

Amenability of semigroup algebras and their second duals

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Modules and derivations

Let A be an algebra, and let E be an A -bimodule. A **derivation** is a linear map $D : A \rightarrow E$ with

$$D(ab) = Da \cdot b + a \cdot Db \quad (a, b \in A).$$

For example, $\delta_x : a \rightarrow a \cdot x - x \cdot a$ is a derivation - called **inner**.

Let A be a Banach algebra, and let E be a Banach A -bimodule, so that

$$\|a \cdot x\|, \|x \cdot a\| \leq \|a\| \|x\| \quad (a \in A, x \in E).$$

The dual module is E' with module operations

$$\langle x, a \cdot \lambda \rangle = \langle x \cdot a, \lambda \rangle, \quad \langle x, \lambda \cdot a \rangle = \langle a \cdot x, \lambda \rangle.$$

A Banach algebra A is **amenable** if every continuous derivation $D : A \rightarrow E'$ is inner for each Banach A -bimodule E , and A is **weakly amenable** if each continuous derivation from A into the special Banach A -bimodule A' is inner.

Diagonals and approximate diagonals

Let A be an algebra with an identity e_A . Then $A \otimes A$ is a bimodule for the actions specified by

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca.$$

There is a linear map $\pi : a \otimes b \mapsto ab, A \otimes A \rightarrow A$. This is a module map.

A **diagonal** for A is an element $u \in A \otimes A$ such that $\pi(u) = e_A$ and

$$a \cdot u = u \cdot a \quad (a \in A).$$

Let A be a Banach algebra with an identity e_A . Then $A \hat{\otimes} A$ is a Banach A -bimodule in a similar way. The projective norm in $A \hat{\otimes} A$ is $\|\cdot\|_\pi$.

A **bounded approximate diagonal** for A is a bounded net $(u_\alpha) \in A \otimes A$ with $\pi(u_\alpha) = e_A$ and

$$\|a \cdot u_\alpha - u_\alpha \cdot a\|_\pi \rightarrow 0 \quad (a \in A).$$

C -amenability

Theorem (Barry Johnson) A unital Banach algebra is amenable if and only if A has a bounded approximate diagonal. In this case the **constant of amenability** is the minimum C such that there is a bounded approximate diagonal (u_α) such that $\sup \|u_\alpha\|_\pi \leq C$, and now A is **C -amenable**. \square

The constant of amenability for A is called $AM(A)$. Always $AM(A) \geq \|e_A\|$.

Example : an amenable C^* -algebra is 1-amenable.

Examples

Example 1 Let $A = (\ell^1(\mathbb{N}_n), \cdot)$. Then a diagonal is

$$\sum_{i=1}^n \delta_i \otimes \delta_i,$$

and $AM(A) = \|e_A\| = n$.

Example 2 For $n \in \mathbb{N}$, set $A = \mathbb{M}_n(\mathbb{C})$, with matrix units E_{ij} . Then a diagonal for A is

$$\Delta = \frac{1}{n} \sum_{i,j=1}^n E_{ij} \otimes E_{ji}.$$

[There are many more.]

Suppose that $\|(a_{ij})\| = \sum_{i,j} |a_{ij}|$. Then

$$AM(A) = \|\Delta\|_{\pi} = \|e_A\| = n.$$

Suppose that $A = \mathcal{B}(\ell_n^p)$. Then

$$AM(A) = \|\Delta\|_{\pi} = \|e_A\| = 1.$$

Group algebras

Let G be a locally compact group. Then $L^1(G)$ is the group algebra of G (with convolution product).

Theorem (Barry Johnson) Let G be a locally compact group.

(i) The following are equivalent:

(a) $L^1(G)$ is amenable;

(b) $L^1(G)$ is 1-amenable;

(c) G is amenable.

(ii) $L^1(G)$ is always weakly amenable. □

Semigroups

Banach algebra theory - a blend of algebra and analysis. We start with some algebraic background.

Semigroup - a non-empty set with an associative binary operation; p is an **idempotent** if $p^2 = p$; the set of these is $E(S)$.

Examples (1) any group G , so $E(S) = \{e_G\}$.

(2) $(\mathbb{N}, +)$ (no idempotents) and (\mathbb{N}, \cdot) (one idempotent).

(3) for $m, n \in \mathbb{N}$, set $m \vee n = \max\{m, n\}$ and $m \wedge n = \min\{m, n\}$; obtain $S = \mathbb{N}_\vee$ and $S = \mathbb{N}_\wedge$; in each case $E(S) = S$.

A **zero** is an element o with $so = os = o^2 = o$.

(4) For each semigroup S , we can adjoin a zero to give S^o .

Rees Semigroups with a Zero

These are semigroups $S^o = \mathcal{M}^o(G, P, m, n)$.

Here $m, n \in \mathbb{N}$, G is a group, and

$$P = (a_{ij}) \in \mathbb{M}_{n,m}(G^o).$$

For $x \in G$, $i \in \mathbb{N}_m$, and $j \in \mathbb{N}_n$, let $(x)_{ij}$ be the element of $\mathbb{M}_{m,n}(G^o)$ with x in the $(i, j)^{\text{th}}$ place and o elsewhere. S consists of the matrices $(x)_{ij}$. Multiplication in S is given by:

$$(x)_{ij}(y)_{kl} = (xa_jky)_{il}$$

for $x, y \in G$, $i, k \in \mathbb{N}_m$, $j, l \in \mathbb{N}_n$. Add o for S^o . The matrix P is the **sandwich** matrix.

The idempotents correspond to the non-zero elements of P . For example, consider

$$P_n = \begin{pmatrix} e & o & o & \cdots & o & o \\ e & e & o & \cdots & o & o \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ e & e & e & \cdots & e & o \\ e & e & e & \cdots & e & e \end{pmatrix}.$$

The number of idempotents is $(n^2 + 2n + 2)/2$.

The structure theorem of Rees - 1941

A semigroup S is **regular** if, for each $s \in S$, there exists $t \in S$ with $sts = s$ (Eg a group.)
An **ideal** I in S is such that $SI \cup IS \subset I$; a minimum ideal is $K(S)$.

Theorem of Rees Let S be an infinite, regular semigroup for which $E(S)$ is finite. Then $K(S)$ exists and S has a principal series

$$S = I_1 \supsetneq I_2 \supsetneq \cdots \supsetneq I_{m-1} \supsetneq I_m = K(S)$$

such that each quotient I_j/I_{j+1} is a Rees semigroup with a zero. Further, at least one of the quotients I_j/I_{j+1} is infinite. \square

Semigroup algebras

Definition Let S be a non-empty set. Then $\ell^1(S)$ is the usual Banach space with the norm

$$\left\| \sum_{s \in S} \alpha_s \delta_s \right\|_1 = \sum_{s \in S} |\alpha_s| < \infty.$$

Definition Let S be a semigroup. Set

$$\left(\sum \alpha_r \delta_r \right) \star \left(\sum \beta_s \delta_s \right) = \sum \left\{ \sum_{rs=t} \alpha_r \beta_s \delta_t : t \in S \right\}.$$

Then $A := (\ell^1(S), \star)$ is the **semigroup algebra** of S . This includes the case of the **group algebra** $\ell^1(G)$ for a group G .

Clearly A is a Banach algebra, containing $s \in S$, identified with $\delta_s \in A$.

Note: if S has an identity e , then so does A , and $\|\delta_e\|_1 = 1$. But A can easily have an identity e_A with norm $\neq 1$. Exercise: give me one with $\|e_A\|_1$ not a rational number.

Duals of semigroup algebras

The dual of $\ell^1(S)$ is $\ell^\infty(S)$. The dual module action of $s \in S$ on $\ell^\infty(S)$ is:

$$(s \cdot \lambda)(t) = \lambda(ts), \quad (\lambda \cdot s)(t) = \lambda(st)$$

for $t \in S$.

The space $\ell^\infty(S)$ is a commutative C^* -algebra for the pointwise operation. We identify the Stone-Cech compactification βS with the character space of $\ell^\infty(S)$, and $\ell^\infty(S)$ with $C(\beta S)$.

Clearly

$$\ell^1(S)'' = C(\beta S)' = M(\beta S),$$

the Banach space of complex-valued, regular Borel measures on βS with $\|\mu\| = |\mu|(\beta S)$.

Amenable semigroups

Definition Let $\mu \in M(\beta S)$. Then μ is a **mean** on S if

$$\|\mu\| = \langle \mu, 1 \rangle = 1,$$

and μ is **invariant** if

$$s \cdot \mu \cdot t = \mu \quad (s, t \in S).$$

The semigroup S is **amenable** if there is an invariant mean on S .

Remark: each abelian semigroup is amenable; a non-amenable group is \mathbb{F}_2 .

The cardinality of the set $\mathfrak{M}(S)$ of invariant means on S is usually large; it is 2^{2^κ} for a group of cardinality κ , but the cardinality can be 1 - for example for a group with zero adjoined.

The union of the supports of the left-invariant means is $L(\beta S)$; there are new results on $L(\beta S)$ in the memoir. **Question:** Is $L(\beta \mathbb{N})$ a closed set?

Amenability of $\ell^1(S)$

Let S be a semigroup. When is $\ell^1(S)$ amenable?

Preliminary intuition : $\ell^1(\mathbb{Z}) = A(\mathbb{T})$ is amenable, but $\ell^1(\mathbb{Z}^+) = A^+(\overline{\mathbb{D}})$ is not amenable - it is not even weakly amenable because it has non-zero point derivations. So $\ell^1(S)$ should not be amenable unless S is a group - or built up from groups in some simple way.

Several partial results were known. For example Duncan and Paterson 1990 use the existence of a bounded approximate diagonal to show the following:

Theorem Suppose that $A = \ell^1(S)$ is amenable. Then: $\ell^1(S)$ is unital; S is an amenable semigroup; S is regular; $E(S)$ is finite. \square

Further results of Esslamzadeh.

Side remark: $\|e_A\|_1$ is rational and $\text{supp } e_A$ is finite for A amenable.

Amenability of $\ell^1(S)$ - the solution

We now utilize the structure theorem of Rees, and the earlier notation. Recall that, for an infinite, regular semigroup S for which $E(S)$ is finite, S has a principal series

$$S = I_1 \supsetneq I_2 \supsetneq \cdots \supsetneq I_{m-1} \supsetneq I_m = K(S)$$

such that each quotient I_j/I_{j+1} is a Rees semigroup with a zero and $K(S)$ is a group.

Theorem Suppose that $\ell^1(S)$ is amenable as a Banach algebra. Then $K(S)$ is an amenable group, and each of the semigroups $\mathcal{M}^o(G, P, m, n)$ that arise in the principal series are such that $m = n$, G is an amenable group, and P is invertible when regarded as a matrix in $\mathbb{M}_n(\ell^1(G))$.

Further, the converse is true. □

The amenability constant of $\ell^1(S)$

Suppose that $\ell^1(S)$ is amenable, with the amenability constant C_S ; the number of idempotents is E_S . What are the possibilities for C_S ? What is its relationship to E_S ?

Theorem (extends Ross Stokke) Let S be a semigroup with $\ell^1(S)$ amenable. Then the following are equivalent:

- (a) $C_S = 1$;
- (b) $E_S = 1$;
- (c) $C_S < 5$;
- (d) S is a group.

Further 5 is the best-possible constant in (c).

So numbers in the range $(1, 5)$ are 'forbidden values' for C_S . Are there any more forbidden values?

More on C_S

Look at Rees matrices of order n .

For $n = 2$, there are only 2 amenable semi-group algebras (up to isometric isomorphism). For one (with P the identity), we have

$$\|e_{\ell^1(S)}\|_1 = E_S = 3 \quad \text{and} \quad C_S = 5.$$

For the other, we have

$$\|e_{\ell^1(S)}\|_1 = 3, \quad E_S = 4, \quad \text{and} \quad C_S = 11.$$

Let the sandwich matrix be the identity matrix, and $n \geq 2$. Then

$$\|e_{\ell^1(S)}\|_1 = 2n - 1 \quad E_S = n + 1, \quad \text{and} \quad C_S = 2n + 1.$$

In general,

$$\|e_{\ell^1(S)}\|_1 \geq 2n - 1 \quad \text{and} \quad C_S \geq 2n + 1.$$

Yet more on C_S

It was conjectured by Duncan and Paterson that always $E_S \leq C_S$. However by using P_n (given above) as a sandwich matrix in a Rees matrix of order n , we see that we can have $E_S \sim n^2$, but with $C_S \sim n$. There is a similar matrix Q_n that shows that we can have $E_S \sim n$, but with $C_S \sim n^2$. So there are no universal bounds relating C_S and E_S .

Inspection of the above numbers shows that numbers in $(5, 7)$ are forbidden values for C_S when S is a Rees matrix. For a more general result we need to solve the following.

Let J be an ideal in a semigroup S . Surely $C_J \leq C_S$ or even $C_J + C_{S/J} \leq C_S$. But we cannot prove this in general. (The obvious estimate gives

$$C_J \leq \|e_J\|_1 C_S,$$

where e_J is the identity of $\ell^1(J)$. I do have a poor bound on C_J that is independent of J .)

Stone-Cech compactifications of semigroups

Let S be a semigroup. Translations are given by $L_s : t \mapsto st$, $R_s : t \mapsto ts$, $S \rightarrow S$.

Now L_s has a continuous extension to a map $L_s : \beta S \rightarrow \beta S$. Define $s \square v = L_s(v)$ ($v \in \beta S$).

The map $R_v : s \mapsto s \square v$, $S \rightarrow \beta S$, has a continuous extension to $R_v : \beta S \rightarrow \beta S$. Define

$$u \square v = R_v(u) \quad (u, v \in \beta S).$$

Let $u, v \in \beta S$, and take nets (s_α) and (t_β) in S with $\lim_\alpha s_\alpha = u$ and $\lim_\beta t_\beta = v$. Then

$$u \square v = \lim_\alpha \lim_\beta s_\alpha t_\beta.$$

Similarly $u \diamond v = \lim_\beta \lim_\alpha s_\alpha t_\beta$.

Warnings: usually $u \square v \neq v \square u$, even if S is abelian; $(\beta G, \square)$ is never a group.

Compact topological semigroups

Theorem $(\beta S, \square)$ is a compact, right topological semigroup. Thus R_v is continuous for each $v \in \beta S$. However L_v is continuous for $v \in S$, and often **only** for $v \in S$. \square

Notation: $S^* = \beta S \setminus S$, the **growth** of S .

Main source of information: Hindman and Strauss; next lecture.

The product in $M(\beta S)$

For $\mu, \nu \in M(\beta S) = \ell^1(S)''$, we have definitions of $\mu \square \nu$ and $\mu \diamond \nu$.

Thus we have the Banach algebra $(M(\beta S), \square)$. It is easily checked that $\delta_u \square \delta_v = \delta_{u \square v}$ for each $u, v \in \beta S$, and so $(\beta S, \square)$ is a subsemigroup of $M(\beta S)$, and $\ell^1(\beta S)$ is a closed subalgebra of $M(\beta S)$.

In the case where S is weakly cancellative, we have

$$M(\beta S) = \ell^1(S) \rtimes M(S^*).$$

Many questions about these algebras are not resolved. For example, is $\ell^1(\beta S)$ semisimple? We know that the radical of $M(\beta S)$ is usually 'large', but cannot characterize it.

Weak amenability of $\ell^1(S)''$

Theorem Let S be an infinite, cancellative semigroup. Then there is a non-zero, continuous point derivation at the discrete augmentation character on $M(\beta S)$, and so $M(\beta S)$ is not weakly amenable. \square

So $M(\beta S)$ is somewhat like $M(G)$ for G locally compact; there is 'residual analytic structure'.

Let $S = \mathbb{N}_V$, as above. Then S is weakly cancellative, but $M(\beta S)$ is weakly amenable.

Idea of the proof There is a non-empty subset V of $S^* \setminus \overline{S_{[2]}^*}$ such that

$$s^{-1} \cdot V \cdot t^{-1} = V \quad (s, t \in S).$$

For $\mu \in M(\beta S)$, define

$$d(\mu) = \sum \{\mu(\{v\}) : v \in V\}$$

Then d is a non-zero, continuous point derivation at the discrete augmentation character of $M(\beta S)$. \square

The amenability of $M(\beta S)$

Suppose that $M(\beta S) = \ell^1(S)''$ is amenable. Then certainly $\ell^1(S)$ is amenable. So we can use the above theory. In fact it is fairly easy to reduce consideration to the case where $S = \mathcal{M}^o(G, P, n)$ is a Rees semigroup with a zero. But now, roughly,

$$\beta S = \mathcal{M}^o(\beta G, P, n)$$

and

$$M(\beta S) \cong \mathbb{M}_n(M(\beta G)).$$

Further, $M(\beta S)$ amenable implies that $M(\beta G)$ is amenable, and yet it is not even weakly amenable for G infinite. Thus:

Theorem Suppose that S is a semigroup such that $M(\beta S)$ is amenable. Then S is finite. \square