

Banach algebras of continuous
functions and measures, and their
second duals

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(proto-paper with To-Ming Lau and Dona
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Laval, July 2007

Second dual algebras

Let A be a Banach algebra.

Then A'' is a Banach A -bimodule for maps

$$(a, \Phi) \mapsto a \cdot \Phi, \quad (a, \Phi) \mapsto \Phi \cdot a$$

from $A \times A''$ to A'' .

There are two products on A'' of A extending the module maps. For $a, b \in A$, and $\lambda \in A'$, define:

$$\langle b, a \cdot \lambda \rangle = \langle ba, \lambda \rangle, \quad \langle b, \lambda \cdot a \rangle = \langle ab, \lambda \rangle.$$

Then, for $a \in A$, $\lambda \in A'$, and $\Phi \in A''$, define:

$$\langle a, \lambda \cdot \Phi \rangle = \langle \Phi, a \cdot \lambda \rangle, \quad \langle a, \Phi \cdot \lambda \rangle = \langle \Phi, \lambda \cdot a \rangle.$$

Let $\Phi, \Psi \in A''$. Then

$$\langle \Phi \square \Psi, \lambda \rangle = \langle \Phi, \Psi \cdot \lambda \rangle, \quad \langle \Phi \diamond \Psi, \lambda \rangle = \langle \Psi, \lambda \cdot \Phi \rangle$$

for each $\lambda \in A'$.

Suppose that $\Phi = \lim_{\alpha} a_{\alpha}$ and $\Psi = \lim_{\beta} b_{\beta}$ for nets (a_{α}) and (b_{β}) in A . Then

$$\Phi \square \Psi = \lim_{\alpha} \lim_{\beta} a_{\alpha} b_{\beta}, \quad \Phi \diamond \Psi = \lim_{\beta} \lim_{\alpha} a_{\alpha} b_{\beta}.$$

Definition The two **topological centres** are

$$\mathfrak{Z}_t^{(\ell)}(A'') = \{ \Phi \in A'' : \Phi \square \Psi = \Phi \diamond \Psi \text{ } (\Psi \in A'') \}$$

and

$$\mathfrak{Z}_t^{(r)}(A'') = \{ \Phi \in A'' : \Psi \square \Phi = \Psi \diamond \Phi \text{ } (\Psi \in A'') \}.$$

The algebra A is **Arens regular** if

$$\mathfrak{Z}_t^{(\ell)}(A'') = \mathfrak{Z}_t^{(r)}(A'') = A'',$$

and **strongly Arens irregular = SAI** if

$$\mathfrak{Z}_t^{(\ell)}(A'') = \mathfrak{Z}_t^{(r)}(A'') = A.$$

Examples Every C^* -algebra is Arens regular. Every group algebra $L^1(G)$ is SAI.

A subset V of A'' is **determining for the topological centre** if $\Phi \in A$ whenever $\Phi \square \Psi = \Phi \diamond \Psi$ for all $\Psi \in V$.

Topological preliminaries

A topological space is *extremely disconnected* if the closure of every open set is itself open. A *Stonian space* is a compact topological space that is extremely disconnected.

Let U be a dense, open subset of a Stonian space Ω . Then $\beta U = \Omega$. Each infinite Stonian space Ω contains a copy of $\beta\mathbb{N}$, and so $|\Omega| \geq 2^{\mathfrak{c}}$.

CCC - countable chain condition on clopen subsets.

Key example \mathbb{Z}_2^κ is the Cantor cube of weight κ (with the product topology).

Borel sets

\mathfrak{B}_Ω denotes the family of Borel subsets of a compact space Ω . (This is the σ -field generated by the open subsets of Ω .)

Recall Let Ω be an uncountable, compact, metrizable space (e.g., \mathbb{I} , $\mathbb{Z}_2^{\aleph_0}$). Then:

(1) $w(\Omega) = \aleph_0$ and $|\Omega| = |\mathfrak{B}_\Omega| = \mathfrak{c}$.

(2) there are \mathfrak{c} pairwise disjoint closed subsets, each homeomorphic to $\mathbb{Z}_2^{\aleph_0}$.

Banach algebra/lattice $C(\Omega)$

Let Ω be a compact space. Then $C(\Omega)$ is the commutative C^* -algebra of all continuous functions on Ω . (This algebra determines Ω .)

$C(\Omega)$ is also a complex Banach lattice - it is the complexification of $C_{\mathbb{R}}(\Omega)$.

The lattice $C_{\mathbb{R}}(\Omega)$ is **complete** if and only if Ω is extremely disconnected, and this holds if and only if $C(\Omega)$ is injective in the category of commutative C^* -algebras and continuous homomorphisms (Gleason's theorem).

Hyper-Stonian spaces

A compact space Ω is **hyper-Stonian** if $C(\Omega)$ is isometrically the dual space of another Banach space (so that $C(\Omega)$ is a von Neumann algebra).

The predual of $C(\Omega)$ is denoted by $C(\Omega)_*$. It is unique as a Banach space.

Measures on Ω

Let Ω be a compact space. Then $M(\Omega)$ is the Banach space of all complex-valued, regular Borel measures on Ω . Here $\|\mu\| = |\mu|(\Omega)$.

We identify $M(\Omega)$ with $C(\Omega)'$: here

$$\langle \lambda, \mu \rangle = \int_{\Omega} \lambda d\mu.$$

Useful fact: Ω metrizable implies that all Borel measures on Ω are regular.

A fixed measure on Ω

Fix a positive measure μ on Ω . (Later it will be Haar measure on a group.) Then $M_{ac}(\Omega)$ is the closed subspace of $M(\Omega)$ consisting of the measures which are absolutely continuous with respect to μ . Thus

$$M_{ac}(\Omega) = L^1(\Omega, \mu).$$

The dual space is $L^1(\Omega, \mu)' = L^\infty(\Omega, \mu)$, and this is a commutative C^* -algebra, with character space Φ_μ , say. The Gel'fand transform is

$$\mathcal{G}_\mu : L^\infty(\Omega, \mu) \rightarrow C(\Phi_\mu).$$

The space Φ_μ is hyper-Stonian.

For example, if μ is counting measure on \mathbb{N} , then Φ_μ is $\beta\mathbb{N}$.

Normal measures

A measure is **normal** if it is order-continuous on $C_{\mathbb{R}}(\Omega)$: $\langle \lambda_{\alpha}, \mu \rangle \rightarrow 0$ whenever (λ_{α}) is a decreasing net in $C_{\mathbb{R}}(\Omega)$ with $\bigwedge_{\alpha} f_{\alpha} = 0$.

Theorem Let Ω be hyper-Stonian. Then the following are equivalent for $\mu \in M(\Omega)^+$:

- (a) μ is normal;
- (b) $\mu \in C(\Omega)_*$;
- (c) $\mu(K) = 0$ for each compact subset K of Ω with $\text{int } K = \emptyset$. □

Theorem Let Ω be Stonian. Then the following are equivalent:

- (a) Ω is hyper-Stonian;
- (b) for each $\lambda \in C(\Omega)^+$ with $\lambda \neq 0$, there exists $\mu \in N(\Omega)^+$ with $\langle \lambda, \mu \rangle \neq 0$;
- (c) $C(\Omega) = L^{\infty}(\Gamma, \nu)$ for some Γ and ν .

Singular measures

A family \mathcal{F} of positive measures on a compact space Ω is **singular** if any two distinct measures in \mathcal{F} are mutually singular.

There is a maximal such family; we may suppose that it contains all point masses.

Fact Let Ω be an uncountable, compact, metrizable space. Then there is such a family \mathcal{F} consisting of just \mathfrak{c} point masses and \mathfrak{c} continuous measures.

Boolean algebras

Let B be a Boolean algebra (e.g., \mathfrak{B}_Ω for a compact space Ω). An **ultrafilter** on B is a subset p that is maximal with respect to the property that

$$b_1, \dots, b_n \in p \Rightarrow b_1 \wedge \dots \wedge b_n \neq 0.$$

The family of ultrafilters on B is the **Stone space** of B , denoted by $S(B)$. A topology on $S(B)$ is defined by taking the sets

$$\{p \in S(B) : b \in p\}$$

for $b \in B$ as a basis of the open sets of $S(B)$. In this way $S(B)$ is a totally disconnected compact space; it is extremely disconnected if and only if B is complete as a Boolean algebra.

Examples of Boolean algebras

(I) $B = \mathcal{P}(\mathbb{N})$, the power set of \mathbb{N} . Then $S(B)$ is just $\beta\mathbb{N}$, the character space of $\ell^\infty = C(\beta\mathbb{N})$.

(II) Ω be a compact space, and fix $\mu \in M(\Omega)^+$. Then \mathfrak{B}_μ is \mathfrak{B}_Ω modulo the μ -null sets. Its Stone space is exactly Φ_μ , described above. So $\varphi \in \Phi_\mu$ is an ultrafilter, and we can consider 'limits along the ultrafilter', say $\lim_{B \rightarrow \varphi}$.

This approach gives us some useful formulae. Indeed,

$$\lim_{B \rightarrow \varphi} \frac{1}{\mu(B)} \int_B \lambda \, d\mu = \mathcal{G}_\mu(\lambda)(\varphi)$$

for each $\lambda \in L^\infty(\Omega, \mu)$.

A characterization of Φ_μ

Theorem Let Ω and μ be as above, and suppose that $|\mathfrak{B}_\mu| = \kappa$, an infinite cardinal. Then Φ_μ is a hyper-Stonian space satisfying CCC. Further,

$$w(\Phi_\mu) = \kappa \quad \text{and} \quad |\Phi_\mu| \leq 2^\kappa. \quad \square$$

Theorem A hyper-Stonian space X has the form Φ_μ for some μ if and only if X satisfies CCC. □

An isomorphism theorem

Let Ω and μ be as above. Then the associated L^1 -space is $(L^1(\Omega, \mu), \|\cdot\|_1)$. The Boolean ring (\mathfrak{B}_μ, μ) is **separable** if the Banach space $L^1(\Omega, \mu)$ is separable.

Classical theorem Suppose that μ is a continuous measure and that (\mathfrak{B}_μ, μ) is separable. Then (\mathfrak{B}_μ, μ) is the same as the special case in which $\Omega = \mathbb{I}$ and μ is Lebesgue measure. \square

In the special case, Φ_μ is \mathbb{H} , called the **hyper-Stonian space of the unit interval**.

Corollary Let Ω be uncountable, compact, and metrizable, and let $\mu \in M_c(\Omega)^+$. Then Φ_μ is homeomorphic to \mathbb{H} .

Theorem \mathbb{H} is the unique topological space X such that X is hyper-Stonian, X has no isolated points, X satisfies CCC, and each weak-* discrete subset of $C(X)_{[1]}$ is countable.

The last clause is not redundant. \square

The second dual of $C(\Omega)$

Let (Ω, τ) be a compact space. Then $C(\Omega)$ is Arens regular, and its second dual $C(\Omega)''$ is a commutative C^* -algebra (with the Arens products denoted by juxtaposition). So $C(\Omega)'' = C(\widetilde{\Omega})$ for a compact, hyper-Stonian space $(\widetilde{\Omega}, \sigma)$, called the **hyper-Stonian cover** of Ω .

There is an embedding $\iota : \Omega \rightarrow \widetilde{\Omega}$ and a continuous projection $\pi : \widetilde{\Omega} \rightarrow \Omega$ such that $\pi \circ \iota$ is the identity on Ω .

The map ι is not usually a homeomorphism. Indeed Ω consists exactly of the isolated points of $(\widetilde{\Omega}, \sigma)$, and so Ω is open in $(\widetilde{\Omega}, \sigma)$. The closure of Ω in $(\widetilde{\Omega}, \sigma)$ is identified with $\beta\Omega_d$.

For $x \in \Omega$, set $\Omega_{\{x\}} = \pi^{-1}(\{x\})$, the **fibre** of x . Then $C(\Omega)$ is identified with the algebra of functions in $C(\widetilde{\Omega})$ that are constant on fibres.

The second dual of $M(\Omega)$

Let Ω be a compact space, and set $E = C(\Omega)$. Then

$$E' = M(\Omega), \quad E'' = C(\tilde{\Omega}),$$

and

$$E''' = M(\Omega)'' = M(\tilde{\Omega}).$$

Clearly $\kappa_{E'}(M(\Omega))$ is just the space $N(\tilde{\Omega})$ of normal measures on $\tilde{\Omega}$.

Query Suppose that X is a compact space such that $C(X) = F''$ for some Banach space F . Does it follow that $X = \tilde{\Omega}$ for some locally compact space Ω ?

This is true if we know that F is a Banach lattice, for then $F = C(\Omega)$ for some Ω , and hence $X = \tilde{\Omega}$.

Bounded Borel functions

Let $B^b(\Omega)$ denote the C^* -algebra of bounded Borel functions on Ω .

For $\lambda \in B^b(\Omega)$, define $\kappa_E(\lambda)$ on $E' = M(\Omega)$ by

$$\langle \kappa_E(\lambda), \mu \rangle = \int_{\Omega} \lambda d\mu \quad (\mu \in M(\Omega)).$$

This extends the canonical embedding.

Thus we identify $B^b(\Omega)$ as a closed subalgebra of $C(\widetilde{\Omega})$. By Stone–Weierstrass, it does not separate the points of $\widetilde{\Omega}$; its character space is $\widetilde{\Omega}/\sim$, a quotient of $\widetilde{\Omega}$. (This space is totally disconnected, but usually not extremely disconnected.)

Submodules of $M(\Omega)$

The $\|\cdot\|$ -closed, $C(\Omega)$ -submodules of $M(\Omega)$ correspond to the clopen subspaces of $\widetilde{\Omega}$.

Thus $M(\Omega) = \ell^1(\Omega) \oplus M_c(\Omega)$ gives a partition

$$\{\beta\Omega_d, \widetilde{\Omega}_c\}$$

of $\widetilde{\Omega}$ into clopen subspaces.

Fix a $\mu \in M(\Omega)^+$. Then

$$\begin{array}{ccc} B^b(\Omega) & \xrightarrow{\kappa_E} & C(\widetilde{\Omega}) \\ q_\mu \downarrow & & \downarrow \rho_\mu \\ L^\infty(\Omega, \mu) & \xrightarrow{\mathcal{G}_\mu} & C(\Phi_\mu) \end{array}$$

is commutative, and $\kappa_E(B^b(\Omega)) \upharpoonright \Phi_\mu = C(\Phi_\mu)$.

Further,

$$M(\Omega) = \ell^1(\Omega) \oplus L^1(\Omega, \mu) \oplus M_s(\Omega, \mu)$$

gives a partition

$$\{\beta\Omega_d, \Phi_\mu, \Phi_{\mu,s}\}$$

of $\widetilde{\Omega}$ into clopen subspaces.

A representation of $M(\Omega)'$

Let $\mathcal{F} = \{\delta_x, \mu, \nu_i : x \in \Omega, i \in I\}$ be a maximal singular family of measures on Ω , where μ and ν_i are continuous. Write Φ_i for Φ_{ν_i} . Then our picture is:

Set $U_{\mathcal{F}} = \{\delta_x : x \in \Omega\} \cup \Phi_{\mu} \cup \cup\{\Phi_i : i \in I\}$. Then $U_{\mathcal{F}}$ is a dense, open subspace of the hyper-Stonian space $\widetilde{\Omega}$, and so $\beta U_{\mathcal{F}} = \widetilde{\Omega}$. Thus

$$M(\Omega)' = C(\beta\Omega_d) \oplus C(\Phi_{\mu}) \oplus \bigoplus_{\infty} \{C(\Phi_i) : i \in I\}.$$

Note that $B^b(\Omega)$ separates the points of $U_{\mathcal{F}}$.

Example I

Let (Ω, τ) be a countable, compact space. Then the hyper-Stonian cover $\widetilde{\Omega}$ is just $\beta\mathbb{N}$, with its usual topology, and $C(\Omega)'' = \ell^\infty(\Omega) = C(\beta\mathbb{N})$.

We cannot recover (Ω, τ) from $\widetilde{\Omega}$. Indeed, consider

$$\Omega = \omega + 1 \quad \text{or} \quad 2 \cdot \omega + 1 \quad \text{or} \quad \omega^\omega + 1,$$

where ω is the first infinite ordinal.

No two of these three spaces are mutually homeomorphic.

We do have $C(\omega + 1) = C(2 \cdot \omega + 1)$ as Banach spaces, but this space is not linearly homeomorphic to $C(\omega^\omega + 1)$.

Example II

Let (Ω, τ) be an uncountable, compact, metrizable space.

Then the hyper-Stonian cover $X = (\widetilde{\Omega}, \sigma)$ has the following properties:

(i) X is a hyper-Stonian space;

(ii) the set S of isolated points of X has cardinality \mathfrak{c} , the closure Y of S in X is a clopen subspace of X , and Y is homeomorphic to βS_d ;

(iii) $X \setminus Y$ contains a family of \mathfrak{c} pairwise disjoint, clopen subspaces, each homeomorphic to \mathbb{H} ;

(iv) the union $U_{\mathcal{F}}$ of the above sets is dense in $X \setminus Y$ and is such that $\beta U_{\mathcal{F}} = X \setminus Y$.

Further, any two spaces X_1 and X_2 satisfying the above properties are mutually homeomorphic.

Groups

Now let G be a compact (or locally compact) group, and let μ be Haar measure. Then $M(G)$ is a unital Banach algebra, called the **measure algebra** of G . The product is specified by

$$\langle \lambda, \mu \star \nu \rangle = \int_G \int_G \lambda(st) d\mu(s) d\nu(t) \quad (\lambda \in C_0(G)).$$

The group algebra $L^1(G)$ is a closed ideal in $M(G)$; write the character space of $L^\infty(G)$ just as Φ . Thus the second duals of $M(G)$ are

$$(M(\tilde{G}), \square) \quad \text{and} \quad (M(\tilde{G}), \diamond),$$

and they contain as respective closed ideals $(L^1(\Phi)'', \square)$ and $(L^1(\Phi)'', \diamond)$.

Aspiration To fully understand these Banach algebras.

The case where G is discrete has been dealt with in earlier talks; now we consider the case where G is non-discrete, but metrizable (e.g., $G = \mathbb{T}$, $G = \mathbb{Z}_2^{\aleph_0}$).

The recovery of G

Suppose that G and H are locally compact groups, and that $(M(\tilde{G}), \square)$ and $(M(\tilde{H}), \square)$ are isomorphic Banach algebras. Is it true that $G \sim H$, in the sense that there is a homeomorphic group isomorphism from G to H ?

As it stands, this is false: there are two finite groups G and H with $\ell^1(G)$ isomorphic to $\ell^1(H)$, but such that G and H are not isomorphic.

However if we insist that $(M(\tilde{G}), \square)$ and $(M(\tilde{H}), \square)$ are isometrically isomorphic as Banach algebras, then it does follow that $G \sim H$,

[Compact case first proved by Ghahramani and McClure. First step: Let $\theta : (M(\tilde{G}), \square) \rightarrow (M(\tilde{H}), \square)$ be an isomorphism. Then $\theta(m_G) = m_H$. Second step: if it is also an isometry, then it preserves pairwise singular measures, and so maps $L^1(G)$ onto $L^1(H)$. Third step: Wendel's theorem. The general locally compact case is longer.]

The structure of (\tilde{G}, \square)

Throughout G is locally compact and non-discrete.

Notation: Let $\varphi, \psi \in \tilde{G}$. Then $\delta_\varphi \square \delta_\psi \in M(\tilde{G})$, and we denote it by $\varphi \square \psi$. This does not imply that it is a point mass \tilde{G} ; we ask about this.

It is easy to see the following.

- 1) If $\varphi \in \beta G_d$ and $\psi \in \tilde{G}$, then $\varphi \square \psi \in \tilde{G}$.
- 2) If $\varphi \in \Phi$ and $\psi \in \tilde{G}$, then $\varphi \square \psi \in \Phi$, and $\varphi \square \psi = \psi$ when $\psi \in \Phi_{\{e\}}$, so that $\Phi_{\{e\}}$ is a left-zero semigroup (Lau-Medghalchi-Pym).

However:

Theorem There exist ('many') $\varphi \in \Phi_s$ and $\psi \in G_d^* \subset \beta G_d$ with $\varphi \square \psi \notin \tilde{G}$.

Method Show this for a few special groups such as \mathbb{T} , $\mathbb{Z}_2^{\mathbb{N}_0}$, and the p -adic integers Δ_p , and then use deep structure theorems.

The structure of (\tilde{G}, \square) -summary

The following table summarizes the inclusions that we have established.

\square	βG_d	Φ	Φ_s
βG_d	$\subset \beta G_d$	$\subset \Phi$	$\subset \Phi_s$
Φ	$\subset \Phi$	$\subset \Phi$	$\subset \Phi$
Φ_s	$\not\subset \tilde{G}$???	???

The topological centre of $L^1(G)$ - history

Take G infinite. It is well-known that $L^1(G)$ is strongly Arens irregular.

Civin and Yood 1961, not Arens regular when G is abelian; Young 1973: same for arbitrary locally compact G ; Ülger 1999 : elegant argument gives this more easily.

$L^1(G)$ is strongly Arens irregular for G abelian (M. Grosser and Losert 1984); for compact G (Işik, Pym, and Ülger 1987); all G (Lau and Losert 1988); shorter proof (Lau and Ülger 1996).

Neufang 2002 : New proof, showing that βG_d is determining for the topological centre.

D, Lau, Strauss 2005: for G discrete (and also for certain semigroups), there is a two-point set that is determining for the topological centre; see also Filali and Salmi.

The topological centre of $L^1(G)$

Theorem There is a subset V of the fibre $\Phi_{\{e\}}$ (plus two other points in the non-compact case) such that V is determining for the topological centre of $L^1(G)$.

Uses: Let $\nu \in M(G)$ be such that $\lambda \cdot \nu$ is continuous for each $\lambda \in L^\infty(G)$. Then we have $\nu \in L^1(G)$.

Query We can get $|V| = \mathfrak{c}$. How much smaller can it be? Finite?

The topological centre of $M(G)$

Conjecture of Ghahramani: $M(G)$ is always SAI.

Neufang (2005) has proved that $M(G)$ is SAI in the case where G is non-compact (and $|G|$ is non-measurable), but the analogous result is open in every case where G is compact (and infinite).

Partial solution Let G be a compact, infinite metrizable group. Then there are four points $\psi_1, \psi_2, \psi_3, \psi_4$ in the fibre $\tilde{G}_{\{e\}}$ with the property that the only measures M in $M(\tilde{G}_{\{e\}})^+$ such that $M \square \delta_{\psi_j} = M \diamond \delta_{\psi_j}$ for $j = 1, 2, 3, 4$ are in $\mathbb{C}\delta_e$.