Basic cohomology

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This is a review of some basic cohomology, all of it now standard. In particular, see:

(1) introductory text ‘Introduction to Banach algebras, harmonic analysis, and operators’ LMS/CUP Student Text;

(2) my book [D] - comprehensive account;

(3) books of Helemskii/Runde;

(4) algebraic account of Weibel;

(5) seminal papers of Johnson on cohomology;

(6) for C^*-algebras, LMS Lecture Notes of Sinclair and Smith.

[Detailed references in my book; they are casual here.]

I begin by following the classical account of Hochschild.

Let A be an algebra (over $\mathbb{C}$), and let E be an A-bimodule. See [D, §1.9].

Examples: take $E = A$ or an ideal in A or a superalgebra of A, with the obvious module operations.

For $n \in \mathbb{Z}^+$, denote by $\mathcal{L}^n(A, E)$ the linear space of n-linear maps from $A^{(n)} = A \times \cdots \times A$ to E, with $\mathcal{L}^0(A, E) = E$.

For $x \in E$, define

$$\delta_x(a) = a \cdot x - x \cdot a \quad (a \in A)\,,$$

so that $\delta_x \in \mathcal{L}^1(A, E)$ and

$$\delta^0 : x \mapsto \delta_x, \quad \mathcal{L}^0(A, E) \to \mathcal{L}^1(A, E),$$

is linear.

For $T \in \mathcal{L}^1(A,E)$, define

$$(\delta^1 T)(a,b) = a \cdot Tb - T(ab) + Ta \cdot b \quad (a,b \in A)\,,$$

so that $\delta^1 T \in \mathcal{L}^2(A,E)$, and the map

$$\delta^1 : T \mapsto \delta^1 T, \quad \mathcal{L}^1(A,E) \to \mathcal{L}^2(A,E)\,,$$

is linear.

For $T \in \mathcal{L}^2(A,E)$, define

$$(\delta^2 T)(a,b,c) = a \cdot T(b,c) - T(ab,c) + T(a,bc) - T(a,b) \cdot c$$

so that $\delta^2 T \in \mathcal{L}^3(A,E)$, and the map

$$\delta^2 : T \mapsto \delta^2 T, \quad \mathcal{L}^2(A,E) \to \mathcal{L}^3(A,E)\,,$$

is linear.

General case: for $T \in \mathcal{L}^n(A,E)$, define

$$
\begin{aligned}
&\mathcal{L}^\bullet(A,E): \\
&\quad (\delta^n T)(a_1,\dots,a_{n+1}) \\
&\qquad = a_1 \cdot T(a_2,\dots,a_{n+1}) \\
&\qquad\quad + \sum_{j=1}^{n} (-1)^j T(a_1,\dots,a_{j-1},a_j a_{j+1},\dots,a_{n+1}) \\
&\qquad\quad + (-1)^{n+1} T(a_1,\dots,a_n) \cdot a_{n+1}\,.
\end{aligned}
$$

Then we get a complex (not necessarily exact)

$$
\begin{aligned}
0 \to E &\xrightarrow{\delta^0} \mathcal{L}(A,E) \xrightarrow{\delta^1} \mathcal{L}^2(A,E) \xrightarrow{\delta^2} \cdots \\
&\xrightarrow{\delta^{n-1}} \mathcal{L}^n(A,E) \xrightarrow{\delta^n} \mathcal{L}^{n+1}(A,E) \xrightarrow{\delta^{n+1}} \cdots
\end{aligned}
$$

Define

$$Z^n(A,E) = \ker \delta^n \quad \text{and} \quad N^n(A,E) = \operatorname{im} \delta^{n-1}\,,$$

so that $N^n(A,E)$ is a subspace of $Z^n(A,E)$. These are the *cocycles* and *coboundaries*, respectively. Then define

$$H^n(A,E) = Z^n(A,E)/N^n(A,E)\,.$$

This is the $n^{\text{th.}}$*-cohomology group of A with coefficients in E.*

A *derivation* from A to E is a map $D \in \mathcal{L}(A,E)$ such that

$$D(ab) = a \cdot Db + Da \cdot b \quad (a,b \in A)\,.$$

Derivations of the form δ_x are the *inner* derivations (and the others are *outer*).

Thus $H^1(A,E) = \{0\}$ iff every derivation from A into E is inner.

Let A be a unital algebra with identity e_A. A *diagonal* for A is an element $u \in A \otimes A$ such that $\pi_A(u) = e_A$ and $a \cdot u = u \cdot a$ $(a \in A)$. Here $\pi_A : A \otimes A \to A$ is the linear map such that $\pi_A(a \otimes b) = ab$ $(a, b \in A)$.

Example: let $A = \mathbb{M}_n(\mathbb{C})$. Then a diagonal in $A \otimes A$ is

$$d = \frac{1}{n} \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji}\,.$$

Theorem [D, 1.9.21] Let A be an algebra. Then the following are equivalent:

(a) $H^1(A, E) = \{0\}$ for every A-bimodule E;

(b) A is unital and has a diagonal;

(c) A is semisimple and finite-dimensional;

(d) $A = \mathbb{M}_n(\mathbb{C})$ for some $n \in \mathbb{N}$.

Let $T \in \mathcal{L}^2(A, E)$. Then $T \in Z^2(A, E)$ iff

$$a \cdot T(b, c) - T(ab, c) + T(a, bc) - T(a, b) \cdot c = 0 \quad (a, b, c \in A).$$

This is the *cocycle identity*.

Example: $A \otimes A$ is an A-bimodule for operations that satisfy

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in A).$$

The map

$$(a, b) \mapsto a \otimes b, \quad A \times A \to A \otimes A,$$

belonds to $Z^2(A, A \otimes A)$. In the case where A is unital, it is a coboundary because $T = \delta^1 S$, where

$$S : a \mapsto \frac{1}{2}(a \otimes e_A + e_A \otimes a).$$

Let A be an algebra. An *extension A by I* is a short exact sequence

$$0 \to I \xrightarrow{\iota} \mathfrak{A} \xrightarrow{q} A \to 0\,,$$

where I is an ideal in the algebra $\mathfrak{A}$ and ι and q are homomorphisms. The extension is *singular* if $I^2 = 0$. It *splits* if there is a homomorphism $\theta : A \to \mathfrak{A}$ such that $q \circ \theta$ is the identity on A; such a map is a *splitting homomorphism*.

Take $T \in \mathcal{L}^2(A, E)$, and set $\mathfrak{A}_T = A \oplus E$ with the product

$$(a, x)(b, y) = (ab, a \cdot y + x \cdot b + T(a, b))\,.$$

Theorem [D, 1.9.5] The map $T \mapsto \mathfrak{A}_T$ induces a bijection from $H^2(A, E)$ onto the family of equivalence classes of singular extensions of A by E.

Corollary $H^2(A, E) = \{0\}$ iff each singular extension of A by E splits.

Let A be an algebra, and let E be an A-bimodule. There is a way of making $\mathcal{L}(A,E)$ into an A-bimodule. Indeed take $a \in A$ and $T \in \mathcal{L}(A,E)$. Then

$$(a \cdot T)(b) = a \cdot Tb, \quad (T \cdot a)(b) = T(ab) - Ta \cdot b$$

for each $b \in A$.

Continue to make $\mathcal{L}^n(A,E)$ into an A-bimodule. Then the basic *reduction of dimension theorem* [D, 1.9.10] is:

Theorem For $k, p \in \mathbb{N}$, we have

$$H^{k+p}(A,E) = H^k(A, \mathcal{L}^p(A,E)) .$$

Corollary Suppose that $H^k(A,E) = \{0\}$ for each E. Then $H^{k+p}(A,F) = \{0\}$ for each F and each $p \in \mathbb{N}$.

Let A be an algebra, and let E be an A-bimodule. Define

$$\mathcal{L}_n(A,E) = A \otimes \cdots \otimes A \otimes E\,.$$

We make each $\mathcal{L}_n(A,E)$ into an A-bimodule, and obtain another complex

$\mathcal{L}_\bullet(A,E)$:

$$0 \to E \xleftarrow{d_0} \mathcal{L}_1(A,E) \xleftarrow{d_1} \mathcal{L}_2(A,E) \xleftarrow{d_2} \cdots$$
$$\xleftarrow{d_{n-1}} \mathcal{L}_n(A,E) \xleftarrow{d_n} \mathcal{L}_{n+1}(A,E) \xleftarrow{d_{n+1}} \cdots$$

Next define

$$H_n(A,E) = \ker d_{n-1}/\operatorname{im} d_n\,.$$

These are the *homology groups of A with coefficients in E*.

Two facts [D, 1.9.18]:

(1) the complex $\mathcal{L}^\bullet(A,E)$ is the dual of the complex $\mathcal{L}_\bullet(A,E)$;

(2) $H^{n+1}(A,E^\times) = H^1(A,\mathcal{L}_n(A,E)^\times)$ as linear spaces.

Now let A be a Banach algebra, and E be a Banach A-bimodule. (See [D, §2.6].) We may suppose that

$$\|a \cdot x\| \le \|a\| \, \|x\| \, , \quad \|x \cdot a\| \le \|a\| \, \|x\|$$

for each $a \in A$ and $x \in E$.

Copy all of the above, but replace $\mathcal{L}^n(A, E)$ by $\mathcal{B}^n(A, E)$, the Banach space of bounded elements in $\mathcal{L}^n(A, E)$, and note that the maps δ^n are all continuous. We obtain a complex called $\mathcal{B}^\bullet(A, E)$.

We now replace $Z^n(A, E)$ and $N^n(A, E)$ by

$$\mathcal{Z}^n(A, E) \quad \text{and} \quad \mathcal{N}^n(A, E) \, ,$$

respectively. These are the *continuous cocycles* and *continuous coboundaries*, respectively. Note that $\mathcal{Z}^n(A, E)$ is always a closed subspace of $\mathcal{B}^n(A, E)$, but that $\mathcal{N}^n(A, E)$ is not necessarily closed. Then define

$$\mathcal{H}^n(A, E) = \mathcal{Z}^n(A, E)/\mathcal{N}^n(A, E) \, .$$

This is the nth. *Banach cohomology group of* A *with coefficients in* E.

We again have a *reduction of dimension theorem*:

Theorem For $k, p \in \mathbb{N}$, we have

$$\mathcal{H}^{k+p}(A, E) = \mathcal{H}^{k}(A, \mathcal{B}^{p}(A, E))\,.$$

Corollary Suppose that $\mathcal{H}^{k}(A, E) = \{0\}$ for each E. Then $\mathcal{H}^{k+p}(A, F) = \{0\}$ for each F and each $p \in \mathbb{N}$.

Nearly as before: $\mathcal{H}^{2}(A, E)$ tells us about splitting and Wedderburn decompositions of Banach algebras. (For quite a lot of calculations of $\mathcal{H}^{2}(A, E)$, and a fair number of open problems, see the memoir of Bade-Dales-Lykova.)

Triviality is nearly the same as before:

Let A be a unital Banach algebra. A *projective diagonal* for A is an element $u \in A \widehat{\otimes} A$ such that $\pi_A(u) = e_A$ and $a \cdot u = u \cdot a$ $(a \in A)$. Here $\pi_A : A \widehat{\otimes} A \to A$ is the continuous linear map such that $\pi_A(a \otimes b) = ab$ $(a, b \in A)$.

Theorem [D, 2.8.48] Let A be an algebra. Consider:

(a) $\mathcal{H}^1(A, E) = \{0\}$ for every Banach A-bimodule E (A is *contractible*);

(b) A is unital and has a projective diagonal;

(c) A is unital and biprojective (see later);

(d) A is semisimple and finite-dimensional.

Then (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d), and all are equivalent if A has CAP as a Banach space.

The contractible Banach algebras are too narrow a class.

Let A be a Banach algebra, and let E be a Banach A-bimodule. Define

$$\mathcal{B}_n(A,E) = A \,\widehat{\otimes}\, \cdots \,\widehat{\otimes} A \,\widehat{\otimes}\, E\,.$$

Then the complex $\mathcal{B}^\bullet(A,E')$ is the (continuous) dual of the complex $\mathcal{B}_\bullet(A,E)$, and we have:

Theorem For $k,p \in \mathbb{N}$, we have

$$\mathcal{H}^{k+p}(A,E') = \mathcal{H}^k(A,\mathcal{B}^p(A,E)')\,.$$

Corollary Suppose that $\mathcal{H}^k(A,E') = \{0\}$ for each E. Then $\mathcal{H}^{k+p}(A,F') = \{0\}$ for each F and each $p \in \mathbb{N}$.

Definition Let A be a Banach algebra. The *weak bidimension* of A, called $\mathrm{db}_w A$, is

$$\min\{n \in \mathbb{Z}^+ : \mathcal{H}^{n+1}(A,E') = \{0\} \text{ for each } E\}\,.$$

The algebra A is *amenable* if $\mathrm{db}_w A = 0$.

Thus A is amenable iff each continuous derivation into a dual Banach A-bimodule is inner.

There is an enormous study of amenability for Banach algebras.

Related concepts:

(1) A is *weakly amenable* if $\mathcal{H}^1(A, A') = \{0\}$;

(2) A is *simplicially trivial* if $\mathcal{H}^n(A, A') = \{0\}$ for each $n \in \mathbb{N}$.

There are many other related concepts (e.g., cyclic cohomology, Connes amenability,...), often given in lectures here.

Proposition Let I be a closed ideal in A.

(i) I and A/I amenable implies A amenable.

(ii) I and A/I weakly amenable implies A weakly amenable.

(iii) Suppose that A is amenable. Then A has a bai, and so A factors, etc.

(iv) Suppose that A is amenable. Then the following are equivalent: (a) I has a bai; (b) I is weakly complemented in A; (c) I is amenable.

Proposition Let A and B be amenable Banach algebras. Then $A \widehat{\otimes} B$ is amenable.

Definition Let A be a Banach algebra. An *approximate diagonal* for A is a bounded net (u_α) in $A \widehat{\otimes} A$ such that, for each $a \in A$, we have

$$\lim_\alpha (u_\alpha \cdot a - a \cdot u_\alpha) = 0, \quad \lim_\alpha \pi_A(u_\alpha)a = a\,.$$

Theorem (Johnson) A Banach algebra A is amenable iff it has an approximate diagonal.

This is often used to show that a particular Banach algebra is amenable.

There are many other characterizations of amenability, perhaps involving other terms from homological algebra.

Theorem (Helemskii) A Banach algebra A is amenable iff it has a bai and is biflat.

When are various Banach algebras is special classes (weakly) amenable?

Theorem (Johnson) The group algebra $L^1(G)$ is amenable iff G is amenable as a locally compact group iff $L^1_0(G)$ has a bai; $L^1(G)$ is always weakly amenable.

Theorem (Dales, Ghahramani, Helemskii) The measure algebra $M(G)$ is amenable iff G is discrete and amenable, and weakly amenable iff G is discrete.

Theorem (Connes, Haagerup, etc) A C^*-algebra is amenable iff it is nuclear iff.... In particular, $\mathcal{B}(\ell^2)$ is not amenable. Each commutative C^*-algebra is amenable. Each C^*-algebra is weakly amenable.

Theorem (Gronbaek) The weighted convolution algebra $\ell^1(\mathbb{Z}, \omega)$ is weakly amenable iff $\inf_{n \in \mathbb{Z}} \omega_n \omega_{-n}/n = 0$.

Theorem (Read) There is a commutative, radical, amenable Banach algebra.

Theorem (Read) $\mathcal{B}(\ell^1)$ is not amenable.

What about $\mathcal{B}(\ell^p)$ for $1 < p < 2$?
Conjecture: $\mathcal{B}(E)$ is only amenable when E is finite-dimensional.

Theorem (Gronbaek, Johnson, Willis) Let E be a Banach space with property $(\mathbb{A})$ (e.g., L^p for $1 \leq p \leq \infty$). Then $\mathcal{K}(E)$ is amenable. But $\mathcal{K}(\ell^r \oplus \ell^s)$ is not amenable whenever $r, s \in (1,2) \cup (2, \infty)$ and $r \neq s$.

Theorem (Blanco, Groenbaek) Conditions for $A(E)$ to be weakly amenable.