Homological properties of modules over group algebras

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Banach modules

Throughout A is a Banach algebra, and E is a Banach left A-module, so $E \in A\text{–}\mathbf{mod}$

(Similar notions for $E \in \mathbf{mod}\text{–}A$ and also for $E \in A\text{–}\mathbf{mod}\text{–}A$)

Example: $E = A$ is a Banach A-bimodule

Notation: E' is the dual space of E

Let $E \in A\text{–}\mathbf{mod}$. Then $E' \in \mathbf{mod}\text{–}A$ for

$$\langle x,\, \lambda \cdot a\rangle = \langle a \cdot x,\, \lambda\rangle \quad (a \in A,\, x \in E,\, \lambda \in E')$$

Morphisms

Let $E, F \in A\text{–}\mathbf{mod}$. Then ${}_A\mathcal{B}(E, F)$ is the closed linear subspace of $\mathcal{B}(E, F)$ consisting of the left A-module morphisms. So

$$T(a \cdot x) = a \cdot Tx \quad (a \in A,\, x \in E)$$

Projectivity

Let $P \in A\text{-}\mathbf{mod}$. The P is *projective* if, for each $E, F \in A\text{-}\mathbf{mod}$, for each admissible epimorphism $T \in {}_A\mathcal{B}(E, F)$, and for each $S \in {}_A\mathcal{B}(P, F)$, there exists $R \in {}_A\mathcal{B}(P, E)$ with $T \circ R = S$. The map R *lifts* S

Notation: $A^\flat$ is A with identity $e^\flat$ adjoined (even if A already has an identity)

Example: Set $P = A^\flat \widehat{\otimes} E$, with module operation specified by

$$a \cdot (b \otimes x) = ab \otimes x \quad (a \in A,\, b \in A^\flat,\, x \in E)\,,$$

so that $A^\flat \widehat{\otimes} E$ is the *free* Banach left A-module. Then P is a projective left A-module

We define $\pi \in {}_A\mathcal{B}(A^\flat \widehat{\otimes} E, E)$ to be such that

$$\pi(a \otimes x) = a \cdot x \quad (a \in A^\flat,\, x \in E)$$

Test: Let $E \in A\text{–}\mathbf{mod}$. Then E is projective if and only if there exists $\rho \in {}_A\mathcal{B}(E, A^\flat \widehat{\otimes} E)$ with $\pi \circ \rho = I_E$ (so that π is a *retraction*)

Eg: A unital. Then take $\rho(a) = a \otimes e_A$. So A is projective.

Injectivity

Let $J \in A\text{–}\mathbf{mod}$. Then J is *injective* if, for each $E, F \in A\text{–}\mathbf{mod}$, for each admissible monomorphism $T \in {}_A\mathcal{B}(E, F)$, and for each $S \in {}_A\mathcal{B}(E, J)$, there exists $R \in {}_A\mathcal{B}(F, J)$ with $R \circ T = S$.

Example: Let E be a Banach space. Set $J = \mathcal{B}(A^\flat, E)$, and define

$$(a \cdot T)(b) = T(ba), \quad (T \cdot a)(b) = T(ab)$$

for $a \in A$, $b \in A^\flat$, and $T \in J$. Then we have $J \in A\text{–}\mathbf{mod}\text{–}A$. In particular $J \in A\text{–}\mathbf{mod}$.

We say that $J = \mathcal{B}(A^\flat, E)$ is a *cofree* Banach left A-module. Then J is injective

Define $\Pi : E \to J = \mathcal{B}(A^\flat, E)$ by

$$\Pi(x)(a) = a \cdot x \quad (a \in A^\flat, \, x \in E)$$

Test: Let $E \in A\text{–}\mathbf{mod}$. Then E is injective if and only if there exists $\rho \in {}_A\mathcal{B}(\mathcal{B}(A^\flat, E), E)$ with $\rho \circ \Pi = I_E$ (so that Π is a *coretraction*)

Connection: Let $E \in A\text{--}\mathbf{mod}$, so that the dual $E' \in \mathbf{mod}\text{--}A$. Then the dual module of $A^\flat \widehat{\otimes} E$ is $\mathcal{B}(A^\flat, E')$ with the prescribed module operations, and the dual of $\pi \in \mathcal{B}(A^\flat \widehat{\otimes} E, E)$ is

$$\pi' = \Pi \in \mathcal{B}(E', \mathcal{B}(A^\flat, E'))\,.$$

It follows that the dual E' of a projective left A-module E is an injective right A-module.

Let $E \in A\text{--}\mathbf{mod}$. Then E is *flat* if E' is injective in $\mathbf{mod}\text{--}A$.

(Original definition different; we say *biflat* in the category $A\text{--}\mathbf{mod}\text{--}A$.)

Amenability

A Banach algebra A is *amenable* if any of the following equivalent conditions hold:

(i) every continuous derivation into each dual Banach A-bimodule is inner (BEJ)

(ii) $A^\flat$ is biflat (AH)

(iii) A has an approximate diagonal.

Fact A: Let A be an amenable Banach algebra, and let $E \in A\text{–}\mathbf{mod}$ or $E \in \mathbf{mod}\text{–}A$. Then E' is injective, equivalently E is flat.

Conjecture:The converse is true.

Later: converse in special cases.

Use of above notions of projectivity, injectivity, and flatness: structure of Banach algebras, Wedderburn theory, cohomology theory, tie in with amenability, which has huge importance

Relations

Let $E \in A\text{–}\mathbf{mod}$. What are relations between 'E is projective', 'E is injective', and 'E is flat'? Only relation:

$$E \text{ is projective} \Rightarrow E \text{ is flat}$$

Most places are easy to fill. Examples:

(1) $A = c_0$. Then A is projective, but not injective, in $A\text{–}\mathbf{mod}$

(2) $A = C([0,1])$, $E = L^2([0,1])$. Then E is injective and flat, but not projective, in $A\text{–}\mathbf{mod}$.

Injective, but not flat, is only non-trivial one.

Group algebras

Let G be a locally compact group, with left Haar measure m, and let $L^1(G)$ be the group algebra of G. This is the Banach algebra of all integrable functions on G, with the norm $\|\cdot\|_1$ specified by

$$\|f\|_1 = \int_G |f(t)| \, \mathrm{d}m(t) \quad (f \in L^1(G))$$

and equipped with the convolution product $\star$, where

$$(f \star g)(t) = \int_G f(s) g(s^{-1}t) \, \mathrm{d}m(s) \quad (t \in G)$$

for $f, g \in L^1(G)$.

Think of the case where G is discrete and the algebra is $\ell^1(G)$, with product defined by

$$\delta_s \star \delta_t = \delta_{st} \quad (s, t \in G)$$

Or : $G = \mathbb{R}$ with Lebesgue measure

Amenability of group algebras

There is always one character on the algebra $L^1(G)$; this is the *augmentation character* φ_G, defined by

$$\varphi_G : f \mapsto \int_G f(t)\,\mathrm{d}m(t), \quad L^1(G) \to \mathbb{C}\,.$$

The space $\mathbb{C}$ is an $L^1(G)$-bimodule for the operation given by

$$f \cdot z = z \cdot f = \varphi_G(f)z \quad (z \in \mathbb{C},\, f \in L^1(G))\,;$$

as such it is denoted by $\mathbb{C}_{\varphi_G}$.

Theorem (Johnson) Let G be a locally compact group. Then the following are equivalent:

(a) the Banach algebra $L^1(G)$ is amenable;

(b) the locally compact group G is amenable;

(c) the module $\mathbb{C}_{\varphi_G}$ is flat in $L^1(G)$–**mod**.

Modules over $L^1(G)$

Set $A = L^1(G)$. Can take the following in the category A–**mod**:

- $E = A$

- $E = L^p(G)$ for $1 < p < \infty$ and convolution product, so that E is a dual module

- $E = A' = L^\infty(G)$, with dual module operation given by

$$(f \cdot \lambda)(t) = \int_G f(s)\lambda(ts)\,\mathrm{d}m(s)$$

- $E = C_0(G)$, a closed submodule of A'

- $E = M(G)$, the measure algebra on G, with product

$$(\mu \star \nu)(B) = \int_G \nu(s^{-1}B)\,\mathrm{d}\mu(s)$$

for each Borel subset B of G, so that A is a closed ideal in $M(G)$, and $M(G) = C_0(G)'$ in A–**mod**–A.

Game to play

Set $A = L^1(G)$, and let $E \in A-\mathbf{mod}$. Give necessary and sufficient conditions on G for E to be projective, injective, or flat.

Some easy or known; some obtained with a struggle; some only partially resolved. I will give some samples, and then summarize the results known.

Case 1 $E = A$ is always projective (Helemskii).

Note that $A \widehat{\otimes} A = L^1(G \times G)$. Take compact K in G with $m(K) = 1$, and define a map $\rho \in {}_A\mathcal{B}(A, A \widehat{\otimes} A)$ by

$$\rho(f)(s,t) = \chi_K(t^{-1}) f(st)$$

for $f \in A$ and $s, t \in G$. Then $\pi \circ \rho = I_A$.

Case 2 $E = M(G)$

Suppose G is amenable. Then $M(G) = C_0(G)'$ is flat by Fact A.

Suppose G is discrete. Then $M(G) = \ell^1(G) = A$ is projective because A is unital, and A is flat by the previous case.

If G is not discrete, $M(G)/A$ is sufficiently 'big' for an argument to show that $M(G)$ is not projective. Hence $M(G)$ is projective if and only if G is discrete.

We **guess** that $M(G)$ is always flat.

Case 3 $E = L^\infty(G)$

Can use the following theorem of Lau and Losert, generalizing Phillips's Lemma.

Theorem For each infinite locally compact group G, the space $C_0(G)$ is not complemented in $L^\infty(G)$.

Theorem Suppose that $L^\infty(G)$ is projective in A–$\mathbf{mod}$. Then G is finite.

Use the fact that π is a retraction to show that $C_0(G)$ is complemented in $L^\infty(G)$.

Theorem Suppose that $L^\infty(G)$ is flat in A–$\mathbf{mod}$. Then G is discrete and contains no infinite, amenable subgroup.

Guess: In fact G must be finite, but Olshanskii's groups (infinite, non-amenable, no finite subgroups) also satisfy the condition.

Case 4 $E = C_0(G)$

Theorem The module $C_0(G)$ is projective in A–**mod** if and only if G is compact.

Note that $A \widehat{\otimes} E = L^1(G, E)$. If G is compact, define $\rho \in {}_A\mathcal{B}(E, L^1(G, E))$ by

$$\rho(\lambda)(s)(t) = \lambda(ts^{-1}) \quad (s, t \in G)$$

for $\lambda \in E$.

If G is not compact, use functions $f \in A$ with $|f|_G = 1$ and $\|f\|_1 = n$ to get a contradiction from the existence of ρ.

Theorem The module $C_0(G)$ is injective in A–**mod** if and only if G is finite.

If G is infinite and $C_0(G)$ is injective, work quite hard to get a map that contradicts Phillips's Lemma

Amenability and injectivity

Definition
Let E be a Banach left $L^1(G)$-module. An element $\lambda \in E'$ is an *augmentation-invariant* functional if

$$\langle f \cdot x, \lambda \rangle = \varphi_G(f) \langle x, \lambda \rangle \quad (f \in L^1(G),\, x \in E)\,.$$

The module E is *augmentation-invariant* if there is a non-zero, augmentation-invariant functional.

Examples (i) $E = L^\infty(G)$ is augmentation-invariant if and only if G is amenable.

(ii) $M(G)$ and $L^\infty(G)'$ are both augmentation-invariant.

The next result is fairly substantial. It uses *Reiter's condition*:

A locally compact group G is amenable if and only if there is a net (h_α) in

$$P(G) := \{f \in L^1(G) : f \geq 0,\, \|f\|_1 = 1\}$$

such that $\lim_\alpha \|L_s h_\alpha - h_\alpha\|_1 = 0$ for each $s \in G$.

Theorem Let E be the dual of a Banach right $L^1(G)$-module. Suppose that E is faithful and augmentation-invariant. Then E is injective if and only if G is amenable.

This gives us:

Theorem Let G be a locally compact group. Then the following are equivalent:

(a) the group G is amenable;

(b) $M(G)$ is injective as a Banach left $L^1(G)$-module;

(c) $L^\infty(G)$ is flat as a Banach left $L^1(G)$-module;

(d) $C_0(G)$ is flat as a Banach left $L^1(G)$-module.

The modules $L^p(G)$, $1 < p < \infty$

The following is similar to earlier results.

Theorem $L^p(G)$ is projective in A–$\mathbf{mod}$ if and only if G is compact.

But when is $L^p(G)$ injective? Certainly this holds whenever G is amenable by Fact A.

Guess: the converse is true.

We can only make progress in the case where G is discrete, so suppose that this holds.

First recall that G is amenable if and only if it satisfies *Følner's condition*: for each $\varepsilon > 0$ and each finite subset F of G, there is a non-empty, finite subset S of G such that

$$|Sx \Delta Sy| < \varepsilon\,|S| \quad (x, y \in F)\,.$$

Let S be a set, and let $n \in \mathbb{N}$. Then $\mathcal{P}_n(S)$ is the family of all subsets of S with cardinality n.

Definition A group G is *pseudo-amenable* if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for each $n \geq n_0$ and each $F \in \mathcal{P}_n(G)$, there exists a non-empty, finite subset S of G such that

$$|SF| < \varepsilon n \, |S|$$

(1) an amenable group is pseudo-amenable;

(2) a group containing $\mathbb{F}_2$ as a subgroup is not pseudo-amenable.

We hope, but do not know, that a pseudo-amenable group is always amenable. We do not know if Olshanskii's groups are pseudo-amenable.

Our theorem is:

Theorem Let G be a group, and take p with $1 < p < \infty$. Suppose that the Banach left $\ell^1(G)$-module $\ell^p(G)$ is injective. Then the group G is pseudo-amenable.

	projective	injective
$L^1(G)$	all G	G discrete and amenable
$C_0(G)$	G compact	G finite
$L^\infty(G)$	G finite	all G
$M(G)$	G discrete	G amenable
$M(G)'$		G amenable $\Rightarrow$ G discrete $\Rightarrow$
$L^1(G)''$	$\Rightarrow$ G discrete and no infinite amenable subgroup	G amenable
$L^p(G)$	G compact	G amenable $\Rightarrow$
$\ell^p(G)$	G finite	$\Rightarrow$ G pseudo-amenable