How to multiply ultrafilters

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Filters and ultrafilters

Let S be a non-empty set. A **filter** on S is a non-empty subset $\mathcal{F}$ of $\mathcal{P}(S)$ such that:

(1) $\emptyset \notin \mathcal{F}$;

(2) $E, F \in \mathcal{F} \Rightarrow E \cap F \in \mathcal{F}$;

(3) $E \in \mathcal{P}(S), F \in \mathcal{F}, E \supset F \Rightarrow E \in \mathcal{F}$.

The family of all filters on S is a partially ordered set; a maximal element in this family is an **ultrafilter**.

A filter $\mathcal{F}$ is an ultrafilter if and only if $E \in \mathcal{F}$ or $F \in \mathcal{F}$ whenever $E, F \in \mathcal{P}(S)$ with $E \cup F = S$.

Example : $\{F \subset S : s \in F\}$ for $s \in S$. This is a **fixed** ultrafilter; the others are **free**.

Fact Let $\kappa = |S|$. Then there are $2^{2^{\kappa}}$ ultrafilters on S.

Special types of ultrafilters

Let S be a non-empty set.

Definition Take κ with $\aleph_0 \leq \kappa \leq |S|$. An ultrafilter $\mathcal{U}$ is **κ-uniform** if $|U| \geq \kappa$ for each $u \in \mathcal{U}$. The set of these is $U_\kappa(S)$. Take $U_S = U_\kappa(S)$ when $\kappa = |S|$.

Definition An ultrafilter $\mathcal{U}$ is **selective** if, for each partition $\{A_k : k \in \mathbb{N}\}$ of S, either there exists $k_0 \in \mathbb{N}$ with $A_{k_0} \in \mathcal{U}$ or there exists $U \in \mathcal{U}$ such that $|U \cap A_k| \leq 1$ for each $k \in \mathbb{N}$.

Do such exist on S when S is countable?

This is implied by CH and MA, but is not a theorem of ZFC.

There are many other special ultrafilters - see Comfort and Negrepontis.

Ultrapowers

Let S be a non-empty set. Then the space $\mathbb{R}^S$ is an algebra (over $\mathbb{R}$).

Let $\mathcal{U}$ be an ultrafilter on the set S, and let $f, g \in \mathbb{R}^S$. Set $f \sim_{\mathcal{U}} g$ if

$$\{s \in S : f(s) = g(s)\} \in \mathcal{U}\,.$$

Then $\sim_{\mathcal{U}}$ is an equivalence relation, and the quotient $\mathbb{R}^S/\mathcal{U}$ of the partially ordered algebra $\mathbb{R}^S$ is a totally ordered field, an **ultrapower**, or a **non-standard model** of $\mathbb{R}$; it contains infinitesimals and infinitely large elements.

Let S be countable. Are the fields $\mathbb{R}^S/\mathcal{U}$ the same as each other? With CH - yes. Without CH - no, and there is a zoo of them.

Bounded functions

Let $\ell^\infty(S)$ be the subspace of $\mathbb{R}^S$ consisting of the bounded functions in $\mathbb{R}^S$. Then $\ell^\infty(S)/\mathcal{U}$ is a subalgebra of $\mathbb{R}^S/\mathcal{U}$ - it consists of the **finite** elements of $\mathbb{R}^S/\mathcal{U}$. It is an integral domain whose quotient field is $\mathbb{R}^S/\mathcal{U}$.

Let A be a (real) algebra. A **character** on A is a homomorphism $\varphi : A \to \mathbb{R}$.

Let A be an algebra which is a Banach space for a norm $\|\cdot\|$. Then $(A, \|\cdot\|)$ is a **Banach algebra** if $\|ab\| \le \|a\| \, \|b\|$ for all $a, b \in A$.

Example Set

$$|\lambda|_S = \sup \{|\lambda(s)| : s \in S\} .$$

Then $|\cdot|_S$ is an algebra norm on $\ell^\infty(S)$, and $(\ell^\infty(S), |\cdot|_S)$ is a commutative Banach algebra.

The Banach algebra $C(\Omega)$

Let Ω be a comapct space. Then $C(\Omega)$ is the space of all continuous functions on Ω. It is an algebra for the pointwise product.

Set

$$|f|_\Omega = \sup\{|f(x)| : x \in \Omega\}$$

Then $(C(\Omega), |\cdot|_\Omega)$ is a commutative Banach algebra. For $x \in \Omega$, the map $f \mapsto f(x)$ is a character on $C(\Omega)$, and it is easy to see that every character arises in this way. Let

$$M_x = \{f \in C(\Omega) : f(x) = 0\}\,.$$

Then M_x is a maximal ideal in $C(\Omega)$, and it is easy to see that every maximal ideal has this form.

The dual space of a Banach space

Let E be a Banach space with a norm $\|\cdot\|$.

The space of continuous linear functionals on E is another Banach space, with norm given by

$$\|\lambda\| = \sup\{|\lambda(x)| : x \in E,\ \|x\| \leq 1\}.$$

This space is called the **dual space**, written E'. The unit ball $E'_{[1]}$ of E' is compact in a different topology - this is the weak-$*$ topology.

The second dual is E'', and there is a canonical embedding of E into E''.

The character space of a Banach algebra

Easy fact Each character on a Banach algebra A is continuous, and so is an element of the dual space A'. The space Φ_A of all characters is a compact Hausdorff space for the weak-$*$ topology.

Let A be a Banach algebra with compact character space Φ_A. Then the **Gelfand transform** $a \mapsto \widehat{a}$, where

$$\widehat{a}(\varphi) = \varphi(a) \quad (\varphi \in \Phi_A)\,,$$

is a continuous homomorphism from $(A, \|\cdot\|)$ into $(C(\Omega), |\cdot|_{\Phi_A})$.

Stone-Cech compactification

Let S be a non-empty set. We denote the collection of ultrafilters on S by βS. The subset of βS of fixed ultrafilters is identified with S. Regard an element of βS as a point u (corresponding to the ultrafilter $\mathcal{U}$). For $T \subset S$, define

$$\overline{T} = \{u \in \beta S : T \in u\}\,.$$

Thus $\overline{S} = \beta S$ and $\overline{\emptyset} = \emptyset$

Basic fact : βS is made into a topological space by taking the family of all sets $\overline{T}$ as the base for a topology.

We obtain a compact Hausdorff space - it is the **Stone-Cech compactification** of S.

So $|\beta S| = 2^{2^{\kappa}}$.

We write $T^* = \overline{T} \setminus T$ for $T \subset S$.

Fact βS is the **Stone space** of the Boolean algebra $\mathcal{P}(\mathcal{S})$.

Extensions of continuous functions

The Stone-Cech theorem Every $f \in \ell^\infty(S)$ has an extension to a **continuous** function

$$\widehat{f} : \beta S \to \mathbb{R}\,,$$

and

$$\left|\widehat{f}\right|_{\beta S} \equiv \text{sup}\,\left\{\left|\widehat{f}(u)\right| : u \in \beta S\right\} = |f|_S\ .$$

Further $\widehat{f}$ is unique.

It follows that $\ell^\infty(S) = C(\beta S)$ as algebras and as Banach spaces.

The space βS is the compact character space of the Banach algebra $\ell^\infty(S)$, and the map

$$f \mapsto \widehat{f}, \quad \ell^\infty(S) \to C(\beta S)\,,$$

is just the Gelfand transform.

Maximal ideals

Let S be a non-empty set. For $f \in \mathbb{R}^S$, set $\mathbf{Z}(f) = \{s \in S : f(s) = 0\}$. Let $u \in \beta S$.

Set
$$M^u = \{f \in \mathbb{R}^S : \mathbf{Z}(f) \in u\}.$$

Then M^u is a maximal ideal in $\mathbb{R}^S$, and every maximal ideal has this form.

Set
$$M_u = \{f \in \ell^\infty(S) : \hat{f}(u) = 0\}.$$

Then M^u is a maximal ideal in $\ell^\infty(S)$, and every maximal ideal has this form.

Set
$$J_u = \{f \in \ell^\infty(S) : \hat{f} = 0 \text{ near } u\}.$$

Then J_u is a prime ideal in $\ell^\infty(S)$, and $\ell^\infty(S)/J_u$ is an integral domain whose quotient field is $\mathbb{R}^S/\mathcal{U}$.

See Dales-Woodin for a discussion of these integral domains - many of their properties depend on the axioms of set theory that are used.

Examples of dual spaces

- $c_0(S)$ is the closed subspace of $\ell^\infty(S)$ consisting of functions f such that

$$\{s \in S : |f(s)| < \varepsilon\}$$

is finite for each $\varepsilon > 0$.

- $\ell^1(S) = \left\{ f \in \mathbb{C}^S : \|f\|_1 = \sum_{s\in S} |f(s)| < \infty \right\}$.

- for a compact space Ω, $M(\Omega)$ is the space of (complex-valued, regular, Borel) measures on Ω, with with $\|\mu\| = |\mu|(\Omega)$.

Then:

$$c_0(S)' = \ell^1(S)\,;$$

$$\ell^1(S)' = \ell^\infty(S) = C(\beta S)\,;$$

$$C(\beta S)' = M(\beta S).$$

Semigroups

Let S be a semigroup with operation denoted by juxtaposition or $\cdot$. For $s \in S$ and $W \subset S$, set

$$s^{-1}W = \{t \in S : st \in W\} .$$

For example, consider $S = (\mathbb{N}, +)$.

A semigroup is **weakly cancellative** if the equations $sx = t$ and $xs = t$ have only finitely many solutions for x for each $s, t \in S$.

For example, $(\mathbb{N}, \vee)$, where $m \vee n = \max\{m, n\}$.

Let S be a semigroup. Then

$$L_s : t \mapsto st, \qquad R_s : t \mapsto ts .$$

Compact right topological semigroups

V is a **compact right topological semigroup** if V is a compact space and a semigroup, and, further,

- R_v is continuous for each $v \in V$.

Let S be a semigroup. We wish to make βS into a compact right topological semigroup for an operation $\Box$ such that $(S, \cdot\,)$ is a dense subset and a subsemigroup of $(\beta S, \Box)$, and

- L_s is continuous for each $s \in S$.

Already βS is a compact space containing S as a dense subset.

The topological method

Let $u, v \in \beta S$, and take nets (s_α) and (t_β) in S with $\lim_\alpha s_\alpha = u$ and $\lim_\beta t_\beta = v$. Then

$$u \Box v = \lim_\alpha \lim_\beta s_\alpha t_\beta \, .$$

Similarly,

$$u \Diamond v = \lim_\beta \lim_\alpha s_\alpha t_\beta \, .$$

Is it obvious that $\Box$ is well-defined and associative, and that $(\beta S, \Box)$ is a right topological semigroup? These facts are true.

The algebraical-logical method

Let $u, v \in \beta S$, and let W be a subset of S. We can show that the following are equivalent:

(a) W belongs to the ultrafilter $u \Box v$;

(b) $\{s \in S : s^{-1}W \in v\} \in u$;

(c) there exists $U \in u$ and a family

$$\{V_s : s \in U\} \subset v$$

with $sV_s \subset W \ \ (s \in U)$.

Is it obvious from this that $\Box$ is associative, and that $(\beta S, \Box)$ is a right topological semigroup?

The functional analytic method

Let A be a Banach algebra.

Then A'' is a Banach A-bimodule for maps

$$(a,\, \Phi) \mapsto a \cdot \Phi, \quad (a,\, \Phi) \mapsto \Phi \cdot a$$

from $A \times A''$ to A''.

There are two products on A'' extending the module maps. For $\lambda \in A'$ and $a, b \in A$, define:

$$\langle b,\, a \cdot \lambda\rangle = \langle ba,\, \lambda\rangle, \quad \langle b,\, \lambda \cdot a\rangle = \langle ab,\, \lambda\rangle\,,$$

and then, for $\Phi \in A''$, we set

$$\langle a,\, \lambda \cdot \Phi\rangle = \langle \Phi,\, a \cdot \lambda\rangle, \quad \langle a,\, \Phi \cdot \lambda\rangle = \langle \Phi,\, \lambda \cdot a\rangle.$$

Let $\Phi, \Psi \in A''$. Then

$$\langle \Phi \,\Box\, \Psi,\, \lambda\rangle = \langle \Phi,\, \Psi \cdot \lambda\rangle, \quad \langle \Phi \diamond \Psi,\, \lambda\rangle = \langle \Psi,\, \lambda \cdot \Phi\rangle$$

for each $\lambda \in A'$.

Theorem Both $(A'', \Box)$ and $(A'', \diamond)$ are Banach algebras containing A as a closed subalgebra. $\Box$

Semigroup algebras

Definition Let S be a semigroup (eg, $S = \mathbb{N}$). Set

$$\left(\sum \alpha_r \delta_r\right) \star \left(\sum \beta_s \delta_s\right) = \sum \left\{ \sum_{rs=t} \alpha_r \beta_s \delta_t : t \in S \right\} .$$

Then $A := (\ell^1(S), \star)$ is the **semigroup algebra** of S. This includes the case of the **group algebra** $\ell^1(G)$ for a group G.

Clearly A is a Banach algebra, containing $s \in S$, identified with $\delta_s \in A$.

The dual module action of $s \in S$ on $\ell^\infty(S)$ is:

$$(s \cdot \lambda)(t) = \lambda(ts), \quad (\lambda \cdot s)(t) = \lambda(st)$$

for $t \in S$. We have

$$A'' = C(\beta S)' = M(\beta S),$$

the measures on βS.

The semigroup $(\beta S, \Box)$ again

Let S be a semigroup. For $\mu, \nu \in M(\beta S)$, we have definitions of $\mu \,\Box\, \nu$ and $\mu \diamond \nu$.

Thus we have the Banach algebra $(M(\beta S), \Box)$. A short calculation shows that $\delta_u \Box \delta_v$ is a character on $C(\beta S)$ for each $u, v \in \beta S$, say at $u \Box v$. Then $(\beta S, \Box)$ is a semigroup containing S as a dense subsemigroup, and $\ell^1(\beta S)$ is a closed subalgebra of $M(\beta S)$.

We may have worked a bit harder, but now it is clear that $\Box$ is associative and that $(\beta S, \Box)$ is a compact right topological semigroup with all the required properties.

Basic properties of the semigroup $(\beta S, \Box)$

The following are well-known and easy. Let S be an infinite semigroup.

- $(\beta S, \Box)$ is never a group - and it is never cancellative.

- $(\beta S, \Box)$ is never abelian, even if S is.

- S^* is an ideal of $(\beta S, \Box)$ if and only if S is weakly cancellative; in this case L_v is continuous on $(\beta S, \Box)$ if and only if $v \in S$.

- for S countable and weakly cancellative, we have $S^*_{[2]}$ nowhere dense in S^*.

- $(\beta S, \Box)$ has a minimum ideal $K(\beta S)$, contained in S^*.

- there are many idempotents in $(\beta S, \Box)$ - including long decreasing chains thereof for the ordering $p \leq q$ iff $pq = qp = p$.

More properties of the semigroup $(\beta S, \Box)$

The following is fairly deep

- there is a copy of $\mathbb{F}_2$ inside $(\beta S, \Box)$ whenever S is cancellative.

The following is very deep.

- The semigroup $(\beta\mathbb{Z}, \Box)$ contains no non-trivial finite groups (Zelenuk).

The following are open.

- Does there exist $u \in \beta\mathbb{N}$ such that $u^2 \neq u$, but $u^3 = u^2$?

- Do there exist increasing chains of idempotents of length $\aleph_1$?

General remarks

(1) Properties of ultrafilters relate to topological properties of βS. For example, selective ultrafilters are P-points in S^*.

(2) Quite a few properties of $(\beta S, \square)$ are independent of ZFC.

(3) The compactness of $(\beta S, \square)$ can be used to give cheap proofs, and to resolve previously open questions, in number theory, combinatorics, Ramsey theory, etc.

Topological centres of semigroups

Definition Let S be a semigroup. The **left** and **right topological centres** of βS are

$$\mathfrak{Z}_t^{(\ell)}(\beta S) = \{s \in \beta S : s \Box t = s \Diamond t \ \ (t \in \beta S)\}$$

and

$$\mathfrak{Z}_t^{(r)}(\beta S) = \{s \in \beta S : t \Box s = t \Diamond s \ \ (t \in \beta S)\}\,,$$

respectively. The semigroup S is
Arens regular if

$$\mathfrak{Z}_t^{(\ell)}(\beta S) = \mathfrak{Z}_t^{(r)}(\beta S) = \beta S\,,$$

and **strongly Arens irregular** if

$$\mathfrak{Z}_t^{(\ell)}(\beta S) = \mathfrak{Z}_t^{(r)}(\beta S) = S\,.$$

Example Take $S = (\mathbb{N}, \vee)$. Then S is weakly cancellative; S^* is an ideal in $(\beta S, \Box)$; as a semigroup, it is the left zero semigroup; S is Arens regular.

Sample new theorems

Theorem Let S be an infinite, cancellative semigroup. Then there exist $a, b \in U_S$ such that the only element $\mu \in M(S^*)$ such that

$$\mu \,\Box\, a = \mu \diamond a \quad \text{and} \quad \mu \,\Box\, b = \mu \diamond b$$

is $\mu = 0$. So ‘two point masses suffice’ to establish that S is strongly Arens irregular. $\Box$

Theorem Let S be an infinite, weakly cancellative semigroup. Then there is a lot of structure on $M(\beta S)$ related to differentiability of measures on βS, and this gives information about products of ultrafilters. $\Box$

The set U_S

Facts Let S be an infinite semigroup. Then:

(1) $M(\beta S \setminus U_S)$ is a $\|\cdot\|$-closed subalgebra of $(M(\beta S), \|\cdot\|)$.

(2) S weakly cancellative $\Rightarrow U_S$ is an ideal in βS.

(3) S weakly cancellative $\Rightarrow M(U_S)$ is a closed ideal in $M(\beta S)$.

Easy for ‘left ideal’; for ‘right ideal’ this solves an open problem.

(4) S cancellative and $|S|$ a regular cardinal $\Rightarrow$ $U_S S^* \cup S^* U_S$ is nowhere dense in U_S.