Stable embeddedness and NIP

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Abstract

We give some sufficient conditions for a predicate P in a complete theory T to be "stably embedded". Let $\mathcal{P}$ be P with its "induced $\emptyset$-definable structure". The conditions are that $\mathcal{P}$ (or rather its theory) is "rosy", P has NIP in T and that P is stably 1-embedded in T. This generalizes a recent result of Hasson and Onshuus [6] which deals with the case where P is o-minimal in T. Our proofs make use of the theory of strict nonforking and weight in NIP theories ([3], [10]).

1 Introduction and preliminaries

The notion of "stable embeddedness" of a predicate (or definable set) in a theory (or structure) is rather important in model theory, and says roughly that no new structure is added by external parameters. The current paper is somewhat technical, and heavily influenced by [6] on the stable embeddability of o-minimal structures, which we wanted to cast in a more general context.

Stable embeddedness usually refers to a structure M which is interpretable (without parameters) in another structure N, and says that any subset of M^n which is definable, with parameters, in N, is definable, with parameters in M. I prefer to think of the universe of M as simply an $\emptyset$-definable set P in N, and to say that P is stably embedded in N if every subset of P^n which is definable (in N) with parameters, is definable (in N) with parameters from P. See below.

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So our general framework is that of a complete theory T in language L and a distinguished predicate or formula or sort $P(x)$. We work in a saturated model N of T, and let P also denote the interpretation of P in N. Unless we say otherwise, "definability" refers to definability with parameters in the ambient structure N. We will also discuss the notion "P is stable in N" just to clarify current notation and relationships.

Definition 1.1. *(i) P is stable in T (or in N) if there do NOT exist a formula $\phi(\bar{x}, y)$ (where $\bar{x}$ is a tuple of variables each of which is of sort P, and y is an arbitrary tuple of variables) and $\bar{a}_i \subset P$ and $b_i \in N^{eq}$ for $i < \omega$ such that $N \models \phi(\bar{a}_i, b_j)$ iff $i \leq j$.*
(ii) P is NIP in T (or N) if there do NOT exist $\phi(\bar{x}, y)$ (with same proviso as before) and $\bar{a}_i \subset P$ for $i < \omega$ and $b_s \in N^{eq}$ for $s \subseteq \omega$ such that $N \models \phi(\bar{a}_i, b_s)$ iff $i \in s$.
(iii) P is stably embedded in T (or N) if for all n every subset of P^n which is definable in N with parameters, is definable in N with parameters from P.
(iv) P is 1-stably embedded, if (iii) holds for $n = 1$.

We let $\mathcal{P}$ denote the structure whose universe is P and whose basic relations are those which are $\emptyset$-definable in N. Note that if P is NIP in N and $\mathcal{P}$ is the structure with universe P and relations all subsets of various P^n which are $\emptyset$-definable in N, then $Th(\mathcal{P})$ has NIP too. Of course if T ($= Th(N)$) has NIP then P has NIP in N, and even in this case there are situations where P is not stably embedded in N. For example when T is the theory of dense pairs of real closed fields and where P is the bottom model.

Remark 1.2. *(i) For P to be stable in N it is enough that Definition 1.1(i) holds in the case where $\bar{x}$ is a single variable x ranging over P. Likewise for P being NIP in N and Definition 1.1(ii). Also if P is stable in N then it is NIP in N.*
(ii) Suppose that $<$ is a distinguished $\emptyset$-definable total ordering on P. Define P to be o-minimal in N if every definable (in N, with parameters) subset of P is a finite union of intervals (with endpoints in P together with plus or minus ∞) and points. Then IF P is o-minimal in N then it is 1-stably embedded in N AND has NIP in N.
(iii) If P is stable in N then P is stably embedded in N.

Comments. (i) (for P being NIP in T) and (ii) were already mentioned in an earlier draft of [6]. For (i) the point is that well-known results reducing stability and NIP to the case of formulas $\phi(x, y)$ where x ranges over elements rather than tuples, adapt to the current "relative" context. (iii) is also well-known.

We now discuss the rosy hypothesis. There is an extensive theory in place around rosy theories and thorn-forking. See [8] where this was initiated, as well as [5], [2], and [4]. So we can define a theory to be rosy if if every finitary complete type $p(x) \in S(B)$ does not thorn-fork over some $A \subseteq B$ of cardinality at most $|T|$, Rather than define thorn forking, we will give an equivalent definition, referring the reader to the above-mentioned papers for further details and discussions. In the following T' is a complete theory, $a, b, c, ..$ range over possibly imaginary elements of a saturated model of T' and $A, B, C, ..$ range over small sets of imaginaries from such a model.

Definition 1.3. *We say that T' is* rosy *if there is an automorphism invariant notion on small subsets (or tuples), A is $*$-independent from B over C, satisfying*
(i) $a \in acl(B \cup C) \setminus acl(C)$ implies a $*$-depends on B over C.
(ii) If a is $*$-independent from B over C and $D \supseteq B$ then there is a' $*$-independent from D over C such that $tp(a'/BC) = tp(a/BC)$.
(iii) for any a and B there is some $C \subseteq B$ of cardinality at most $|T|$ such that a is $*$-independent from B over C.
(iv) If $B \subseteq C \subseteq D$ then a is $*$-independent from D over B if it is $*$-independent from D over C and from C over B.
(v) A is $*$-independent from B over C if a is $*$-independent from B over C for all finite tuples a from A.
(vi) a is $*$-independent from b over C iff b is $*$-independent from a over C.

Of course, a basic example of a rosy theory is an o-minimal one, in which case we are forced to take $*$-independence as given by the pregeometry from $acl(-)$.

We can now state our main result, reverting to our earlier notation (T,P,$\mathcal{P}$, etc.).

Theorem 1.4. *Suppose that $Th(\mathcal{P})$ is rosy, P has NIP in T and P is 1-stably embedded in N. Then P is stably embedded in N.*

Our previous discussion shows that if P is o-minimal in T then all the hypotheses of Theorem 1.4 are satisfied. Let us note immediately that Theorem

1.4 needs the "NIP in N" hypothesis on P. For example let N be the structure with two disjoint unary predicates P, Q, and a random bi-partite graph relation R between $P^{(2)}$ (unordered pairs from P) and Q. Then one checks that P is 1-stably embedded in N but not stably embedded.

Hrushovski suggested the following example showing that the rosy hypothesis is needed. Let T be some completion of the theory of proper dense elementary pairs of algebraically closed valued fields $F_1 < F_2$, which have the same residue field and value group. Then F_1 is 1-stably embedded in F_2. But if $a \in F_2 \setminus F_1$, then the function taking $x \in F_1$ to $v(x - a)$ cannot be definable with parameters from F_1.

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2 Forking

In this section we fix a complete theory T and work in a saturated model $\bar{M}$. As usual $A, B, ..$ denote small subsets of $\bar{M}$. Likewise for small elementary substructures M, M_0 etc. $a, b, ..$ usually denote elements of $\bar{M}^{eq}$ unless we say otherwise. Likewise for variables. Dividing and forking are meant in the sense of Shelah. Namely a formula $\phi(x, b)$ divides over A if some A-indiscernible sequence $(\phi(x, b_i) : i < \omega)$ (with $b_0 = b$) is inconsistent. And a partial type forks over A if it implies a finite disjunction of formulas each of which divides over A. By a global type we mean a complete type over $\bar{M}$ (or over a sufficiently saturated M). Note that for a global type $p(x)$, p does not fork over A iff p does not divide over A (i.e. every formula in p does not divide over A). Also any partial type does not fork over A if and only if it has an extension to a global type which does not divide (fork) over A. Let M_0 denote a small model (elementary substructure of $\bar{M}$).

Fact 2.1. *(i) Suppose that T has NIP. Let $p(x)$ be a global type. Then p does not fork over M_0 if and only if p is $Aut(\bar{M}/M_0)$-invariant.*
(ii) If the global type $p(x)$ is $Aut(\bar{M}/M_0)$-invariant, and $(a_i : i < \omega)$ are such that a_{n+1} realizes $p|(M_0 a_0..a_n)$ for all n, then $(a_i : i < \omega)$ is indiscerinible over M_0 and its type over M_0 depends only on p. $(a_i : i < \omega)$ is called a Morley sequence in p over M_0.

Comment. (i) appears explicitly in [1], and also in [7] but is implicit in [10]. (ii) is well-known, but see Chapter 12 of [9] for a nice account.

Definition 2.2. *(i) Let $p(x)$ be a global type (or complete type over a saturated model), realized by c. We say that p strictly does not fork over A if p does not fork over A and $tp(\bar{M}/Ac)$ does not fork over A (namely for each small B containing A and realization c' of $p|B$, $tp(B/Ac)$ does not fork over A).*
(ii) Let $A \subseteq B$, and $p(x) \in S(B)$. Then p strictly does not fork over A (or p is a strict nonforking extension of $p|A$) if p has an extension to a global complete type $q(x)$ which strictly does not fork over A.

Fact 2.3. *Assume that T has NIP and let M_0 be a model.*
(i) For any formula $\phi(x,b)$, $\phi(x,b)$ divides over M_0 iff $\phi(x,b)$ forks over M_0.
(ii) Any $p(x) \in S(M_0)$ has a global strict nonforking extension.
(iii) Suppose $p(x)$ is a global complete type which strictly does not fork over M_0. Let $(c_i : i < \omega)$ be a Morley sequence in p over M_0. Suppose $\phi(x,c_0)$ divides over M_0 (where $\phi(x,y)$ is over M_0). Then $\{\phi(x,c_i) : i < \omega\}$ is inconsistent.

Proof. This is all contained in [3], where essentially only the property NTP_2 (implied by NIP) is used. (i) is Theorem 1.2 there (as any model is an "extension base" for nonforking). (ii) is Theorem 3.29 there. And (iii) is Claim 3.14 there.

Corollary 2.4. *Suppose that T has NIP and M_0 is a model of cardinality $\kappa \geq |T|$. Let $q(y)$ be a global complete type which strictly does not fork over M_0 and let $(c_\alpha : \alpha < \kappa^+)$ be a Morley sequence in q over M_0. Then for any (finite) tuple a, there is $\alpha < \kappa^+$ such that $tp(a/M_0c_\alpha)$ does not fork over M_0.*

Proof. Suppose not. So, using Fact 2.3(i), for each α there is some formula $\phi_\alpha(x,y)$ with parameters from M_0 such that $\models \phi_\alpha(a,c_\alpha)$ and $\phi_\alpha(x,c_\alpha)$ divides over M_0. As M_0 has cardinality $\kappa \geq |T|$ there is $\phi(x,y)$ over M_0 and $\alpha_0 < \alpha_1 < \alpha_2 < ... < \kappa^+$ such that $\phi(x,y) = \phi_{\alpha_i}(x,y)$ for all $i < \omega$. By Fact 2.3 (iii) $\{\phi(x,c_{\alpha_i}) : i < \omega\}$ is inconsistent, which is a contradiction as this set of formulas is supposed to be realized by a.

Some remarks are in order concerning the notions introduced above and the last corollary. Strict nonforking in the NIP context was introduced by Shelah [10] (where the study of forking in NIP theories was also initiated) and all the results above are closely connected in one way or another with Shelah's work. A version of Corollary 2.4, which on the face of it is incorrect due possibly to typographical errors, appears in [10] as Claim 5.19. A better version of Corollary 2.4 appears in [11].

3 Proof of Theorem 1.4

We revert to the context of section 1, and Theorem 1.4. Namely T is an arbitrary theory, N a saturated model, and P a $\emptyset$-definable set in N. As there $\mathcal{P}$ is the structure with universe P and relations subsets of various P^n which are $\emptyset$-definable in N. So if for example a and b are tuples from P (or even from $\mathcal{P}^{eq}$) which have the same type in $\mathcal{P}$ then they have the same type in N. We will assume that $Th(\mathcal{P})$ is rosy, as in Definition 1.3, witnessed by $*$-independence. We assume that P has NIP in N (from which it follows that any sort in $\mathcal{P}^{eq}$ has NIP in N and also the structure $\mathcal{P}$ has NIP in its own right). We also assume that P is 1-stably embedded in N. Note that any sort (or definable set) in $\mathcal{P}^{eq}$ is also a sort (or definable set) in N^{eq}. If S is a sort of $\mathcal{P}^{eq}$, and X is a subset of S which is definable (with parameters) in N, we will say, hopefully without ambiguity, that X is *coded in P* if X can be defined in N with parameters from P, which is equivalent to saying that a canonical parameter for X can be chosen in $\mathcal{P}^{eq}$, and is also of course equivalent to saying that X is definable with parameters in the structure $\mathcal{P}$. Our aim is to prove that for any n, any subset of P^n definable in N is coded in P. The 1-stable embeddedness of P states that any subset of P definable with parameters in N is coded in P. We first aim towards the following key proposition, which extends this to definable functions on P.

Proposition 3.1. *Let Z be a sort in $\mathcal{P}^{eq}$. Let $f : P \to Z$ be a function definable in N. Then f is coded in P.*

Proof of Proposition 3.1.
This will go through a couple of steps. The first is an adaptation of ideas from [6], where the rosy hypothesis on $\mathcal{P}$ comes in to play.

Lemma 3.2. *(In the situation of Proposition 3.1) There is a relation $R(x, z)$ which is definable in N with parameters from P, and some $k < \omega$, such that $N \models (\forall x \in P)(R(x, f(x)) \wedge (\exists^{\leq k} z \in Z)R(x, z))$*

Proof. Note that we are free to add parameters from P to the language (but not, of course, from outside P). Suppose f is definable in N with parameter e (which of course need not be in P). We write $f = f_e$.

We try to construct $a_\alpha \in P, b_\alpha \in Z$ for α an ordinal, such that
(i) $b_\alpha = f(a_\alpha)$, and
(ii) $b_\alpha \notin acl((a_i b_i)_{i<\alpha}, a_\alpha)$, for all α.

(Of course we allow ourselves to work in larger and larger saturated models N.) If at stage α we cannot continue the construction it means that for all $a \in P$, $f(a) \in acl((a_i b_i)_{i<\alpha}, a)$. Compactness will yield the required relation $R(x, z)$ (defined with parameters from P).
So we assume that the construction can be carried out and aim for a contradiction.

Claim I. We may assume that the sequence $(a_\alpha b_\alpha)_\alpha$ is indiscernible over e in N.
Proof. This is by Erdös-Rado. Namely Erdös-Rado gives us a long sequence $(c_i d_i)_i$ which is indiscernible over e in N, and such that for all n $tp(c_0 d_0, .., c_n d_n/e) = tp(a_{\alpha_0} b_{\alpha_0} .... a_{\alpha_n} b_{\alpha_n}/e)$ for some $\alpha_0 < ... < \alpha_n$. Note then that $f_e(c_n) = d_n$ and $d_n \notin acl(c_0 d_0, .., c_{n-1} d_{n-1}, c_n)$ for all n. In particular for all j $f_e(c_j) = d_j$) and $d_j \notin acl((c_i d_i)_{i<j}, c_j)$ for all j.

Claim II. For some γ (in fact $< |T|^+$), the sequence $(a_\alpha b_\alpha)_{\gamma \le \alpha < ..}$ is $*$-independent over $(a_\beta b_\beta)_{\beta<\gamma}$, in the rosy structure $\mathcal{P}$.
Proof. We just use indiscernibility of the sequence $(a_\alpha b_\alpha)_\alpha$ over $\emptyset$ in the structure $\mathcal{P}$. If the claim fails we can find a long increasing sequence of ordinals $(\alpha_i)_i$ such that for all i, $(a_{\alpha_i} b_{\alpha_i})$ $*$-depends on $(a_\beta b_\beta)_{\beta<\alpha_i}$ over $(a_\beta b_\beta)_{\beta<\alpha_j}$ for all $j < i$. By indiscernibility we contradict rosiness, namely we find a "$*$-forking chain" of length $\ge |T|^+$.

So after adding constants (in P!) for $(a_i b_i)_{i<\gamma}$, and relabelling, we have, working in the rosy theory $Th(\mathcal{P})$,
(iii) $\{(a_0 b_0), (a_1 b_1), (a_2 b_2), ...\}$ is a "$*$-independent" set of tuples (i.e. for each i, $a_i b_i$ is $*$-independent from $(a_j b_j)_{j \ne i}$ over $\emptyset$).

Of course we still have that $b_i \notin acl(a_i)$ and $b_i = f_e(a_i)$ for all i.
Claim. For each $S \subseteq \omega$ there are $b_i^S \in Z$ for $i < \omega$ such that
(a) For $i \in \omega$, $i \in S$ iff $b_i^S = b_i$.
(b) $tp((a_i b_i)_{i<\omega}) = tp((a_i b_i^S)_{i<\omega})$.
Proof of claim. Fix $n \notin S$. By (iii) above, as well as the fact that $b_n \notin acl(a_n)$ and properties of $*$-independence, one sees that $b_n \notin acl(a_n, (a_i b_i)_{i \ne n})$. So

choosing b_n^S such that $tp(b_n^S/a_n,(a_ib_i)_{i\neq n}) = tp(b_n/a_n,(a_ib_i)_{i\neq n})$ and b_n^S is $*$-independent from $(a_ib_i)_{i<\omega})$ over a_n (or equivalently over $(a_n,(a_ib_i)_{i\neq n})$ we see that $b_n^S \neq b_n$ and $tp((a_nb_n),(a_ib_i)_{i\neq n}) = tp(a_nb_n^S),(a_ib_i)_{i\neq n})$ and $b_n^S \neq b_n$. Iterate this to obtain the claim.

By (b) of the claim, and automorphism, for each $S \subseteq \omega$, there is e_S such that $f_{e_S}(a_n) = b_n^S$ for all $n \in \omega$. By (a) of the claim, we have that $f_{e_S}(a_n) = b_n$ iff $n \in S$. This contradicts P having NIP in T, and proves Lemma 3.2

Note that if $Th(\mathcal{P})$ had Skolem functions, or even Skolem functions for "algebraic" formulas, we could quickly deduce Proposition 3.1 from Lemma 3.2. (And this is how it works in the o-minimal case.) Likewise if we can choose R in Lemma 3.2 such that $k = 1$, we would be finished. So let us choose R in Lemma 3.2 such that k is minimized, and work towards showing that $k = 1$. Here we use nonforking in the NIP theory $Th(\mathcal{P})$. Let us fix a small elementary substructure M_0 of $\mathcal{P}$ which contains the parameters from R. So for any $a \in P$, $f(a) \in acl(M_0, a)$, and this will be used all the time.

Lemma 3.3. *There is a finite tuple c from $\mathcal{P}^{eq}$ such that whenever $a \in P$ and $b = f(a)$ and $tp(a/M_0c)$ does not fork over M_0 (in the structure $\mathcal{P}$) then $b \in dcl(M_0, c, a)$ (in $\mathcal{P}$ or equivalently in N).*

Proof. Let us suppose the lemma fails (and aim for a contradiction). We will first find inductively a_n, b_n, d_n for $n < \omega$ such that writing $c_n = (a_n, b_n, d_n)$ we have
(i) $a_n \in P$, $b_n \neq d_n$ are in Z and $b_n = f(a_n)$.
(ii) $tp(a_nb_n/M_0c_0..c_{n-1})$ does not fork over M_0, and
(iii) $tp(a_nb_n/M_0c_0...c_{n-1}) = tp(a_nd_n/M_0c_0...c_{n-1})$.

Suppose have found a_i, b_i, d_i for $i < n$. Put $c = c_0...c_{n-1}$ (so if $n = 0$, c is the empty tuple.) As the claim fails for c, we find $a \in P$, such that $f(a) = b$, $tp(a/M_0c)$ does not fork over M_0, and $b \notin dcl(M_0ca)$. Noting that $b \in acl(M_0, a)$ we also have that $tp(ab/M_0c)$ does not fork over M_0. Let d realize the same type as b over M_0ca with $d \neq b$. Put $a_n = a$, $b_n = b$, $d_n = d$. So the construction can be accomplished.

Claim. Let $S \subseteq \omega$. Define $g_n = b_n$ if $n \in S$ and $g_n = d_n$ if $n \notin S$. Then $tp((a_ig_i)_{i<\omega}/M_0) = tp((a_ib_i)_{i<\omega}/M_0$ (in $\mathcal{P}$ or equivalently in N).
Proof of claim. Assume, by induction, that
(*) $tp((a_ig_i)_{i<n}/M_0) = tp((a_ib_i)_{i<n}/M_0)$,

and we want the same thing with $n+1$ in place of n. By (iii) above it suffices to prove that $tp((a_i g_i)_{i<n}(a_n b_n)/M_0) = tp((a_i b_i)_{i<n}(a_n b_n)/M_0)$. This is true by (*) and Fact 2.1(i). (Namely as $tp(a_n b_n/M_0 c)$ does not fork over M_0 it has a global complete extension q say which does not fork over M_0 so by 2.1(i) q is fixed by automorphisms which fix M_0 pointwise. So whether or not a formula $\phi(x,z,d)$ say is in q or not depends just on $tp(d/M_0)$, so the same is true for $tp(a_n b_n/M_0 c)$.) The claim is proved.

As earlier we write f as f_e where e is a parameter in N over which f is defined. By the claim, and automorphism, for each $S \subseteq \omega$ we can find e_S in N such that $f_{e_S}(a_i) = g_i$ for all $i < \omega$. In particular, as $d_i \neq b_i$ for all i, $f_{e_S}(a_i) = b_i$ if and only if $i \in S$, showing that P has the independence property in N, a contradiction. Lemma 3.3 is proved.

We now complete the proof of Proposition 3.1 by showing that $k = 1$. Let us assume $k > 1$ and get a contradiction. Let $c \in \mathcal{P}^{eq}$ be as given by the Lemma 3.3. Then, (by the previous two lemmas), whenever $a \in P$ and $tp(a/M_0 c)$ does not fork over M_0, THEN there is $d \in Z$ such that $\models R(a,d)$ and $d \in dcl(M_0, a, c)$. Note that f is not mentioned here, so this statement is purely about $\mathcal{P}$. Hence:
(*) whenever $c' \in \mathcal{P}$ has the same type over M_0 as c, then whenever $a \in P$ and $tp(a/M_0 c')$ does not fork over M_0 then $d \in dcl(M_0, a, c)$ for some $d \in Z$ such that $\models R(a,d)$.

We work in $\mathcal{P}$. By Fact 2.3 (ii) let $q(y)$ be a global strict nonforking extension of $tp(c/M_0)$, and let $(c_\alpha : i < |T|^+)$ be a Morley sequence in q over M_0. By Corollary 2.4, for any $a \in P$, there is α such that $tp(a/M_0 c_\alpha)$ does not fork over M_0, and hence by (*), there is d such that $\models R(a,d)$ and $d \in dcl(M_0, a, c_\alpha)$. By compactness there is a partial function $g(-)$ defined over $M_0 \cup \{c_\alpha : \alpha < |T|^+\}$ such that
(**) for all $a \in P$, there is $d \in Z$ such that $\models R(a,d)$ and $g(a) = d$.

Now by our assumption that P is 1-stable embedded, $\{a \in P : g(a) = f(a)\}$ is definable over parameters from P by some formula $\psi(x)$ say. Let $R'(x,z)$ be $(\psi(x) \wedge z = g(x)) \vee (\neg\psi(x) \wedge R(x,z) \wedge z \neq g(x))$. Then clearly R' satisfies Lemma 3.2 with $k-1$. This contradiction proves Proposition 3.1.

Proof of Theorem 1.4.
There is no harm in adding a few constants for elements of P (so that we

can do definition by cases). We will prove by induction on n that any subset of P^n which is definable in N with parameters, is coded in $\mathcal{P}^{eq}$. For $n = 1$ this is the 1-stable embeddednes hypothesis. Assume true for n and we'll prove for $n+1$. Let $\phi(x_1, .., x_n, x_{n+1}, e)$ be a formula, with parameter $e \in N$ defining the set $X \subseteq P^{n+1}$. For each $a \in P$, let X_a be the subset of P^n defined by $\phi(x_1, x_2, .., x_n, a, e)$. By induction hypothesis X_a is coded in P, namely has a canonical parameter c_a say in $\mathcal{P}^{eq}$. By compactness, we assume that there is a formula $\psi(x_1, .., x_n, z)$ (of L), where z ranges over a sort Z of $\mathcal{P}^{eq}$, such that for each $a \in P$, X_a is defined by $\psi(x_1, .., x_n, c_a)$ (and c_a is still a canonical parameter for X_a). The map taking a to c_a is clearly an e-definable function f say from P to Z. By Proposition 3.1, f is coded in $\mathcal{P}$ (i.e. can be defined with parameters from P). As the original set $X \subseteq P^{n+1}$ is defined by the "formula" $\psi(x_1, .., x_n, f(x_{n+1}))$, it follows that X is also coded in P. The proof of Theorem 1.4 is complete.

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