T(a)T(b)| < \delta\,|T(a)|\,|T(b)| \quad (a, b \in A)$. Then $\|T\| \le 1 + \delta$. Let A and B be subalgebras of a Banach algbera C such that A and B are 'geometrically close'. Then it often follows (see [12]) that A and B are isomorphic. Similarly, let a Banach space A have two multiplications that make it into an algebra and are such that the two products are close as bilinear maps. Then the two algebras so formed often have common algebraic properties. These ideas a! re related to that of *approximately multiplicative maps*, as studied by Johnson in [13] and [14]. A pair (A, B) of Banach algebras is AMNM if, for each $\varepsilon > 0$ and each $K > 0$, there exists $\delta > 0$ such that, for each bounded linear map $T : A \to B$ such that $\|T\| \le K$ and $\|T(ab) - T(a)T(b)\| \le \delta\,\|a\|\,\|b\|$ for all $a, b \in A$, there is a bounded homomorphism $\theta : A \to B$ such that $\|T - \theta\| \le \varepsilon$. The theorems of Johnson discuss which Banach algebras are AMNM; the proofs and conditions involve the cohomology theory of Banach algebras. Related matters involving approximate cohomology groups and approximate amenability are discussed in [18].

In this paper, following [6], we define multi-normed spaces, and investigate some properties of multi-bounded mappings on multi-normed spaces. Moreover, we prove a generalized

Hyers–Ulam–Rassias stability theorem associated to the Cauchy additive equation for mappings from linear spaces into multi-normed spaces by using a fixed point approach in the style of [4, 11, 17]. The theory of multi-normed spaces and of multi-Banach algebras was originated in [6]. Our expectation is that our Theorem 3.5 will have similar applications to the above in the theory of multi-normed spaces and algebras.

Let $(E, \|\cdot\|)$ be a complex linear space, and let $k \in \mathbb{N}$. We denote by E^k the linear space $E \oplus \cdots \oplus E$ consisting of k-tuples $(x_1, \dots, x_k)$, where $x_1, \dots, x_k \in E$. The linear operations on E^k are defined coordinatewise. When we write $(0, \dots, 0, x_i, 0, \dots, 0)$ for an element in E^k, we understand that x_i appears in the i^{th} coordinate. The zero element of either E or E^k is denoted by 0. We denote by $\mathbb{N}_k$ the set $\{1, 2, \dots, k\}$ and by $\mathfrak{S}_k$ the group of permutations on k symbols; we set $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

2. Multi-normed spaces and multi-bounded operators

We start this section by recalling the notion of a multi-normed space from [6]. Throughout this section $(E, \|\cdot\|)$ denotes a complex normed space.

Definition 2.1. A *multi-norm* on $\{E^n : n \in \mathbb{N}\}$ is a sequence

$$(\|\cdot\|_n) = (\|\cdot\|_n : n \in \mathbb{N})$$

such that $\|\cdot\|_n$ is a norm on E^n for each $n \in \mathbb{N}$, such that $\|x\|_1 = \|x\|$ for each $x \in E$, and such that the following axioms are satisfied for each $n \in \mathbb{N}$ with $n \geq 2$:

(A1)

$$\left\|(x_{\sigma(1)}, \dots, x_{\sigma(n)})\right\|_n = \|(x_1, \dots, x_n)\|_n \quad (\sigma \in \mathfrak{S}_n,\ x_1, \dots, x_n \in E)\,;$$

(A2)

$$\|(\alpha_1 x_1, \dots, \alpha_n x_n)\|_n \leq (\max_{i \in \mathbb{N}_n} |\alpha_i|)\, \|(x_1, \dots, x_n)\|_n \quad (\alpha_1, \dots, \alpha_n \in \mathbb{C},\ x_1, \dots, x_n \in E)\,;$$

(A3)

$$\|(x_1, \dots, x_{n-1}, 0)\|_n = \|(x_1, \dots, x_{n-1})\|_{n-1} \quad (x_1, \dots, x_{n-1} \in E)\,;$$

(A4)

$$\|(x_1, \dots, x_{n-1}, x_{n-1})\|_n = \|(x_1, \dots, x_{n-1})\|_{n-1} \quad (\, x_1, \dots, x_{n-1} \in E)\,.$$

In this case, we say that $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a *multi-normed space.*

Motivations for the study of multi-normed spaces (and multi-normed algebras) and many examples are given in [6].

Suppose that $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a multi-normed space. The following properties are almost immediate consequences of the axioms.

(a)

$$\|(x, \dots, x)\|_k = \|x\| \quad (k \in \mathbb{N},\, x \in E);$$

(b) Let $j, l \in \mathbb{N}$ with $j \leq l$, and let $x_1, \dots, x_j, y_1, \dots, y_l \in E$ be such that $\{x_1, \dots, x_j\} \subset \{y_1, \dots, y_l\}$. Then

$$\|(x_1, \dots, x_j)\|_j \leq \|(y_1, \dots, y_l)\|_l \ ;$$

(c)

$$\max_{i \in \mathbb{N}_k} \|x_i\| \leq \|(x_1, \dots, x_k)\|_k \leq \sum_{i=1}^{k} \|x_i\| \leq k \max_{i \in \mathbb{N}_k} \|x_i\| \quad (k \in \mathbb{N},\, x_1, \dots, x_k \in E).$$

It follows from (c) that, if $(E, \|\cdot\|)$ is a Banach space, then $(E^k, \|\cdot\|_k)$ is a Banach space for each $k \in \mathbb{N}$; in this case $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a *multi-Banach space.*

Now we recall three important examples of multi-norms for an arbitrary normed space E; see [6] for details and many other examples.

Example 2.2. The sequence $(\|\cdot\|_n : n \in \mathbb{N})$ on $\{E^n : n \in \mathbb{N}\}$ defined by

$$\|(x_1, \dots, x_n)\|_n := \max_{i \in \mathbb{N}_n} \|x_i\| \quad (x_1, \dots, x_n \in E)$$

is a multi-norm called *the minimum multi-norm.* The terminology 'minimum' is justified by property (c). □

Example 2.3. Let $\{(\|\cdot\|_n^\alpha : n \in \mathbb{N}) : \alpha \in A\}$ be the (non-empty) family of all multi-norms on $\{E^n : n \in \mathbb{N}\}$. For $n \in \mathbb{N}$, set

$$|||(x_1, \dots, x_n)|||_n := \sup_{\alpha \in A} \|(x_1, \dots, x_n)\|_n^\alpha \quad (x_1, \dots, x_n \in E).$$

Then $(|||\cdot|||_n : n \in \mathbb{N})$ is a multi-norm on $\{E^n : n \in \mathbb{N}\}$, called *the maximum multi-norm.* □

Example 2.4. Let $(E, \|\cdot\|)$ be a Banach lattice, and define

$$\|(x_1, \dots, x_n)\|_n := \|\, |x_1| \vee \cdots \vee |x_n| \,\| \quad (x_1, \dots, x_n \in E).$$

Then $(\|\cdot\|_n : n \in \mathbb{N})$ is a multi-Banach space. □

Definition 2.5. Let $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-normed space. A sequence (x_n) in E is a *multi-null sequence* if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{k\in\mathbb{N}} \|(x_n, \dots, x_{n+k-1})\|_k < \varepsilon \quad (n \geq n_0).$$

Let $x \in E$. Then

$$\operatorname*{Lim}_{n\to\infty} x_n = x$$

if $(x_n - x)$ is a multi-null sequence; in this case, the sequence (x_n) is *multi-convergent* to x in E.

Definition 2.6. Let $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ and $((F^k, \|\cdot\|_k) : k \in \mathbb{N})$ be multi-normed spaces, and let $x \in E$. A mapping $f : E \to F$ is *multi-continuous* at the point x if $\operatorname*{Lim}_{n\to\infty} f(x_n) = f(x)$ in F whenever $\operatorname*{Lim}_{n\to\infty} x_n = x$ in E.

Now we give the definition from [6] of a multi-bounded operator between multi-normed spaces; this is similar to the notion of bounded linear operator between two normed spaces.

Definition 2.7. Let $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ and $((F^k, \|\cdot\|_k) : k \in \mathbb{N})$ be multi-normed spaces. A linear operator $T : E \to F$ is *multi-bounded* if

$$\|T\|_{mb} := \sup_n \, \sup \left\{ \frac{\|(Tx_1, \dots, Tx_n)\|_n}{\|(x_1, \dots, x_n)\|_n} : (x_1, \dots, x_n) \neq 0 \right\} < \infty.$$

The collection of multi-bounded operators from E to F is denoted by $\mathcal{M}(E, F)$. It is easy to see that $(\mathcal{M}(E, F), \|\cdot\|_{mb})$ is a Banach subspace of the Banach space $\mathcal{B}(E, F)$ of all bounded linear operators from E to F, and that $\mathcal{M}(E, F)$ contains all nuclear operators. In fact, $\mathcal{M}(E, F)$ is also a multi-Banach space in a canonical way; see [6, Chapter 4].

Theorem 2.8. *Let $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ and $((F^k, \|\cdot\|_k) : k \in \mathbb{N})$ be multi-normed spaces. Then a linear mapping $T : E \to F$ is multi-continuous if and only if it is multi-bounded.*

Proof. Suppose that T is multi-bounded, and let (x_n) be a multi-null sequence in E. Then, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{k\in\mathbb{N}} \|(x_n, \dots, x_{n+k-1})\|_k < \varepsilon \quad (n \geq n_0).$$

But now

$$\sup_{k\in\mathbb{N}} \|(Tx_n, \dots, Tx_{n+k-1})\|_k \leq \|T\|_{mb}\, \varepsilon \quad (n \geq n_0)\,,$$

and so (Tx_n) is a multi-null sequence in F. Thus T is multi-continuous.

Conversely, suppose that T is not multi-bounded. Then, for each $n \in \mathbb{N}$, there exists the positive integer $k_n \geq n$ such that

$$\sup\left\{\frac{\|(Tx_1,\dots,Tx_{k_n})\|_{k_n}}{\|(x_1,\dots,x_{k_n})\|_{k_n}} : (x_1,\dots,x_{k_n}) \neq 0\right\} > n^2\,.$$

Hence there exists $(y_{1,n},\dots,y_{k_n,n}) \in E^{k_n}$ such that

$$\frac{\|(Ty_{1,n},\dots,Ty_{k_n,n})\|_{k_n}}{\|(y_{1,n},\dots,y_{k_n,n})\|_{k_n}} > n^2\,.$$

Now set

$$x_{j,n} = \frac{y_{j,n}}{n^2\,\|(y_{1,n},\dots,y_{k_n,n})\|_{k_n}} \quad (j \in \mathbb{N}_{k_n})\,.$$

Then

$$\|(x_{1,n},\dots,x_{k_n,n})\|_{k_n} < \frac{1}{n^2} \quad \text{and} \quad \|(Tx_{1,n},\dots,Tx_{k_n,n})\|_{k_n} > 1\,.$$

Consider the sequence

$$(z_i) = (x_{1,1},\dots,x_{k_1,1},x_{1,2},\dots,x_{k_2,2},\dots,x_{1,n},\dots,x_{k_n,n},\dots)$$

in E. We claim that the sequence (z_i) is multi-null in E. Indeed, given $\varepsilon > 0$, there exists $j \in \mathbb{N}$ with $j \geq 2$ such that $\sum_{i=j}^{\infty} 1/i^2 < \varepsilon$, and then

$$\begin{aligned}
\left\|(x_{1,j},\dots,x_{k_j,j},\dots,x_{1,j+n},\dots,x_{k_{j+n},j+n})\right\|_{k_j+\cdots+k_{j+n}} &\leq \sum_{\ell=j}^{j+n}\|(x_{1,\ell},\dots,x_{k_\ell,\ell}\|_{k_\ell} \\
&\leq \sum_{\ell=j}^{j+n}\frac{1}{\ell^2} < \varepsilon \quad (n \in \mathbb{N})\,. \qquad (2.1)
\end{aligned}$$

Fix $n_0 = k_1 + \cdots + k_{j-1} + 1$. For $n, k \in \mathbb{N}$ with $n \geq n_0$, choose $m \in \mathbb{N}$ with

$$k_j + \cdots + k_{j+m} > (n - k_1 - \cdots - k_{j-1}) + k - 1\,.$$

Then $\{z_n,\dots,z_{n+k-1}\} \subseteq \{x_{1,j},\dots,x_{k_j,j},\dots,x_{1,j+m},\dots,x_{k_{j+m},j+m}\}$, and so it follows from (2.1) and property (b) that

$$\|(z_n,\dots,z_{n+k-1})\|_k \leq \left\|(x_{1,j},\dots,x_{k_j,j},\dots,x_{1,j+m},\dots,x_{k_{j+m},j+m})\right\|_{k_j+\cdots+k_{j+m}} \leq \varepsilon\,.$$

Hence $\sup_{k\in\mathbb{N}}\|(z_n,\dots,z_{n+k-1})\|_k \leq \varepsilon$ $(n \geq n_0)$, and so (z_i) is multi-null. The sequence (Tz_i) is clearly not a multi-null sequence in F, and so T is not multi-continuous. $\square$

3. Generalized Hyers–Ulam–Rassias stability

We begin this section with 'the alternative of fixed point theorem' [16]. A generalized metric space is a pair (X, d), where X is a non-empty set and $d : X \times X \to [0, \infty]$ satisfies the usual axioms. A map $J : X \to X$ is strictly contractive if there exists a 'Lipschitz constant' $0 \le L < 1$ such that $d(Jx, Jy) \le Ld(x, y) \quad (x \in X)$.

Theorem 3.1. *Let (X, d) be a complete generalized metric space, and let $J : X \to X$ be a strictly contractive mapping with a Lipschitz constant $L < 1$. Then, for each element $x \in X$, either*
(F1) $d(J^n x, J^{n+1} x) = \infty \quad (n \ge 0)$, or
(F2) *there exists $n_0 \ge 0$ such that:*
(F20) $d(J^n x, J^{n+1} x) < \infty \quad (n \ge n_0)$ *;*
(F21) *the sequence $(J^n x)$ converges to a fixed point y^* of J ;*
(F22) *y^* is the unique fixed point of J in the set $U = \{y \in X : d(J^{n_0} x, y) < \infty\}$;*
(F23) $d(y, y^*) \le d(y, Jy)/(1 - L) \quad (y \in U)$. □

The following lemma gives a useful strictly contractive mapping. We adopt the convention that $\inf \emptyset = \infty$.

Lemma 3.2. *Let $k \in \mathbb{N}$, and let E and F be linear spaces such that $(F^k, \|\cdot\|)$ is a Banach space. Suppose that there exist $0 \le M < 1$, $\lambda > 0$, and a map $\psi : E^k \to [0, \infty)$ such that*

$$\psi(\lambda x_1, \dots, \lambda x_k) \le \lambda M \psi(x_1, \dots, x_k) \quad (x_1, \dots, x_k \in E).$$

Set $X := \{g : E \to F : g(0) = 0\}$, and define $d : X \times X \to [0, \infty]$ by

$$\begin{aligned} d(g, h) = \inf\{c > 0 : \; & \| (g(x_1) - h(x_1), \dots, g(x_k) - h(x_k)) \| \\ \le \; & c\psi(x_1, \dots, x_k) \text{ for all } x_1, \dots, x_k \in E\}. \end{aligned}$$

Then (X, d) is a complete generalized metric space, and the mapping $J : X \to X$ defined by $(Jg)(x) := g(\lambda x)/\lambda \quad (x \in X)$ is a strictly contractive mapping.

Proof. It is easy to see that (X, d) is complete generalized metric space.

For elements $g, h \in X$ with $d(g,h) < \infty$, take $c > d(g,h)$. Then, for each $x_1, \dots, x_k \in E$, we have $\| (g(x_1) - h(x_1), \dots, g(x_k) - h(x_k)) \| \leq c\psi(x_1, \dots, x_k)$, and so

$$\begin{aligned}\left\| \left(\frac{1}{\lambda} g(\lambda x_1) - \frac{1}{\lambda} h(\lambda x_1), \dots, \frac{1}{\lambda} g(\lambda x_k) - \frac{1}{\lambda} h(\lambda x_k) \right) \right\| &\leq \frac{1}{\lambda} c\psi(\lambda x_1, \dots, \lambda x_k) \\ &\leq cM\psi(x_1, \dots, x_k) .\end{aligned}$$

Thus $d(Jg, Jh) \leq cM$. Hence $d(Jg, Jh) \leq Md(g,h)$ for all $g, h \in X$, and so J is a strictly contractive mapping on X with Lipschitz constant M. $\square$

Proposition 3.3. *Let E be a linear space, and let $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-Banach space. Let $L, k \in \mathbb{N}$ with $L \geq 2$, and let there exist $0 \leq M_0 < 1$ and a function $\varphi : E^{2k} \to [0, \infty)$ satisfying*

$$\varphi(Lx_1, Ly_1, \dots, Lx_k, Ly_k) \leq LM_0 \varphi(x_1, y_1 \dots, x_k, y_k) \tag{3.1}$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in E$. Suppose that $f : E \to F$ is a mapping satisfying $f(0) = 0$ and

$$\begin{aligned}&\| (f(\mu x_1 + \mu y_1) - \mu f(x_1) - \mu f(y_1), \dots, f(\mu x_k + \mu y_k) - \mu f(x_k) - \mu f(y_k)) \|_k \\ &\qquad \leq \varphi(x_1, y_1, \dots, x_k, y_k)\end{aligned} \tag{3.2}$$

for all $\mu \in \mathbb{T}$ and $x_1, \dots, x_k, y_1, \dots, y_k \in E$. Then there exists a unique linear mapping $T : E \to F$ such that

$$\| (f(x_1) - Tx_1, \dots, f(x_k) - Tx_k) \|_k \leq \frac{M_0}{1 - M_0} \psi(x_1, \dots, x_k) , \tag{3.3}$$

where

$$\psi(x_1, \dots, x_k) := \sum_{j=1}^{L-1} \varphi(jx_1/L, x_1/L, jx_2/L, x_2/L, \dots, jx_k/L, x_k/L)$$

for all $x_1, \dots, x_k \in E$.

Proof. Let $x_1, \dots, x_k \in E$. Setting $\mu = 1, y_1 = x_1, \dots, y_k = x_k$ in (3.2), we obtain

$$\| (f(2x_1) - 2f(x_1), \dots, f(2x_k) - 2f(x_k)) \|_k \leq \varphi(x_1, x_1, x_2, x_2, \dots, x_k, x_k) .$$

By using induction one can easily see that

$$\| (f(Lx_1) - Lf(x_1), \dots, f(Lx_k) - Lf(x_k)) \|_k \leq \psi(Lx_1, \dots, Lx_k) . \tag{3.4}$$

It follows from (3.1) that $\psi(Lx_1,\dots,Lx_k)\le LM_0\psi(x_1,\dots,x_k)$, and so, by (3.4), we have

$$\begin{aligned}\left\|\left(\frac{1}{L}f(Lx_1)-f(x_1),\dots,\frac{1}{L}f(Lx_k)-f(x_k)\right)\right\|_k &\le \frac{1}{L}\psi(Lx_1,\dots,Lx_k)\\ &\le M_0\psi(x_1,\dots,x_k)\,.\end{aligned}\tag{3.5}$$

Let $X:=\{g:E\to F:g(0)=0\}$ and d be as in Lemma 3.2, with M and λ replaced by M_0 and L, respectively. Define the mapping $J_0:X\to X$ by $(J_0g)(x):=g(Lx)/L\quad(x\in X)$. Then J_0 is a strictly contractive mapping. By (3.5), we have $d(f,J_0f)\le M_0<\infty$. By Theorem 3.1 (with f for x), there exists $n_0\in\mathbb{N}$ such that the sequence (J_0^nf) converges to a fixed point T of J_0 (so that $T(Lx)=LT(x)\quad(x\in E)$), T is the unique fixed point of J_0 in the set $U_0=\{g\in X:d(J_0^{n_0}f,g)<\infty\}$, and $d(g,T)\le d(g,J_0f)/(1-M_0)\quad(g\in U_0)$.

Since $\lim_{n\to\infty}d(J_0^nf,T)=0$, we easily conclude that

$$\lim_{n\to\infty}\frac{f(L^nx)}{L^n}=T(x)\quad(x\in E)\,,$$

where we are using property (a). Clearly $d(f,J_0^{n_0})<\infty$, and so $f\in U_0$. Hence

$$d(f,T)\le\frac{1}{1-M_0}d(f,J_0f)\le\frac{M_0}{1-M_0}\,,$$

and so

$$\|\,(f(x_1)-T(x_1),\dots,f(x_k)-T(x_k))\,\|_k\le\frac{M_0}{1-M_0}\psi(x_1,\dots,x_k)\quad(x_1,\dots,x_k\in E).$$

Fix $x,y\in E$ and $\mu\in\mathbb{T}$. Let us replace all $x_1,\dots,x_k$ by L^nx and all $y_1,\dots,y_k$ by L^ny in (3.2), divide both sides by L^n, and pass to the limit as $n\to\infty$. Then, using property (a) again, we obtain

$$\begin{aligned}\|T(\mu x+\mu y)-\mu T(x)-\mu T(y)\| &\le \limsup_{n\to\infty}\frac{1}{L^n}\varphi(L^nx,L^ny,\dots,L^nx,L^ny)\\ &\le \limsup_{n\to\infty}M_0^n\varphi(x,y,\dots,x,y)=0\,.\end{aligned}$$

Thus $T(\mu x+\mu y)=\mu T(x)+\mu T(y)$ for all $\mu\in\mathbb{T}$ and $x,y\in E$.

Next, let $\xi\in\mathbb{C}$, and take $K\in\mathbb{N}$ with $K>|\xi|$. By an easily geometric argument, one can see that there are $\mu_1,\mu_2\in\mathbb{T}$ such that $2\xi=K(\mu_1+\mu_2)$. Using the additivity of T, we obtain

$$\begin{aligned} T(\xi x) &= T\left(\frac{K}{2}\cdot 2\cdot\frac{\xi}{K}x\right) = KT\left(\frac{1}{2}\cdot 2\cdot\frac{\xi}{K}x\right) = \frac{K}{2}T\left(2\cdot\frac{\xi}{K}x\right) \\ &= \frac{K}{2}T(\mu_1 x+\mu_2 x) = \frac{K}{2}(T(\mu_1 x)+T(\mu_2 x)) \\ &= \frac{K}{2}(\mu_1+\mu_2)T(x) = \frac{K}{2}\cdot 2\cdot\frac{\xi}{K} = \xi T(x) \end{aligned}$$

for all $x \in E$. Thus the mapping T is linear. If S is another linear mapping satisfying (3.3), then $S(Lx) = LS(x)$ and so S is a fixed point of J_0. Moreover,

$$\left\|\left(\frac{f(L^{n_0}x_1)}{L^{n_0}} - S(x_1),\dots,\frac{f(L^{n_0}x_k)}{L^{n_0}} - S(x_k)\right)\right\|_k \le \frac{M_0}{1-M_0}\psi(x_1,\dots,x_k)$$

for all $x_1,\dots,x_k \in E$. Hence S is a fixed point of J_0 in the set U_0, and so $T = S$. □

Corollary 3.4. *Let $(E,\|\cdot\|)$ be a normed space, and let $((F^n,\|\cdot\|_n) : n \in \mathbb{N})$ be a multi-Banach space. Let $k \in \mathbb{N}$, $p \in [0,1)$, $\alpha,\beta \ge 0$, and let $f : E \to F$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} &\|\,(f(\mu x_1+\mu y_1) - \mu f(x_1) - \mu f(y_1),\dots,f(\mu x_k+\mu y_k) - \mu f(x_k) - \mu f(y_k))\,\|_k \\ &\qquad\qquad \le \alpha + \beta(\|x_1\|^p + \|y_1\|^p + \cdots + \|x_k\|^p + \|y_k\|^p) \end{aligned}$$

for all $\mu \in \mathbb{T}$, and $x_1,\dots,x_k,y_1,\dots,y_k \in E$. Then there exists a unique linear mapping $T : E \to F$ such that

$$\|\,(f(x_1) - Tx_1,\dots,f(x_k) - Tx_k)\,\|_k \le \frac{2^{p-1}\alpha + \beta(\|x_1\|^p + \cdots + \|x_k\|^p)}{1-2^{p-1}}$$

for all $x_1,\dots,x_k \in E$.

Proof. Set $L = 2$, $M_0 = 2^{p-1} < 1$, and

$$\varphi(x_1,y_1,\dots,x_k,y_k) = \alpha + \beta(\|x_1\|^p + \|y_1\|^p + \cdots + \|x_k\|^p + \|y_k\|^p)\,,$$

so that φ satisfies (3.1) in Proposition 3.3. We have

$$\psi(x_1,\dots,x_k) = \alpha + 2^{1-p}\beta(\|x_1\|^p + \cdots + \|x_k\|^p) \quad (x_1,\dots,x_k \in E)\,,$$

and so there exists a linear mapping T with the required property. □

Theorem 3.5. *Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, let $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-Banach space, let $\alpha \geq 0$, and let $f : E \to F$ be a mapping satisfying $f(0) = 0$ and*

$$\|\,(f(\mu x_1 + \mu y_1) - \mu f(x_1) - \mu f(y_1), \ldots, f(\mu x_k + \mu y_k) - \mu f(x_k) - \mu f(y_k))\,\|_k \leq \alpha$$

for all $k \in \mathbb{N}$, $\mu \in \mathbb{T}$ and $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$. Suppose that f is multi-continuous at some point of E. Then there exists a unique multi-bounded linear mapping $T : E \to F$ such that

$$\|f(x) - Tx\| \leq \alpha$$

for all $x \in E$.

Proof. By Corollary 3.4 with $p = \beta = 0$, for each $k \in \mathbb{N}$, there exists a unique linear mapping $T_k : E \to F$ such that

$$\|\,(f(x_1) - T_k x_1, \ldots, f(x_k) - T_k x_k)\,\|_k \leq \alpha \quad (x_1, \ldots, x_k \in E).$$

By (A3), in fact $T_k = T_1$ $(k \in \mathbb{N})$, say $T = T_1$. Therefore $\|f(x) - Tx\| \leq \alpha \quad (x \in E)$.

We shall prove that T is multi-continuous. Indeed, suppose that f is multi-continuous at $x_0 \in E$, and assume towards a contradiction that T is not multi-continuous at 0. Then there exist a sequence (x_n) in E and $\eta > 0$ such that $\operatorname*{Lim}_{n\to\infty} x_n = 0$ and

$$\limsup_{n\to\infty} \sup_k \|\,(Tx_n, \ldots, Tx_{n+k-1})\,\|_k > \eta\,.$$

Set $\varepsilon > 2\alpha/\eta$. Then

$$\limsup_{n\to\infty} \sup_k \|(T(\varepsilon x_n + x_0) - T(x_0), \ldots, T(\varepsilon x_{n+k-1} + x_0) - T(x_0))\|_k > 2\alpha\,.$$

For $n \in \mathbb{N}$, we have

$$\begin{aligned}
\sup_k \|(T(\varepsilon x_n + x_0) &- T(x_0), \ldots, T(\varepsilon x_{n+k-1} + x_0) - T(x_0))\|_k \\
&\leq \sup_k \|(T(\varepsilon x_n + x_0) - f(\varepsilon x_n + x_0), \\
&\qquad \ldots, T(\varepsilon x_{n+k-1} + x_0) - f(\varepsilon x_{n+k-1} + x_0))\|_k \\
&\quad + \sup_k \|(f(\varepsilon x_n + x_0) - f(x_0), \ldots, f(\varepsilon x_{n+k-1} + x_0) - f(x_0))\|_k \\
&\quad + \|(f(x_0) - T(x_0), \ldots, f(x_0) - T(x_0))\|_k\,.
\end{aligned}$$

Since $\operatorname*{Lim}_{n\to\infty} f(\varepsilon x_n + x_0) = f(x_0)$, we have

$$\limsup_{n\to\infty}\sup_k \|(T(\varepsilon x_n + x_0) - T(x_0), \dots, T(\varepsilon x_{n+k-1} + x_0) - T(x_0))\|_k \leq 2\alpha\,,$$

a contradiction. Hence T is multi-continuous. By Theorem 2.8, T is multi-bounded. $\square$

Proposition 3.6. *Let E be a linear space, and let $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-Banach space. Let $L, k \in \mathbb{N}$ with $L \geq 2$, and let there exist $0 \leq M_1 < 1$ and a function $\varphi : E^{2k} \to [0, \infty)$ satisfying*

$$\varphi(x_1, y_1 \dots, x_k, y_k) \leq \frac{1}{L} M_1 \varphi(Lx_1, Ly_1, \dots, Lx_k, Ly_k)$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in E$. Suppose that $f : E \to F$ is a mapping satisfying $f(0) = 0$ and

$$\begin{aligned} &\|\,(f(\mu x_1 + \mu y_1) - \mu f(x_1) - \mu f(y_1), \dots, f(\mu x_k + \mu y_k) - \mu f(x_k) - \mu f(y_k))\,\|_k \\ &\qquad \leq \varphi(x_1, y_1, \dots, x_k, y_k) \end{aligned} \tag{3.6}$$

for $\mu = 1, \mathrm{i}$ and for all $x_1, \dots, x_k, y_1, \dots, y_k \in E$. Further, suppose that, for each fixed $x \in E$, the function $t \mapsto f(tx)$ is continuous on $\mathbb{R}$. Then there exists a unique linear mapping $T : E \to F$ such that

$$\|\,(f(x_1) - Tx_1, \dots, f(x_k) - Tx_k)\,\|_k \leq \frac{1}{1 - M_1}\psi(x_1, \dots, x_k) \tag{3.7}$$

where

$$\psi(x_1, \dots, x_k) := \sum_{j=1}^{L-1} \varphi\,(jx_1/L, x_1/L, jx_2/L, x_2/L, \dots, jx_k/L, x_k/L)$$

for all $x_1, \dots, x_k \in E$.

Proof. Let $x_1, \dots, x_k \in E$. Setting $\mu = 1, y_1 = x_1, \dots, y_k = x_k$ in (3.6), we obtain

$$\|\,(f(2x_1) - 2f(x_1), \dots, f(2x_k) - 2f(x_k))\,\|_k \leq \varphi(x_1, x_1, x_2, x_2, \dots, x_k, x_k)\,.$$

By using induction, one can see that

$$\begin{aligned} \|(f(Lx_1) - Lf(x_1), \quad \dots \quad &, f(Lx_k) - Lf(x_k))\|_k \\ \leq \quad &\sum_{j=1}^{L-1} \varphi(jx_1, x_1, jx_2, x_2, \dots, jx_k, x_k)\,. \end{aligned} \tag{3.8}$$

Replacing x_j $(1 \le j \le k)$ by x_j/L in (3.8), we obtain

$$\|(f(x_1) - Lf(x_1/L), \ldots, f(x_k) - Lf(x_k/L))\|_k \le \psi(x_1, \ldots, x_k)\,. \tag{3.9}$$

Let $X := \{g : E \to F : g(0) = 0\}$ and d be as in Lemma 3.2, with M and λ replacing M_0 and $1/L$, respectively. Define the mapping $J_1 : X \to X$ by $(J_1 g)(x) := Lg\,(x/L)$. Then J_1 is a strictly contractive mapping. By (3.9), we have $d(f, J_1 f) \le 1 < \infty$. Essentially as in the proof of Proposition 3.3, we see that there exists a mapping $T : E \to F$ such that $T(\mu x + \mu y) = \mu T(x) + \mu T(y)$ for all $x, y \in E$ and $\mu = 1, \mathrm{i}$, and that T is unique among maps satisfying (3.7).

Now we use the strategy of [19] to show that T is linear. Fix $x_0 \in E$ and $\rho \in F'$. Then the mapping $\Phi : \mathbb{R} \to \mathbb{R}$ defined by $\Phi(t) := \rho(T(tx_0)) = \lim_{n\to\infty} L^n \rho(f(L^{-n}tx_0))$ is additive. Since Φ is the pointwise limit of continuous functions $L^n\rho(f(L^{-n}tx_0))$, it is measurable. Thus the additive mapping Φ is continuous, and hence $\mathbb{R}$-linear. For all $\xi = a_1 + \mathrm{i}a_2 \in \mathbb{C}$, where $a_1, a_2 \in \mathbb{R}$, we then have

$$T(\xi x) = T(a_1 x) + \mathrm{i}T(a_2 x) = a_1 T(x) + \mathrm{i}a_2 T(x) = \xi T(x).$$

Therefore T is a linear mapping. $\square$

Corollary 3.7. *Let $(E, \|\cdot\|)$ be a normed space and let $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-Banach space. Let $k \in \mathbb{N}$, $p > 1$, $\beta \ge 0$, and let $f : E \to F$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned}\|\,(f(\mu x_1 + \mu y_1) - \mu f(x_1) - \mu f(y_1), \ldots, f(\mu x_k + \mu y_k) - \mu f(x_k) - \mu f(y_k))\,\|_k \\ \le \beta(\|x_1\|^p + \|y_1\|^p + \cdots + \|x_k\|^p + \|y_k\|^p)\end{aligned}$$

for $\mu = 1, \mathrm{i}$ and for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$. If for each fixed $x \in E$ the function $t \mapsto f(tx)$ is continuous on $\mathbb{R}$, then there exists a unique linear mapping $T : E \to F$ such that

$$\|\,(f(x_1) - T(x_1), \ldots, f(x_k) - T(x_k))\,\|_k \le \frac{\beta}{2^{p-1} - 1}(\|x_1\|^p + \cdots + \|x_k\|^p)$$

for all $x_1, \ldots, x_k \in E$.

Proof. Set

$$\varphi(x_1, y_1, \ldots, x_k, y_k) = \beta(\|x_1\|^p + \|y_1\|^p + \cdots + \|x_k\|^p + \|y_k\|^p)\,,$$

and let $M_1 = 2^{1-p} < 1$ in Proposition 3.6. Then

$$\psi(x_1, \ldots, x_k) = 2^{1-p}\beta(\|x_1\|^p + \cdots + \|x_k\|^p) \quad (x_1, \ldots, x_k \in E)\,,$$

and there exists a unique linear mapping T with the required property. $\square$

Acknowledgement. This work was written whilst the second author was visiting the University of Leeds during his sabbatical leave in 2006; he would like to thank sincerely Professor H. G. Dales for his warm hospitality and members of the School of Mathematics at Leeds for their kindness.

References

[1] J. Aczél, *A short course on functional equations*, D. Reidel Publ. Co., Dordrecht, 1987.

[2] M. Amyari and M. S. Moslehian, Approximately ternary semigroup homomorphisms, *Lett. Math. Phys.* **77** (2006), 1–9.

[3] C. Baak and M. S. Moslehian, Stability of J^*-homomorphisms, *Nonlinear Anal.* **63** (2005), 42–48.

[4] L. Cădariu and V. Radu, Fixed points and the stability of Jensen's functional equation, *J. Inequal. Pure Appl. Math.* **4** (2003), no. 1, art. 4, 7 pp.

[5] S. Czerwik, *Functional equations and inequalities in several variables*, World Scientific, New Jersey, London, Singapore, Hong Kong, 2002.

[6] H. G. Dales and M. E. Polyakov, Multi-normed spaces and multi-Banach algebras (preprint).

[7] Z. Gajda, On stability of additive mappings, *Internat. J. Math. Math. Sci.* **14** (1991), 431–434.

[8] P. Găvruta, A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.* **184** (1994), 431–436.

[9] D. H. Hyers, On the stability of the linear functional equation, *Proc. National Academy Sciences U.S.A.* **27** (1941), 222–224.

[10] D. H. Hyers, G. Isac and Th. M. Rassias, *Stability of functional equations in several variables*, Birkhäuser, Basel, 1998.

[11] G Isac and Th. M. Rassias, Stability of ψ-additive mappings: Applications to nonlinear analysis, *Internat. J. Math. Math. Sci.* **19** (1996), 219–228.

[12] K. Jarosz, *Perturbations of Banach algebras*, Lecture Notes in Mathematics, Volume 1120, Springer–Verlag, Berlin, 1985.

[13] B. E. Johnson, Approximately multiplicative functionals, *J. London Math. Soc.* (2), **34** (1986), 489–510.

[14] B. E. Johnson, Approximately multiplicative maps between Banach algebras, *J. London Math. Soc.* (2), **37** (1988), 294–316.

[15] S.-M. Jung, *Hyers–Ulam–Rassias stability of functional equations in mathematical analysis*, Hadronic Press, Palm Harbor, Florida, 2001.

[16] B. Margolis and J. B. Diaz, A fixed point theorem of the alternative for contractions on a generalized complete metric space, *Bull. American Math. Soc.* **126** (1968), 305–309.

[17] M. Mirzavaziri and M. S. Moslehian, A fixed point approach to stability of a quadratic equation, *Bull. Brazilian Math. Soc.* **37** (2006), no.3, 361–376.

[18] M. S. Moslehian, Approximately vanishing of topological cohomology groups, *J. Math. Anal. Appl.*, **318** (2006), 758–771.

[19] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. American Math. Soc.* **72** (1978), 297–300.

[20] Th. M. Rassias (ed.), *Functional equations, inequalities and applications*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2003.

[21] Th. M. Rassias and P. Šemrl, On the behaviour of mappings which do not satisfy Hyers–Ulam stability, *Proc. American Math. Soc.* **114** (1992), 989–993.

[22] S. M. Ulam, *Problems in modern mathematics*, Chapter VI, Science Editions, Wiley, New York, 1964.

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