

Differential Groups and Differential Relations

Michael F. Singer
Department of Mathematics
North Carolina State University
Raleigh, NC 27695-8205 USA

<http://www.math.ncsu.edu/~singer>

Theorem: (Hölder, 1887) The Gamma function $\Gamma(x + 1) = x\Gamma(x)$ satisfies no polynomial differential equation.

Goal: Prove this using *differential* algebraic groups and generalize

Ex. If $y_1(x)$ and $y_2(x)$ are lin. indep. solutions of

$$y(x + 2) - xy(x + 1) + y(x) = 0$$

then $y_1(x)$, $y_1(x + 1)$ and $y_2(x)$ satisfy no polynomial differential equation.

- Galois Theory of Difference Equations
- Differential Galois Theory of Difference Equations
- Differential Relations Among Solutions of
Linear Difference Equations
- Inverse Problems
- General Theory
- Final Comments

Galois Theory of Difference Equations

k - field, σ - an automorphism Ex. $\mathbb{C}(x)$, $\sigma(x) = x + 1$, $\sigma(x) = qx$

Difference Equation: $\sigma(Y) = AY$ $A \in \text{GL}_n(k)$

Splitting Ring: $k[Y, \frac{1}{\det(Y)}]$, $Y = (y_{i,j})$ indeterminates, $\sigma(Y) = AY$,

$M = \max \sigma\text{-ideal}$

$$R = k[Y, \frac{1}{\det(Y)}] / M = k[Z, \frac{1}{\det(Z)}] = \sigma\text{-Picard-Vessiot Ring}$$

- M is radical $\Rightarrow R$ is reduced
- If $C = k^\sigma = \{c \in k \mid \sigma c = c\}$ is alg closed $\Rightarrow R$ is unique and $R^\sigma = C$

Ex.

$$k = \mathbb{C} \quad \sigma(y) = -y$$
$$R = \mathbb{C}[y, \frac{1}{y}] / (y^2 - 1)$$

σ -Galois Group: $\text{Gal}_\sigma(R/k) = \{\phi : R \rightarrow R \mid \phi \text{ is a } \sigma \text{ } k\text{-automorphism}\}$

Ex.

$$k = \mathbb{C} \quad \sigma(y) = -y \Rightarrow R = \mathbb{C}[y, \frac{1}{y}]/(y^2 - 1)$$

$$\text{Gal}_\sigma(R/k) = \mathbb{Z}/2\mathbb{Z}$$

Ex.

$$k = \mathbb{C}(x), \sigma(x) = x + 1$$

$$\sigma^2 y - x\sigma y + y = 0 \Rightarrow \sigma Y = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} Y$$

$$R = k[Y, \frac{1}{\det(Y)}]/(\det(Y) - 1), \quad \text{Gal}_\sigma = \text{SL}_2(\mathbb{C})$$

Ex.

$$\sigma(y) - y = f, \quad f \in k \Leftrightarrow \sigma \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$$

$$\phi \in \text{Gal}_\sigma \Rightarrow \phi(y) = y + c_\phi, \quad c_\phi \in C$$

$$\text{Gal}_\sigma = (C, +) \text{ or } \{0\}$$

- $\phi \in \text{Gal}_\sigma$, $\sigma(\mathbf{Z}) = \mathbf{AZ} \Rightarrow \phi(\mathbf{Z}) = \mathbf{Z}[\phi]$, $[\phi] \in \text{GL}_n(\mathbf{C})$

$\text{Gal}_\sigma \hookrightarrow \text{GL}_n(\mathbf{C})$ and the image is Zariski closed

$$\text{Gal}_\sigma = G(\mathbf{C}), G \text{ a lin. alg. gp. } / \mathbf{C}.$$

- $R =$ coord. ring of a G -torsor

$$R^{\text{Gal}_\sigma} = k$$

$$\dim(G) = \text{Krull dim.}_k R (\simeq \text{trans. deg. of quotient field})$$

The size of $\text{Gal}(K/k)$ measures algebraic relations among the solutions .

Ex.

$$\sigma^2 y - x \sigma y + y = 0 \Rightarrow \sigma Y = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} Y$$

$$\text{Gal}_\sigma = \text{SL}_2(\mathbb{C})$$

$$\begin{aligned} 3 &= \dim \text{SL}_2(\mathbb{C}) = \text{tr. deg.}_k k(y_1, y_2, \sigma(y_1), \sigma(y_2)) \\ &\Rightarrow y_1, y_2, \sigma(y_1) \text{ alg. indep. over } k \end{aligned}$$

Ex. $f_1, \dots, f_n \in k$, k a difference field w. alg. closed const.

$$\begin{aligned}\sigma(y_1) - y_1 &= f_1 \\ &\vdots \\ \sigma(y_n) - y_n &= f_n\end{aligned}$$

Picard-Vessiot ring = $k[y_1, \dots, y_n]$

Prop. y_1, \dots, y_n alg. dep. over k
if and only if

$\exists g \in k$ and a const coeff. linear form L s.t. $L(y_1, \dots, y_n) = g$
(equiv., $c_1 f_1 + \dots + c_n f_n = \sigma(g) - g$)

Proof. $\text{Gal}_\sigma \subset (C, +)^n$.

\Rightarrow) Alg. dependent $\Rightarrow \text{Gal}_\sigma \subsetneq (C, +)^n$
 $\Rightarrow \exists L$ s.t. $\text{Gal}_\sigma \subset \{(c_1, \dots, c_n) \mid L(c_1, \dots, c_n) = 0\}$

$$\begin{aligned}\phi \in \text{Gal}_\sigma, \phi(L(y_1, \dots, y_n)) &= L(y_1 + c_1, \dots, y_n + c_n) \\ &= L(y_1, \dots, y_n) + L(c_1, \dots, c_n) = L(y_1, \dots, y_n)\end{aligned}$$

So, $L(y_1, \dots, y_n) = g \in k$. □

Ex. $y(x+1) - y(x) = \frac{1}{x} \Rightarrow y(x)$ is not alg. over $\mathbb{C}(x)$.

Differential Galois Theory of Difference Equations

k - field, σ - an automorphism δ - a derivation s.t. $\sigma\delta = \delta\sigma$

Ex. $\mathbb{C}(x) : \sigma(x) = x + 1, \delta = \frac{d}{dx}$
 $\sigma(x) = qx, \delta = x \frac{d}{dx}$
 $\mathbb{C}(x, t) : \sigma(x) = x + 1, \delta = \frac{\partial}{\partial t}$

Difference Equation: $\sigma(Y) = AY \quad A \in \text{GL}_n(k)$

Splitting Ring: $k\{Y, \frac{1}{\det(Y)}\} = k[Y, \delta Y, \delta^2 Y, \dots, \frac{1}{\det(Y)}]$

$Y = (y_{i,j})$ differential indeterminates

$$\sigma(Y) = AY, \sigma(\delta Y) = A(\delta Y) + (\delta A)Y, \dots$$

$$M = \max \sigma\delta\text{-ideal}$$

$$R = k\{Y, \frac{1}{\det(Y)}\} / M = k\{Z, \frac{1}{\det(Z)}\} = \sigma\delta\text{-Picard-Vessiot Ring}$$

k - $\sigma\delta$ field

$$\sigma(Y) = AY, A \in \mathrm{GL}_n(k)$$

$R = k\{Z, \frac{1}{\det(Z)}\}$ - $\sigma\delta$ -Picard-Vessiot ring

- R is reduced
- If $C = k^\sigma = \{c \in k \mid \sigma c = c\}$ is *differentially closed*
 $\Rightarrow R$ is unique and $R^\sigma = C$

$\sigma\delta$ -Galois Group: $\text{Gal}_{\sigma\delta}(R/k) = \{\phi : R \rightarrow R \mid \phi \text{ is a } \sigma\delta \text{ } k\text{-automorphism}\}$

- $\phi \in \text{Gal}_{\sigma\delta} \sigma(Z) = AZ \Rightarrow \phi(Z) = Z[\phi], [\phi] \in \text{GL}_n(C)$

$\text{Gal}_{\sigma\delta} \hookrightarrow \text{GL}_n(C)$ and the image is Kolchin closed

$\text{Gal}_{\sigma\delta} = G(C), G$ a lin. *differential* alg. gp. / C .

- $\text{Gal}_{\sigma\delta}$ is Zariski dense in Gal_{σ}

- $R =$ coord. ring of a G -torsor

- $R^{\text{Gal}_{\sigma\delta}} = k$

- Assume G connected. Then $\text{diff. dim.}_C(G) = \text{diff. tr. deg}_k F$

where F is the quotient field of R .

Ex.

$$k = \tilde{\mathbb{C}} \quad \sigma(y) = -y \Rightarrow R = k[y, \frac{1}{y}] / (y^2 - 1)$$

$$\text{Gal}_{\sigma\delta}(R/k) = \mathbb{Z}/2\mathbb{Z}$$

Ex.

$$\begin{aligned} \sigma(y) - y = f, \quad f \in k, \quad \text{Gal}_{\sigma\delta} \subset \mathbb{G}_a \\ \Rightarrow \text{Gal}_{\sigma\delta} = \{c \in R^\sigma \mid L(c) = 0\} \text{ for some } L \in R^\sigma[\delta]. \end{aligned}$$

Ex.

$$\begin{aligned} k = \tilde{\mathbb{C}}(x), \quad \sigma(x) = x + 1, \quad \delta(x) = 1 \\ \sigma^2 y - xy + y = 0 \Rightarrow \sigma Y = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} Y \end{aligned}$$

$$\text{Will show: } R = k\{Y, \frac{1}{\det(Y)}\} / \{\det(Y) - 1\}$$

$$\text{Gal}_{\delta\sigma} = \text{SL}_2(\tilde{\mathbb{C}})$$

Differential Relations Among Solutions of Linear Difference Equations

Groups Measure Relations

k - $\sigma\delta$ - field, $C = k^\sigma$ differentially closed.

Differential subgroups of $G_a^n(k) = (k^n, +)$ are all of the form

$$G_a^L = \{(z_1, \dots, z_n) \in k^n \mid L(z_1, \dots, z_n) = 0\}$$

where L is a linear homogeneous differential polynomial.



Proposition. Let R be a $\sigma\delta$ -Picard-Vessiot extension of k containing z_1, \dots, z_n such that

$$\sigma(z_i) - z_i = f_i, \quad i = 1, \dots, n.$$

with $f_i \in k$. Then z_1, \dots, z_n are differentially dependent over k if and only if there is a homogeneous linear differential polynomial L over C such that

$$L(z_1, \dots, z_n) = g \quad g \in k$$

Equivalently, $L(f_1, \dots, f_n) = \sigma(g) - g$.

Corollary. Let $f_1, \dots, f_n \in \mathbb{C}(x)$, $\sigma(x) = x + 1$, $\delta = \frac{d}{dx}$ and let z_1, \dots, z_n satisfy

$$\sigma(z_i) - z_i = f_i, \quad i = 1, \dots, n.$$

Then z_1, \dots, z_n are differentially dependent over $\mathcal{F}(x)$ (\mathcal{F} is the field of 1-periodic functions) if and only if there is a homogeneous linear differential polynomial L over \mathbb{C} such that

$$L(z_1, \dots, z_n) = g \quad g \in \mathbb{C}(x)$$

Equivalently, $L(f_1, \dots, f_n) = \sigma(g) - g$.

- Similar result for q -difference equations. Also for $\sigma y_i = f_i y_i$
- C. Hardouin proved (using difference Galois theory) similar result for $\sigma y_i = f_i y_i$ and gave criterion in terms of divisors of the f_i . Simplified by M. van der Put

The Gamma function is hypertranscendental.

- $z(x) = \Gamma'(x)/\Gamma(x)$ satisfies $\sigma(z) - z = \frac{1}{x}$.

- If $z(x)$ satisfies a polynomial differential equation, then

$$\exists L \in \mathbb{C}\left[\frac{d}{dx}\right], g(x) \in \mathbb{C}(x) \text{ s.t. } L\left(\frac{1}{x}\right) = g(x+1) - g(x)$$

- $L\left(\frac{1}{x}\right)$ has a pole $\Rightarrow g(x)$ has a pole.
- If $g(x)$ has a pole then $g(x+1) - g(x)$ has at least two poles but $L\left(\frac{1}{x}\right)$ has exactly one pole .

If H a Zariski-dense proper differential subgroup of $G \subset \mathrm{GL}_n(k)$, a simple algebraic group defined over C

$\Rightarrow \exists g \in \mathrm{GL}_n(k)$ such that $gHg^{-1} = G(C)$, $C = \ker(\delta)$.

\Downarrow

Proposition. Let $A \in \mathrm{GL}_n(k)$ and assume that the σ -Galois group of $\sigma(Y) = AY$ is a simple (noncommutative) linear algebraic group G of dimension t . Let $R = k\{Z, \frac{1}{\det Z}\}$ be the $\sigma\delta$ -PV ring.

The differential trans. deg. of R over k is less than t

\Updownarrow

$\exists B \in \mathfrak{gl}_n(k)$ s.t. $\sigma(B) = ABA^{-1} + \delta(A)A^{-1}$
 (in which case, $(\delta Z - BZ)Z^{-1} \in \mathfrak{gl}_n(k^\sigma)$)

Ex.

$$k = \mathbb{C}(x), \sigma(x) = x + 1$$

$$\sigma^2 y - x \sigma y + y = 0 \Rightarrow \sigma Y = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} Y$$

$$R = k[Y, \frac{1}{\det(Y)}] / (\det(Y) - 1), \text{ Gal}_\sigma = \text{SL}_2(\mathbb{C})$$

$y_1(x), y_2(x)$ linearly independent solutions.

$y_1(x), y_2(x), y_1(x + 1)$ are differentially dependent over $\mathbb{C}(x)$

\Updownarrow

$$\exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{gl}_2(\mathbb{C}(x)) \text{ s.t.}$$

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}' \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}^{-1} + \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}^{-1}$$

This 4th order inhomogeneous equation has no such solutions

$\Rightarrow y_1(x), y_2(x), y_1(x + 1)$ are differentially independent over $\mathbb{C}(x)$

Inverse Problems

Given a $\sigma\delta$ -field k , which δ -groups occur as $\sigma\delta$ -Galois groups?

Ex. $k = C(x)$, $\sigma(x) = x + 1$, $\delta = \frac{d}{dx}$

Which subgroups of $\mathbb{G}_a = (C, +)$ occur as $\sigma\delta$ -Galois groups?

Prop. (Hardouin) $G \subset \mathbb{G}_a$ occurs as a $\sigma\delta$ -Galois group
 \Updownarrow

$$G = \begin{cases} \{0\}, & \text{or} \\ \mathbb{G}_a, & \text{or} \\ \{c \in \mathbb{G}_a \mid L(c) = 0\}, & \text{where } L \text{ is of } \underline{\text{order 1}} \end{cases}$$

General Theory

k , a field,

- $\Delta = \{\delta_1, \delta_2, \dots, \delta_d\}$ derivations
- $\Sigma = \{\sigma_1, \sigma_2, \dots, \sigma_s\}$ automorphisms
- $\Pi = \{\partial_1, \partial_2, \dots, \partial_p\}$ another set of derivations

Assume: $\Delta \cup \Sigma \cup \Pi$ commute. Consider the system

$$\begin{aligned}\delta_i Y &= D_i Y, & D_i &\in \mathfrak{gl}_n(k), & i &= 1, \dots, d \\ \sigma_i Y &= S_i Y, & S_i &\in \mathrm{GL}_n(k), & i &= 1, \dots, s\end{aligned}$$

assumed to be *integrable*, i.e. $[D_i, D_j] = \delta_i(D_j) - \delta_j(D_i)$, $S_i S_j = S_j S_i$ and $\sigma_i(D_j) = \delta_j(S_i) + S_i D_j$

Ex. 1 $\Pi = \emptyset \implies$ Usual PV theories.

Ex. 2 $k = C(x)$, $\Delta = \emptyset$, $\Sigma = \{\text{shift}\}$, $\Pi = \left\{\frac{d}{dx}\right\}$

Ex. 3 $k = C(x, t)$, $\Delta = \left\{\frac{d}{dx}\right\}$, $\Sigma = \emptyset$, $\Pi = \left\{\frac{d}{dt}\right\}$.

$$\frac{dY}{dx} = \frac{t}{x} Y$$

We want to measure the differential properties (wrt the derivations of Π) of the solutions of

$$\delta_i Y = D_i Y, \quad D_i \in \mathfrak{gl}_n(k), \quad i = 1, \dots, d \quad (1)$$

$$\sigma_i Y = S_i Y, \quad S_i \in \mathrm{GL}_n(k), \quad i = 1, \dots, s \quad (2)$$

Assume that $C = k^{\Delta\Sigma} = \{c \in k \mid \delta(c) = 0 = \sigma(c) \forall \delta \in \Delta, \sigma \in \Sigma\}$ is a Π -differentially closed field. Then:

- There exists a $\Delta\Sigma\Pi$ -simple ring $R = k[Z, \frac{1}{\det Z}]$, Z a solution of (1) and (2), with no new $\Delta\Sigma$ -constants
- The group G of $\Delta\Sigma\Pi$ -automorphisms of R over k is a Π -linear differential algebraic group
- R is a torsor for G and there is a Galois theory.

Can use this to develop parameterized theory of differential and difference equations. Also show

$$\gamma(x, t) = \int_1^t u^x e^{-u} du \Rightarrow \gamma, \gamma_x, \gamma_{xx}, \dots \text{ alg. indep } / \mathbb{C}(x, t)$$

Final Comments

- q -hypergeometric functions ${}_2\phi_1(a, b; c; x)$ satisfy

$$\phi(q^2x) - \frac{(a-b)x - (1+c/q)}{abx - c/q} \phi(qx) + \frac{x-1}{abx - c/q} \phi(x) = 0$$

Classify differential dependence among these.

- Inverse problem
- Nonlinear equations
- Characteristic p
- Relation to model theory of $\sigma\delta$ -fields