Weighted convolution algebras on the rationals

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A second dual algebra

Let A be a commutative Banach algebra.

Then A'' is a Banach A-module for map $(a, \Phi) \mapsto a \cdot \Phi$ from $A \times A''$ to A''.

There is a product $\Box$ on A'' of A extending the module map. For $a, b \in A$, $\lambda \in A'$, and $\Phi \in A''$, define:

$$\langle b,\, a \,\cdot\, \lambda\rangle = \langle ba,\, \lambda\rangle, \quad \langle a,\, \lambda \cdot \Phi\rangle = \langle \Phi,\, a \,\cdot\, \lambda\rangle\,.$$

Let $\Phi, \Psi \in A''$. Then

$$\langle \Phi \Box \Psi,\, \lambda\rangle = \langle \Phi,\, \Psi \,\cdot\, \lambda\rangle \quad (\lambda \in A')\,.$$

Suppose that $\Phi = \lim_\alpha a_\alpha$ and $\Psi = \lim_\beta b_\beta$ for nets (a_α) and (b_β) in A. Then

$$\Phi \Box \Psi = \lim_\alpha \lim_\beta a_\alpha b_\beta\,.$$

The centre

Definition The **(topological) centre** is
$$\mathfrak{Z}(A'') = \left\{ \Phi \in A'' : \Phi \,\Box\, \Psi = \Psi \,\Box\, \Phi \;\; (\Psi \in A'') \right\} .$$

The algebra A is **Arens regular** if $\mathfrak{Z}(A'') = A''$ and **strongly Arens irregular = SAI** if $\mathfrak{Z}(A'') = A$.

Fact (Pym) A is Arens regular if and only if, for each pair $\{(a_m), (b_n)\}$ of bounded sequences in A and each $\lambda \in A'$, the two repeated limits

$$\lim_{m\to\infty} \lim_{n\to\infty} \langle a_m b_n, \lambda\rangle \quad \text{and} \quad \lim_{n\to\infty} \lim_{m\to\infty} \langle a_m b_n, \lambda\rangle$$

are equal whenever they both exist.

A subset V of A'' is **determining for the topological centre** if $\Phi \in A$ whenever $\Phi \,\Box\, \Psi = \Psi \,\Box\, \Phi$ for all $\Psi \in V$.

Similar definitions, involving the left and right topological centres arise when A is not necessarily commutative.

Weighted ℓ^1-spaces

Let $\omega : S \to (0, \infty)$ be a function. Then $A_\omega = \ell^1(S, \omega)$ with

$$\|f\|_\omega = \sum_{s \in S} |f(s)| \, \omega(s) < \infty$$

Clearly $(\ell^1(S, \omega), \|\cdot\|_\omega)$ is a Banach space. The dual is $A'_\omega = \ell^\infty(S, 1/\omega)$, with the norm also denoted by $\|\cdot\|_\omega$, so that

$$\|\lambda\|_\omega = \sup\{|\lambda(s)| \, / \omega(s) : s \in S\} \, .$$

Set $E_\omega := c_0(S, 1/\omega)$. As a Banach space,

$$A''_\omega = A_\omega \oplus E^\circ_\omega \, .$$

For $s \in S$, the *normalized point mass* at s is $\widetilde{\delta}_s = \delta_s / \omega(s)$.

There is an obvious map

$$\theta_\omega : f \mapsto f/\omega \, , \quad \ell^1(S) \to A_\omega \, .$$

Recall that $\ell^1(S)' = \ell^\infty(S) = C(\beta S)$ and that $\ell^1(S)'' = M(\beta S)$.

Weighted semigroup algebras

Let S be an abelian semigroup. A **weight** on S is a map $\omega : S \to (0, \infty)$ such that

$$\omega(s+t) \leq \omega(s)\omega(t) \quad (s, t \in S)\,.$$

Then $(A_\omega, \|\cdot\|_\omega)$ is a commutative Banach algebra.

Set $\mathbb{Q}^{+\bullet} = \{p/q : p, q \in \mathbb{N},$ coprime$\}$.

On $\mathbb{Q}^{+\bullet}$, etc., we have we have

$$(A''_\omega, \Box) = A_\omega \ltimes E^\circ_\omega\,.$$

The character space of $\ell^1(S)$ is Φ_S, identified with the space of semi-characters, i.e., non-zero homomorphisms $\varphi : S \to (\overline{\mathbb{D}}, \,\cdot\,)$.

We say that ω is semisimple/Arens regular/SAI, etc. if A_ω has the corresponding property.

Define $\quad \nu_s = \inf\{\omega(s^n)^{1/n} : n \in \mathbb{N}\}$ $(s \in S)$.

Elementary facts

1) S abelian. Then ω is semisimple iff, for each $s \neq t$, there exists $\varphi \in \Phi_S$ with $\varphi(s) \neq \varphi(t)$, and $\nu_s > 0$ for all $s \in S$.

2) For $S = \mathbb{Q}^{+\bullet}$, ω is semisimple iff

$$\nu_s = \inf\{\omega(t)^{1/t} : t \in \mathbb{Q}^{+\bullet}\} \quad (s \in \mathbb{Q}^{+\bullet})\,.$$

Each weight on $\mathbb{Q}^{+\bullet}$ is radical or semisimple.

3) Let ω be a weight on $\mathbb{Q}^{+\bullet}$. Then the following are equivalent:

(a) there exist $a, b \in \mathbb{Q}^{+\bullet}$ with $a < b$ such that $\inf\{\omega(s) : s \in \mathbb{Q}^{+\bullet} \cap (a, b)\} > 0$;

(b) there exists $c \in \mathbb{Q}^{+\bullet}$ such that $\inf\{\omega(s) : s \in \mathbb{Q}^{+\bullet} \cap (0, c)\} > 0$;

(c) $\liminf_{s \to 0+} \omega(s) \geq 1$.

Are there any weights on $\mathbb{Q}^{+\bullet}$ which fail these conditions?

Notation

For a weight ω on a (non-abelian) S, set

$$\Omega(s,t) = \frac{\omega(st)}{\omega(s)\omega(t)} \quad (s,t \in S)\,.$$

Thus $0 < \Omega(s,t) \le 1 \;\; (s,t \in S)$.

Let $u,v \in \beta S$, say $u = \lim_\alpha s_\alpha$ and $v = \lim_\beta t_\beta$ (weak-$*$). Then define

$$\Omega(u,v) = \lim_\alpha \lim_\beta \Omega(s_\alpha, t_\beta)\,.$$

For $\ell^1(\mathbb{Z})$, $\Omega(u,v) = 1$ for $u,v \in \beta\mathbb{Z}$ and $\ell^1(\mathbb{Z})$ is strongly Arens irregular. But take the weight to be

$$\omega_\alpha(n) = (1+|n|)^\alpha \quad (n \in \mathbb{Z}),$$

where $\alpha > 0$. Then $\Omega(u,v) = 0 \;\; (u,v \in \beta\mathbb{Z} \setminus \mathbb{Z})$ and $\ell^1(\mathbb{Z}, \omega_\alpha)$ is Arens regular.

Weakly diagonally bounded weights

Let ω be a weight on a semigroup S, and take $T \subset S$.

Then ω is **DB = diagonally bounded** on T if

$$\Omega(s,t) \geq 1/d_T > 0 \quad (s \in S, t \in T)\,.$$

For a group, this is equivalent to the condition that $\omega(t)\omega(t^{-1}) < d_T \quad (t \in T)$.

Now ω is **WDB = weakly diagonally bounded** on T if

$$\Omega(u,v) \geq 1/c_T > 0 \quad (u \in S^*, v \in T^*)\,.$$

Explicitly: for each $\varepsilon > 0$, there is a cofinite subset $S_0 \subset S$ such that, for each $s \in S_0$, there is a cofinite subset $T_s \subset T$ such that

$$\Omega(s,t) \geq (1-\varepsilon)/c_T \quad (t \in T_s)\,.$$

This is definitely a weaker condition than DB, as we shall see later.

A theorem on the topological centre

Write T_ω^* for the weak-$*$ closure of $\{\widetilde{\delta}_t : t \in T\}$.

Theorem Let S be an infinite, countable, cancellative, abelian semigroup (e.g., $\mathbb{Q}^{+\bullet}$), and suppose that ω is a weight S that is WDB on an infinite subset T, with bound $c_T \geq 1$. Take $n \in \mathbb{N}$ with $n > c_T$. Then there is a subset V of T_ω^* with $|V| = n$ that is determining for the topological centre.

The proof is a variation of one in a memoir with Lau and Strauss for the group algebras $\ell^1(G)$.

Actually we need a somewhat more technical version of the above.

Is there an example where we cannot take $c_T = 1$? Can we always find 2 points that determine the centre?

Similar results when ω is DB by Neufang and by Filali and Salmi.

Sketch of the proof

There are $a_1, \ldots, a_n \in T^*$ such that $a_1, \ldots, a_n$ are right cancellative in βS and such that

$$(\beta S \,\Box\, a_i) \cap (\beta S \,\Box\, a_j) = \emptyset \quad (i \neq j)\,.$$

Set $v_i = \theta_\omega''(a_j)$ and $V = \{v_1, \ldots, v_n\}$.

Take $\Phi \in A''$, norm 1, with $\Phi \,\Box_\omega v = v \,\Box_\omega \Phi$ for all $v \in V$.

There are $\lambda_i \in \ell^\infty$, with norm 1, disjoint supports K_i, so that $(1-\varepsilon)/c_T < |\langle \Phi \,\Box_\omega v_i,\, \lambda_i \omega\rangle|$.

For each f in the linear span of the point masses on βS, norm 1, split f as $f_1 + \cdots + f_n$ with f_i on K_i. There exists j with $\left\| f_j \right\| \leq 1/n$. Take some limits. Then

$$\frac{1-\varepsilon}{c_T} < \left|\langle \Phi \,\Box_\omega v_j,\, \lambda_j \omega\rangle\right| = \left|\langle v_j \,\Box_\omega \Phi,\, \lambda_j \omega\rangle\right| \leq \frac{1}{n}\,,$$

a contradiction for suitably small $\varepsilon > 0$. $\square$

An example on $\mathbb{Z}$

Each $n \in \mathbb{Z}$ with $n \neq 0$ can be written as

$$n = \sum_{j=1}^{r} \varepsilon_j 2^{a_j}\,,$$

where $\varepsilon_j \in \{-1, 1\}$ and $a_j \in \mathbb{Z}^+$ for each $j \in \mathbb{N}$, and where $a_1 \geq a_2 \geq \cdots \geq a_r$. Define $\eta(n)$ to be the minimum such r, and set $\omega = \exp(\eta)$.

Set $T = \{2^k : k \in \mathbb{N}\}$. Then ω is DB on T with $d_T = e^2$, but ω is WDB on T with $c_T = 1$. So the centre of $\ell^1(\mathbb{Z}, \omega)''$ is specified by 2 points.

[Fix $n \in \mathbb{Z}$ as above. For each $k \in \mathbb{N}$ with $k > a_1 + 1$, we have $\Omega(n, 2^k) = 1$, so ω is WDB on T.]

[Is $(\ell^1(\mathbb{Z}, \omega)'', \Box)$ semismple?]

Examples on $\mathbb{Q}$

Take $p/q \in \mathbb{Q}$ (with $q \in \mathbb{N}$ and p, q coprime).

Take $\omega(p/q)$ to be

$$q \quad \text{or} \quad 1 + \log q \quad \text{or} \quad 1 + |p| + q\,.$$

Then ω is a weight on $\mathbb{Q}$.

Let $T = \{1/r : r \in \mathbb{P}\}$. Then our weights are all WDB by 1, but not necessarily DB, on T.

[Set $\omega(p/q) = q$. For $s = p/q \in \mathbb{Q}$, take

$$T_s = \{1/r : r \in \mathbb{P}, r > q\}\,;$$

since $pr + q$ and qr are coprime, $\Omega(s,t) = 1$ when $t \in T_s$.]

All the weights are SAI, and 2 points determine the centre.

A radical example on $\mathbb{N}$

A radical weight on $\mathbb{N}$ cannot be DB. But there is a radical weight on $\mathbb{N}$ that is WDB. One needs an inductive construction of a rapidly increasing sequence (m_k) and a weight with special values on the points m_k. Then ω is radical and $\ell^1(\mathbb{N}, \omega)$ is SAI.

Continuous weights on $\mathbb{R}^+$

Let ω be a continuous weight on $\mathbb{R}^+$, and restrict it to $\mathbb{Q}^{+\bullet}$, so that ω is weight on $\mathbb{Q}^{+\bullet}$. Then ω is SAI on $\mathbb{Q}^{+\bullet}$. For example, set

$$\omega(s) = \exp(-s^2) \quad (s \in \mathbb{Q}^{+\bullet})$$

to get a radical example.

Conditions for Arens regularity

The following is an old result of Craw and Young (1974).

Theorem Let ω be a weight on a cancellative, abelian semigroup S. Then ω is Arens regular if and only if Ω 0-clusters on $S \times S$, in the sense that

$$\lim_{m\to\infty}\lim_{n\to\infty}\Omega(x_m, y_n) = 0$$

whenever (x_m) and (y_n) are sequences of distinct elements of S.

For example, let ω be a weight on $\mathbb{Q}^{+\bullet}$ such that ω is bounded on $\mathbb{Q}^{+\bullet} \cap (a, b)$ for some $a, b \in \mathbb{R}$ with $0 < a < b$. Then ω is **not** Arens regular.

But

$$\omega(n) = \exp(-n^2) \quad (n \in \mathbb{N})$$

is a radical Arens regular weight on $\mathbb{N}$. (*cf.* this weight on $\mathbb{Q}^{+\bullet}$.)

More weights on $\mathbb{Q}^{+\bullet}$

We look for (radical) weights on $\mathbb{Q}^{+\bullet}$ that are neither Arens regular nor SAI.

This is the case for $\omega(s) = 1/(m+1)!$ whenever $s \in \mathbb{Q}^{+\bullet}$ and $m < s \leq m+1$ for some $m \in \mathbb{Z}^{+}$.

Also $\omega_1(s) = \exp(-s^2)$ $(s \in \mathbb{Q}^{+\bullet})$ is the restriction of a continuous weight on $\mathbb{R}^{+}$, and so SAI, and $\omega_2(p/q) = 1 + \log q$ is a SAI weight, but their product $\omega_1\omega_2$ is neither Arens regular nor SAI.

Let

$$\omega_1(s) = \exp\left(-s^2 + \frac{1}{s}\right) \quad (s \in \mathbb{Q}^{+\bullet})$$

and $\omega_2(p/q) = 1 + \log q$. Then $\omega_1\omega_2$ is an explicit Arens regular weight on $\mathbb{Q}^{+\bullet}$.

[Can you find a weight ω_1 which is the restriction of a continuous weight on $\mathbb{R}^{+}$ to $\mathbb{Q}^{+\bullet}$ and an SAI weight on ω_2 on $\mathbb{Q}^{+\bullet}$ such that $\omega_1\omega_2$ is Arens regular?]

An intermediate weight on $\mathbb{N}$

It seems to be quite hard to construct a weight ω on $\mathbb{N}$ such that ω is neither Arens regular nor SAI.

We fix a very rapidly increasing sequence (α_k), and then define

$$\eta(2^k) = \alpha_k \quad (k \in \mathbb{Z}^+)$$

and

$$\eta(2^k + 2^j) = \alpha_k + \alpha_j \quad (k \in \mathbb{N},\ j = 0, \dots, k-1)\,.$$

Then we make η linear between these points, save that η is linear between

$$(2^k + 2^{k-1}, \alpha_k + \alpha_{k-1}) \quad \text{and} \quad (2^{k+1}, \beta_k)\,,$$

where β_k is much bigger than $\alpha_k + \alpha_{k-1}$, but much smaller than α_{k+1}.

An intermediate weight on $\mathbb{N}$ -continued

We must check that $\eta(s+t) \geq \eta(s) + \eta(t)$ for $s, t \in \mathbb{N}$ (a little complicated).

Set $\omega = \exp(-\eta)$, a weight on $\mathbb{N}$.

Since $\eta(2^k)/2^k > 2^k$, the weight is radical.

Since $\Omega(2^m, 2^n) = 1$ for all $m, n \in \mathbb{N}$, the weight is not Arens regular.

Let $C_{y,k} = \eta(2^k + 2^{k-1} + y) - \eta(2^k + 2^{k-1}) - \eta(y)$. Then

$$\lim_{y\to\infty} \lim_{k\to\infty} C_{y,k} = \lim_{k\to\infty} \liminf_{y\to\infty} C_{y,k} = \infty\,,$$

which is enough to show that the weight is not SAI.

A strange weight on $\mathbb{Q}^{+\bullet}$

In my book, page 159, there is a failed attempt to exhibit a weight on $\mathbb{Q}^{+\bullet}$ with

$$\liminf_{s\to 0+} \omega(s) = 0\,.$$

Theorem There is a weight ω on $\mathbb{Q}^{+\bullet}$ with the following properties:

(i) ω is radical, but not uniformly radical;

(ii) $\limsup_{s\to 0+} \omega(s) = \infty$ and $\liminf_{s\to 0+} \omega(s) = 0$;

(iii) $\inf\{\omega(s) : s \in \mathbb{Q}^{+\bullet} \cap (a,b)\} = 0$ and $\sup\{\omega(s) : s \in \mathbb{Q}^{+\bullet} \cap (a,b)\} = \infty$ for each a, b with $0 < a < b$;

(iv) the only compact element in $\ell^{\,1}(\mathbb{Q}^{+\bullet}, \omega)$ is 0;

(v) ω is strongly Arens irregular.

The construction

Choose a strictly increasing sequence (q_j) of prime numbers with $q_1 \geq 3$ such that

$$q_j/q_{j+1} \searrow 0 \quad \text{and} \quad q_{j+1} > (j+1)q_j \ (j \in \mathbb{N}) \, .$$

Given $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\beta = (\beta_1, \ldots, \beta_n)$ in $(\mathbb{Z}^+)^{<\omega}$, set

$$N(\alpha, \beta) = \sum_{i=1}^{m} \alpha_i \mathrm{e}^i - \sum_{j=1}^{n} j\beta_j$$

and

$$\theta(\alpha, \beta) = \sum_{i=1}^{m} \frac{\alpha_i}{i} + \sum_{j=1}^{n} \beta_j \frac{q_j}{q_{j+1}} \, .$$

Some calculations show that, for each $x \in \mathbb{Q}^{+\bullet}$, there exists $f(x) \in \mathbb{R}$ such that $N(\alpha, \beta) \geq f(x)$ whenever $\theta(\alpha, \beta) = x$.

Now for each $x \in \mathbb{Q}^{+\bullet}$, set

$$\eta(x) = \inf\{N(\alpha, \beta) : \theta(\alpha, \beta) = x\} \, ,$$

so that η is well-defined and subadditive.

The construction - conclusion

We set $\omega = \exp \eta$, so that ω is a weight on $\mathbb{Q}^{+\bullet}$.

We see that

$$\eta\left(\frac{1}{k}\right) = \mathrm{e}^k\,,$$

so that $\limsup_{s\to 0+} \omega(s) = \infty$.

Also

$$\eta\left(\frac{q_k}{q_{k+1}}\right) = -k\,,$$

and so $\liminf_{s\to 0+} \omega(s) = 0$.

We can also check that ω is WDB on the infinite set $T = \{q_k/q_{k+1} : k \in \mathbb{N}\}$ with bound $c_T = 1$ (but it is not DB on any infinite set). Thus ω is SAI, and again 2 points determine the centre.