Banach function algebras

H. G. Dales, Leeds

(Joint work with Joel Feinstein, Nottingham)

Teheran

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Let X be a compact (Hausdorff) space. Then $C(X)$ is the commutative algebra of all continuous functions on X (with pointwise operations).

For $f \in C(X)$, set $|f|_X = \sup\{|f(x)| : x \in X\}$. Then $|\cdot|_X$ is the uniform norm on X.

$(C(X), |\cdot|_X)$ is a commutative Banach algebra.

A **Banach function algebra** $(A, \|\cdot\|)$ on X is a subalgebra A of $C(X)$ such that:

1) A contains the constant functions;

2) for each $x \neq y$ in X, there exists $f \in A$ such that $f(x) \neq f(y)$;

3) A is a Banach algebra with respect to the norm $\|\cdot\|$.

A is natural

Let $(A, \|\cdot\|)$ be a Banach function algebra on X. Necessarily $\|f\| \geq |f|_X$ for each $f \in A$.

The **character space** of A is called Φ_A.

For $x \in X$, define $\varepsilon_x : f \mapsto f(x)$. The map

$$x \mapsto \varepsilon_x, \ X \to \Phi_A \, ,$$

is a continuous embedding.

The algebra is **natural** if this map is a surjection, and so $X = \Phi_A$. It is also standard that every maximal ideal of A has the form $\ker \varphi$ for some $\varphi \in \Phi_A$, and so Φ_A is also called the **maximal ideal space** of A.

Of course $(C(X), |\cdot|_X)$ is a natural Banach function algebra on X.

Every semisimple, commutative Banach algebra can be regarded as a natural Banach function algebra via the Gelfand transform.

Uniform algebras

A Banach function algebra A is a **uniform algebra** on X if A is closed in $(C(X), |\cdot|_X)$.

Uniform algebras have been much studied, and there are many results – especially around 1960-1975. (Gamelin, Stout).

But there are many open questions on their structure.

Example Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the open unit disc. The **disc algebra** is

$$A(\overline{\mathbb{D}}) = \{f \in C(\overline{\mathbb{D}})P : f \mid \mathbb{D} \text{ is analytic}\}.$$

Of course $A(\overline{\mathbb{D}}) \subsetneq C(\overline{\mathbb{D}})$.

Standard examples

Let X be a compact set in $\mathbb{C}^n$. Then $P(X)$ (respectively, $R(X)$) is the uniform algebra consisting of the limits in $|\cdot|_X$ of the restrictions to X of the polynomials (respectively, of the rational functions which are analytic near X).

The **polynomially convex hull** of X is

$\widehat{X} = \{z \in \mathbb{C}^n : |p(z)| \leq |p|_X$ all polynomials $p\}$.
So $\widehat{X} = \Phi_{P(X)}$. Similarly $h_r(X)$ is defined to be the rationally convex hull, and $h_r(X) = \Phi_{R(X)}$.

A uniform algebra A on X is **approximately regular** if, for each proper, closed $F \subset X$ and each $x \in X \setminus F$ there exists $f \in A$ with $f(x) = 1$ and $|f|_F < 1$.

For compact $X \subset \mathbb{C}$, $\widehat{X}$ is the union of X and the bounded components of $\mathbb{C} \setminus X$, and $h_r(X) = X$. Further, $R(X)$ is approximately regular whenever $\operatorname{int}_{\mathbb{C}} X = \emptyset$.

Analytic structure

Let A be a natural Banach function algebra. Then Φ_A **contains analytic structure** if there is a continuous injection $\tau : \mathbb{D} \to \Phi_A$ such that $f \circ \tau$ is analytic on $\mathbb{D}$ for each $f \in A$.

First rough guess: a uniform algebra A is either $C(X)$ or Φ_A contains analytic structure.

A **boundary** for A is a closed subset B of Φ_A such that, for each $f \in A$, there exists $x \in B$ such that $|f(x)| = |f|_{\Phi_A}$. The **Shilov boundary** Γ_A is the minimum closed boundary; it always exists.

Second guess: for a uniform algebra A with $\Gamma_A \neq \Phi_A$, Φ_A contains analytic structure.

This is not true (Stolzenberg 1963). Further example: Cole 1968 showed that it can be that every point in Φ_A is a singleton Gleason part, but $\Gamma(A) \neq \Phi_A$.

Various other examples of Feinstein.

The invertibles

Let A be a unital algebra. Then $a \in A$ is **invertible** if there exist $b \in A$ with $ab = ba = e_A$.

The invertibles are Inv A. They are an open set when A is a Banach algebra.

Easy to see: if Φ_A has analytic structure, then $\overline{\text{Inv } A} \neq A$.

Revised conjecture : A has dense invertibles (so $\overline{\text{Inv } A} = A$) implies that $\Gamma_A = \Phi_A$.

We shall show that even this is false.

Remark: an algebra $C(X)$ has dense invertibles if and only if dim X is 0 or 1, so this does not hold for $C(\overline{\mathbb{D}})$.

Topological stable rank

A generalized version of 'A has dense invertibles' involves the **topological stable rank** tsr$A$ of a Banach algebra A. For a Banach function algebra A, set

$$U_n(A) = \{(f_1, \ldots, f_n) \in A^n : \bigcap_{j=1}^{n} Z(f_j) = \emptyset\},$$

and tsr$A = n$ means that n is the minimum number such that $U_n(A)$ is dense in A^n. So A has dense invertibles if and only if tsr$A \leq 1$.

Let $d = \dim X$. Then tsr$C(X) = [d/2] + 1$.

Theorem For approximately regular Banach function algebras A, we have tsr$A \geq$ tsr$C(\Phi_A)$.

Open : Is it always true that tsr$A \leq$ tsr$C(\Phi_A)$, and so tsr$A =$ tsr$C(\Phi_A)$ when A is approximately regular?

The example

The topological boundary of $X \subset \mathbb{C}^n$ is ∂X. Write $\underline{z} = (z, w)$ for a point of $\mathbb{C}^2$.

Theorem There is a compact set $Y \subset \partial\overline{\mathbb{D}}^2$ in $\mathbb{C}^2$ such that $(0, 0) \in \widehat{Y}$, yet $P(Y)$ has dense invertibles. Set $X = \widehat{Y}$. Then $P(X)$ is natural and has dense invertibles, yet $\Gamma_{P(X)} \subset \partial\overline{\mathbb{D}}^2$.

Idea of proof Let $\mathcal{F}$ be the set of all non-constant polynomials $p = p(z, w)$ with coefficients in $\mathbb{Q} + i\mathbb{Q}$ such that $p(\overline{\mathbb{D}}^2) \subset \overline{\mathbb{D}}$. Then $\mathcal{F}$ is countable.

We find $Y \subset \partial\overline{\mathbb{D}}^2$ with $(0, 0) \in \widehat{Y}$ such that the spectrum of $p \mid Y$ with respect to $P(Y)$ has empty interior in $\mathbb{C}$ for all $p \in \mathcal{F}$. It follows easily that Y has the required properties.

More details

Choose a countable dense set $\{\zeta_i : i \in \mathbb{N}\}$ of $\mathbb{D}$ such that the set is disjoint from the countable set $\{p(0,0) : p \in \mathcal{F}\}$.

Define $E_{i,p}$ for $i \in \mathbb{N}$ and $p \in \mathcal{F}$ by

$$E_{i,p} = \{\underline{z} \in \overline{\mathbb{D}}^2 : p(\underline{z}) = \zeta_i\}\,.$$

Each set is compact, and there are only countably many. Enumerate those that are non-empty, and set $K_j = E_{i_j,p_j}$ in an obvious way.

Inductively construct a sequence of successively more complicated analytic varieties W_n through $(0,0)$, each of which is a level set of a certain non-constant entire function on $\mathbb{C}^2$, and some closed sets $M_j \subset \overline{\mathbb{D}}^2$ such that $\widehat{M_j} \cap K_j = \emptyset$ and $W_n \cap \overline{\mathbb{D}}^2 \subset M_j$ whenever $n \geq j$.

Set $V_n = W_n \cap \overline{\mathbb{D}}^2$ for each $n \in \mathbb{N}$ - then p_j does not take the value ζ_j on $\overline{V_n}$ for $n \geq j$.

More details - continued

As n increases, the interiors of the sets $p(\overline{V_n})$ are ‘gradually eliminated’.

The sequence (V_n) of compact sets has a subsequence which converges in the Hausdorff metric to a non-empty, compact subset V of $\overline{\mathbb{D}}^2$.

The set $Y = V \cap \partial\overline{\mathbb{D}}^2$ has the required properties.

Indeed, for each $p \in \mathcal{F}$, the set $p(\overline{V})$ does not meet the set $\{\zeta_i : i \in \mathbb{N}\}$, which is dense in $\overline{\mathbb{D}}$.

Note that Y is rationally convex and that $P(Y) = R(Y)$.

What is the dimension of X? We have dim $X < 4$ because int $X = \emptyset$, and dim $X > 0$ because $P(X) \neq C(X)$. So dim X is 1 or 2 or 3.

Open questions

Question 1 Set $\exp A = \{\exp a : a \in A\}$ for a unital Banach algebra A, so that $\exp A$ is the component of Inv A containing e_A.

Suppose that A is uniform and $\exp A$ is dense in A. Show that $\Gamma_A = \Phi_A$. Perhaps necessarily $A = C(\Phi_A)$.

Question 2 Suppose that A is uniform and $C(\Phi_A)$ has dense invertibles. Does A have dense invertibles? Is the converse always true? Does $C(X)$ have dense invertibles in our example?

Question 3 Let K be a compact subset of $\mathbb{C}^n$ with $\widehat{K} \neq K$. Does $\widehat{K} \setminus K$ always contain a homeomorphic copy of $\mathbb{D}$?

Warning

Assume that you think that you can prove that a uniform algebra A has dense invertibles whenever A is approximately regular and $C(\Phi_A)$ has dense invertibles.

Then you would also have solved the following very famous question of Gelfand : Is there a natural uniform algebra A on $[0,1]$ not equal to $C([0,1])$?

Indeed each such A is approximately regular, and $C(\Phi_A)$ has dense invertibles. So, by the assumed result, A has dense invertibles. But this implies that $A = C([0,1])$.

Differentiable functions

Throughout X is a non-empty, perfect, compact subset of $\mathbb{C}$.

For f on X, we say that f is **differentiable** at $a \in X$ if the limit

$$f'(a) = \lim \left\{ \frac{f(z) - f(a)}{z - a} : z \to a,\, z \in X \right\}$$

exists. In particular, f' is analytic on $\operatorname{int}_{\mathbb{C}} X$.

Thus we have 'continuously differentiable on X', 'infinitely-differentiable on X', etc.

The space of continuously differentiable functions on X is $D^{(1)}(X)$.

[Similarly $D^{(n)}(X)$ and $D(X, (M_k))$ for some sequences (M_k) - see the work of Honary et al.]

Continuously differentiable functions

It is immediate that $D^{(1)}(X)$ is an algebra, and that it is a normed algebra with respect to the norm $\|\cdot\|$ given by

$$\|f\| = |f|_X + \left|f'\right|_X \quad (f \in D^{(1)}(X)).$$

Basic question When is $(D^{(1)}(X), \|\cdot\|)$ complete, and hence a Banach function algebra on X? What are its other properties?

Suppose that X has infinitely many components. Then $D^{(1)}(X)$ is not complete. So we consider only the case where X is connected.

Question Is $D^{(1)}(X) \subset R(X)$ for all perfect, compact plane sets? For all pointwise regular X?

Paths

A **path** in X is (the image of) a continuous function $\gamma : [a, b] \to X$. (A Jordan arc is a path such that γ is injective.) The **end points** of γ are

$$\gamma^- = \gamma(a) \quad \text{and} \quad \gamma^+ = \gamma(b)\,.$$

The length of a rectifiable path γ is $|\gamma|$.

Recall the fundamental theorem of the calculus: Let γ be such a rectifiable path in X. Then

$$\int_\gamma f'(z)\,\mathrm{d}z = f(\gamma^+) - f(\gamma^-) \quad (f \in D^{(1)}(X))\,.$$

The set X is **regular** at $z \in X$ if there is a constant k_z such that, for each $w \in X$, there is a rectifiable path $\gamma : [a, b] \to X$ with $\gamma^- = z$ and $\gamma^+ = w$ and $|\gamma| \le k_z\,|z - w|$.

The set X is **pointwise regular** if it is regular at each point of X.

Completeness

Honary, extending Dales and Davie, showed that $(D^{(1)}(X), \|\cdot\|)$ is complete whenever X is a pointwise regular set.

Question 1 Is the converse true?

Question 2 The completion of $(D^{(1)}(X), \|\cdot\|)$ is called $\widetilde{D}^{(1)}(X)$. What is it when $D^{(1)}(X)$ is not complete? When is $\widetilde{D}^{(1)}(X)$ a Banach function algebra?

Proposition The map

$$f \mapsto (f, f')\,, \quad D^{(1)}(X) \to C(X) \ltimes C(X)\,,$$

is an isometric embedding of $(D^{(1)}(X), \|\cdot\|)$ in the semidirect product $C(X) \ltimes C(X)$. The latter is a Banach algebra, and so $\widetilde{D}^{(1)}(X)$ is its closure in $C(X) \ltimes C(X)$. $\square$

Partial results

Example There is a Jordan arc J (not rectifiable) such that $\widetilde{D}^{(1)}(J) = C(J) \ltimes C(J)$, and so $\widetilde{D}^{(1)}(J)$ is far from semisimple. $\square$

Example There is a rectifiable Jordan arc J such that $D^{(1)}(J)$ is not complete.

Another Banach function algebra

Definition Let X be as usual and such that $\operatorname{int}_{\mathbb{C}} X$ is dense in X. Then $A^{(1)}(X)$ is the set of functions $f \in C(X)$ such that f' is analytic on $\operatorname{int}_{\mathbb{C}} X$ and such that f' has a [necessarily unique] extension to a function in $C(X)$. The norm is as before. So $A^{(1)}(X)$ is a Banach function algebra on X.

In this case $D^{(1)}(X)$ embeds isometrically in $A^{(1)}(X)$, and so $\widetilde{D}^{(1)}(X)$ is just the closure of $D^{(1)}(X)$ in $A^{(1)}(X)$.

Example - see below It may be that, even in this case, $D^{(1)}(X)$ is not complete. Further, its completion $\widetilde{D}^{(1)}(X)$ may be a proper subset of $A^{(1)}(X)$. $\square$

Question Suppose that $\operatorname{int}_{\mathbb{C}} X$ is dense in X. Is $A^{(1)}(X) \subset R(X)$?

First we generalize the above idea.

Yet another Banach function algebra

Definition Let X be as usual. A family $\mathcal{F}$ of paths in X is **useful** if the following conditions are satisfied:

1) every path in $\mathcal{F}$ is rectifiable, and there are no constant subpaths;

2) every subpath of a path in $\mathcal{F}$ is also in $\mathcal{F}$;

3) the union of the paths is dense in X.

In this case, a function $g \in C(X)$ is an $\mathcal{F}$-**derivative** of $f \in C(X)$ if

$$\int_\gamma g(z)\,\mathrm{d}z = f(\gamma^+) - f(\gamma^-) \quad (f \in D^{(1)}(X))$$

for each $\gamma \in \mathcal{F}$. (*cf.* the fundamental theorem)

Another Banach function algebra -continued

Definition Now $D^1_{\mathcal{F}}(X)$ is the set of functions $f \in C(X)$ which have such an $\mathcal{F}$-derivative. Take the usual norm.

It is not too hard to show that $D^1_{\mathcal{F}}(X)$ is a Banach function algebra on X.

So $\widetilde{D}^{(1)}(X)$ is just the closure of $D^{(1)}(X)$ in $D^1_{\mathcal{F}}(X)$.

Theorem In the above setting, $\widetilde{D}^{(1)}(X)$ is a Banach function algebra.

In many cases, we have $\widetilde{D}^{(1)}(X) = D^1_{\mathcal{F}}(X)$, but we do not know if this is always true.

Conditions for completeness of $D^{(1)}(X)$

Theorem For the usual X with X polynomially convex and the union of paths in $\mathcal{F}$ dense in X, the following are equivalent:

a) $D^{(1)}(X)$ is complete:

b) $D^{(1)}(X) = D^1_{\mathcal{F}}(X)$;

c) for each $z \in X$, there is a constant $C_z > 0$ such that

$$|p(w) - p(z)| \leq C_z \left| p' \right|_X |w - z|$$

for each $w \in X$ and each polynomial p. $\square$

Example 1

There is polynomially convex, compact set with dense interior such that

$$D^{(1)}(X) \neq A^{(1)}(X)\,.$$

In particular the polynomials are not dense in $A^{(1)}(X)$.

Example 2

There is a compact plane set X with dense interior such that $D^{(1)}(X)$ is not complete.

In the picture, $w_n = \mathrm{i}/n^3$, $z_n = 1+\mathrm{i}/n$, U_n is the open sector with vertex z_n and lines through w_n and w_{n+1}, V_n is the open disc with centre z_n and radius c/n^2 for suitable constant $c > 0$. Next β_n is the angle between the centre of the sector and the real axis. We use

$$L_n(w) = \log(\mathrm{e}^{-\mathrm{i}\beta_n}(w - z_n))\,.$$

Then $|L_n(w_n) - L_n(0)| \to 2\pi$, but there is no constant satisfying a version of c) because $n^2/\,|w_n| \to \infty$.

Picture of Example 2

Example 3

Set $r_n = 1/8\sqrt{n}$, $\alpha_n = \pi/4n^2$,

$$w_n = \mathrm{e}^{\mathrm{i}\alpha_n} \, ,$$

$$\beta_n = (\alpha_n + \alpha_{n+1})/2 \, ,$$

$$a_n = (1 - 4r_n)\mathrm{e}^{\mathrm{i}\beta_n} \, ,$$

$$z_n = (1 - 2r_n)\mathrm{e}^{\mathrm{i}\beta_n} \, .$$

Use

$$L_n(w) = \log(\mathrm{e}^{(\pi - \beta_n)}(w - z_n)) \, .$$

The set X is **radially self-absorbing**, in the sense that $X \subset \mathrm{int}_C(rX)$ for all $r > 1$.

Picture of Example 3

Test question

Fix a sequence (w_n) with $w_n \to 0$, and another sequence (ε_n) so that $w_n - \varepsilon_n > w_{n+1}$. The set X is as below. Is $D^{(1)}(X)$ always incomplete?