Generic stability, regularity, and quasi-minimality

Anand Pillay[*]
University of Leeds

Predrag Tanović[†]
Mathematical Institute SANU, Belgrade

December 31, 2009

Abstract

We study the notions generic stability, regularity, homogeneous pregeometries, quasiminimality, and their mutual relations, in arbitrary first order theories. We prove that "infinite-dimensional homogeneous pregeometries" coincide with generically stable strongly regular types $(p(x), x = x)$. We prove that in a theory without the strict order property, regular types are generically stable, and prove analogous results for quasiminimal structures. We prove that the "generic type" of a quasiminimal structure is "locally strongly regular".

1 Introduction

The first author was motivated partly by hearing Wilkie's talks on his program for proving Zilber's conjecture that the complex exponential field is quasiminimal (definable subsets are countable or co-countable), and wondering about the first order (rather than infinitary) consequences of the approach. The second author was partly motivated by his interest in adapting

---

[*] Supported by EPSRC grant EP/F009712/1

[†] Supported by Ministry of Science and Technology of Serbia

his study of minimal structures (definable subsets are finite or cofinite) and his dichotomy theorems ([6]), to the quasiminimal context.

Zilber's conjecture (and the approach to it outlined by Wilkie) is closely related to the existence and properties of a canonical pregeometry on the complex exponential field. See the end of section 4 for a more detailed discussion. Also in section 5 of this paper we discuss to what extent a pregeometry can be recovered just from quasi-minimality, sometimes assuming the presence of a definable group structure. This continues in a sense an earlier study of the general model theory of quasi-minimality by Itai, Tsuboi, and Wakai [2].

A pregeometry is a closure relation on subsets of a not necessarily saturated structure M satisfying usual properties (including exchange). We will also assume "homogeneity" ($\mathrm{tp}(b/A)$ is unique for $b \notin \mathrm{cl}(A)$) and "infinite-dimensionality" ($\dim(M)$ is infinite). One of the points of this paper then is that the canonical "generic type" p of the pregeometry is "generically stable" and regular. This includes the statement that on realizations of p, the closure operation is precisely forking in the sense of Shelah. See Theorem 3 of section 4.

Generic stability, the stable-like behavior of a given complete type vis-a-vis forking, was studied in several papers including [5] and [1], but mainly in the context of theories with NIP (i.e. without the independence property). Here we take the opportunity, in section 2, to give appropriate definitions in an arbitrary ambient theory T, as well as discussing generically stable (strongly) regular types.

The notion of a regular type is central in stability theory and classification theory, where the counting of models of superstable theories is related to dimensions of regular types. Here (section 3) we give appropriate generalizations of (strong) regularity for an arbitrary theory T (although it does not agree with the established definitions for simple theories). In section 6 a local version of regularity is given and applied to the analysis of quasiminimal structures (see Corollaries 2 and 3).

The current paper is a revised and expanded version of a preprint "Remarks on generic stability, pregeometries, and quasiminimality" by the first author, which was written and circulated in June 2009.

The first author would like to thank Clifton Ealy, Krzysztof Krupinski, and Alex Usvyatsov for various helpful conversations and comments. After seeing the first author's preliminary results (and talk at at a meeting in Lyon in July 2009), Ealy pointed out that the commutativity of invariant-regular groups is

problematic (see our Question at the end of section 3) and suggested possible directions towards a counterexample. Krupinski independently came up with examples such as Example 3 of the current paper. And after a talk by the first author on the same subject in Bedlewo in August 2009, Usvyatsov pointed out examples such as Example 1 and 2 of the current paper. Both authors would also like to thank Jonathan Kirby for helpful comments and questions, and for clarifying the connection to exponential algebraicity (see the end of section 4).

We now give our conventions and give a few basic definitions relevant to the paper.

T denotes an arbitrary complete 1-sorted theory in a language L and $\bar{M}$ denotes a saturated (monster) model of T. As a rule $a, b, c, ...$ denote elements of $\bar{M}$, and $\bar{a}, \bar{b}, \bar{c}$ denote finite tuples of elements. (But in some situations $a, b, ..$ may denote elements of $\bar{M}^{eq}$.) A, B, C denote small subsets, and $M, M_0, ...$ denote small elementary submodels of $\bar{M}$. A global type $p(\bar{x}) \in S(\bar{M})$ is said to be A-invariant if p is $\mathrm{Aut}(\bar{M}/A)$-invariant; p is invariant if it is A-invariant for some small A. For A-invariant p we can form a Morley sequence $I = (\bar{a}_i : i < \omega)$ of p over A, by letting $\bar{a}_n \models p|(A, \bar{a}_0, ..., \bar{a}_{n-1})$, similarly for any ordinal λ in place of ω. The A-invariance of $p(\bar{x})$ implies that Morley sequences are indiscernible and that $\mathrm{tp}(I/A)$ depends only on $p(\bar{x})$ and A; in particular, $p^{(n)}(\bar{x}_1, ..., \bar{x}_n)$ is well defined as a global type of a Morley sequences in p of length n. Note that the $p^{(n)}$'s are all invariant. We will say that p is *symmetric* if $p^{(2)}(\bar{x}_1, \bar{x}_2) = p^{(2)}(\bar{x}_2, \bar{x}_1)$, otherwise p is *asymmetric*. The symmetry of p is equivalent to: every Morley sequence in p over (any small) A is totally indiscernible, in which case all the $p^{(n)}$'s are invariant under permutations of (tuples) of variables.

Recall that (P, cl), where cl is an operation on subsets of P, is a *pregeometry* (or cl is a pregeometry on P) if for all $A, B \subseteq P$ and $a, b \in P$ the following holds:

Monotonicity: $A \subseteq B$ implies $A \subseteq \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$;

Finite character $\mathrm{cl}(A) = \bigcup\{\mathrm{cl}(A_0) \mid A_0 \subseteq A \text{ finite}\}$;

Transitivity $\mathrm{cl}(A) = \mathrm{cl}(\mathrm{cl}(A))$, and

Exchange (symmetry) $a \in \mathrm{cl}(A \cup \{b\}) \setminus \mathrm{cl}(A)$ implies $b \in \mathrm{cl}(A \cup \{a\})$.

cl is called a *closure operator* if it satisfies the first three conditions..

2 Generic stability

Definition 1. A non-algebraic global type $p(\bar{x}) \in S(\bar{M})$ is generically stable if, for some small A, it is A-invariant and:

if $(\bar{a}_i : i < \omega)$ is a Morley sequence in p over A then for any formula $\phi(\bar{x})$ (with parameters from $\bar{M}$) $\{i : \models \phi(\bar{a}_i)\}$ is either finite or co-finite.

Remark 1. If p is generically stable then as a witness-set A in the definition we can take any small A such that P is A-invariant.

Proposition 1. *Let $p(\bar{x}) \in S(\bar{M})$ be generically stable and A-invariant. Then:*

(i) For any formula $\phi(\bar{x}, \bar{y}) \in L$ there is n_ϕ such that for any Morley sequence $(\bar{a}_i : i < \omega)$ of p over A, and any $\bar{b}$:

$$\phi(\bar{x}, \bar{b}) \in p \quad \textit{iff} \quad \models \bigvee_{w \subset \{0,1,\ldots 2n_\phi\},\, |w| = n_\phi + 1} \bigwedge_{i \in w} \phi(\bar{a}_i, \bar{b}).$$

(ii) p is definable over A and almost finitely satisfiable in A.

(iii) Any Morley sequence of p over A is totally indiscernible.

(iv) p is the unique global nonforking extension of $p|A$.

Proof. This is a slight elaboration of the proof of Proposition 3.2 from [1].

(i) By compactness, for any $\phi(\bar{x}, \bar{y}) \in L$ there is n_ϕ such that for any Morley sequence $(\bar{a}_i : i < \omega)$ in p over A, and any $\bar{b}$ either at most n_ϕ many $\bar{a}_i$'s satisfy $\phi(\bar{x}, \bar{b})$ or at most n_ϕ many $\bar{a}_i$'s satisfy $\neg\phi(\bar{x}, \bar{b})$.

(ii) By (i) p is definable (over a Morley sequence), so by A-invariance p is definable over A. Let M be a model containing A, $\phi(\bar{x}, \bar{c}) \in p$ and let $I = (\bar{a}_i : i < \omega)$ be a Morley sequence in p over A such that $\mathrm{tp}(I/M\bar{c})$ is finitely satisfiable in M. Then, by (i), $\phi(\bar{x}, \bar{c})$ is satisfied by some $\bar{a}_i$ hence also by an element of M.

(iii) This follows from (i) exactly as in the proof of the Proposition 3.2 of [1] (where NIP was not used).

(iv) It is clearly enough to prove it for $B = A$. Let $q(\bar{x})$ be a global nonforking extension of $p|A$. We will prove that $q = p$.

Claim. Suppose $(\bar{a}_0, ..., \bar{a}_n, \bar{b})$ are such that $\bar{a}_0 \models q|A$, $\bar{a}_{i+1} \models q|(A, \bar{a}_0, ..., \bar{a}_i)$ and $b \models q|(A, \bar{a}_0, ..., \bar{a}_n)$. Then $(\bar{a}_0, ..., \bar{a}_n, \bar{b})$ is a Morley sequence in p over A.

Proof. We prove it by induction. Suppose we have chosen $\bar{a}_0, ..., \bar{a}_n$ as in the claim and we know (induction hypothesis) that $(\bar{a}_0, ..., \bar{a}_n)$ begins a Morley sequence $I = (\bar{a}_i \mid i < \omega)$ in p over A. Suppose that $\phi(\bar{a}_0, ..., \bar{a}_n, \bar{x}) \in q$. Then we claim that $\phi(\bar{a}_0, ..., \bar{a}_{n-1}, \bar{a}_i, x) \in q$ for all $i > n$. For otherwise, without loss of generality $\neg\phi(\bar{a}_0, ..., \bar{a}_{n-1}, \bar{a}_{n+1}, \bar{x}) \in q$. But then, by indiscernibility of $\{\bar{a}_i\bar{a}_{i+1} : i = n, n+2, n+4, ...\}$ over $(A, \bar{a}_0, ..., \bar{a}_{n-1})$, and the nondividing of q over A, we have that

$$\{\phi(\bar{a}_0, ..., \bar{a}_{n-1}, \bar{a}_i, x) \wedge \neg\phi(\bar{a}_0, ..., \bar{a}_{n-1}, \bar{a}_{i+1}, \bar{x}) \ : \ i = n, n+2, n+4, ...\}$$

is consistent, which contradicts Definition 1. Hence $\models \phi(\bar{a}_0, ..., \bar{a}_{n-1}, \bar{a}_i, \bar{b})$ for all $i > n$. So by part (i), $\phi(\bar{a}_0, ..., \bar{a}_{n-1}, \bar{x}, \bar{b}) \in p(\bar{x})$. The inductive assumption gives that $\mathrm{tp}(\bar{a}_0, ..., \bar{a}_{n-1}, \bar{a}_n/A) = \mathrm{tp}(\bar{a}_0, ..., \bar{a}_{n-1}, \bar{b}/A)$, so by A-invariance of p, $\phi(\bar{a}_0, ..., \bar{a}_{n-1}, \bar{x}, \bar{a}_n) \in p(\bar{x})$. Thus $\models \phi(\bar{a}_0, ..., \bar{a}_{n-1}, \bar{a}_{n+1}, \bar{a}_n)$ and, by total indiscernibility of I over A (part (iii)), $\models \phi(\bar{a}_0, ..., \bar{a}_{n-1}, \bar{a}_n, \bar{a}_{n+1})$. We have shown that $q|(A, \bar{a}_0, ..., \bar{a}_n) = p|(A, \bar{a}_0, ..., \bar{a}_n)$, which allows the induction process to continue. The claim is proved.

Now suppose that $\phi(\bar{x}, \bar{c}) \in q(\bar{x})$. Let $\bar{a}_i$ for $i < \omega$ be such that $\bar{a}_i$ realizes $q|(A, \bar{c}, \bar{a}_0, ..., \bar{a}_{i-1})$ for all i. By the claim $(\bar{a}_i \, ; \, i < \omega)$ is a Morley sequence of p over A. But $\phi(\bar{a}_i, \bar{c})$ for all i, hence by the proof of (i), $\phi(\bar{x}, \bar{c}) \in p$. So $q = p$. □

Let us note for the record that for a global type p, generically stable implies definable implies invariant, and these are strict implications.

Let us restate the notion of generic stability for groups from [1] in the context of connected groups. Recall that a definable (or even type-definable) group G is connected if it has no relatively definable subgroup of finite index. A type $p(x) \in S_G(\bar{M})$ is left G-invariant iff for all $g \in G$:

$$(g \cdot p)(x) =^{def} \{\phi(g^{-1} \cdot x) \ : \phi(x) \in p(x)\} = p(x);$$

likewise for right G-invariant.

Definition 2. Let G be a definable (or even type-definable) connected group in $\bar{M}$. G is *generically stable* if there is a global complete 1-type $p(x)$ extending '$x \in G$' such that p is generically stable and left G-invariant.

As in the NIP context, we show that a generically stable left invariant type is also right invariant and is unique such (and we will call it the generic type):

Lemma 1. *Suppose that G is generically stable, witnessed by $p(x)$. Then $p(x)$ is the unique left-invariant and also the unique right-invariant type.*

Proof. First we prove that p is right invariant. Let (a, b) be a Morley sequence in p over (any small) A. By left invariance $g = a^{-1} \cdot b \models p|A$. By total indiscernibility we have $\mathrm{tp}(a, b) = \mathrm{tp}(b, a)$, so $g^{-1} = b^{-1} \cdot a \models p|A$. This proves that $p = p^{-1}$. Now, for any $g \in G$ we have

$$p = g^{-1} \cdot p = p^{-1} = (g^{-1} \cdot p)^{-1} = p^{-1} \cdot g = p \cdot g \ ,$$

so p is also right-invariant.

For the uniqueness, suppose that q is a left invariant global type and we prove that $p = q$. Let $\phi(x) \in q$ be over A, let $I = (a_i : i \in \omega)$ be a Morley sequence in p over A, and let $b \models q|(A, I)$. Then, by left invariance of q, for all $i \in \omega$ we have $\models \phi(a_i \cdot b)$. By Proposition 1(i) we get $\phi(x \cdot b) \in p(x)$ and, by right invariance, $\phi(x) \in p(x)$. Thus $p = q$. ☐

3 Global regularity

For stable T, a stationary type $p(x) \in S(A)$ is said to be regular if for any $B \supseteq A$, and b realizing a forking extension of p over B, $p|B$ (the unique nonforking extension of p over B) has a unique complete extension over B, b. Appropriate versions of regularity (where we do not have stationarity) have been given for simple theories for example. Here we give somewhat different versions for global "invariant" types in arbitrary theories.

Definition 3. Let $p(\bar{x})$ be a global non-algebraic type.

(i) $p(\bar{x})$ is said to be *invariant-regular* if, for some small A, it is A-invariant and for any $B \supseteq A$ and $\bar{a} \models p|A$: either $\bar{a} \models p|B$ or $p|B \vdash p|B\bar{a}$.

(ii) Suppose $\phi(\bar{x}) \in p$. We say that $(p(\bar{x}), \phi(\bar{x}))$ is *invariant-strongly regular* if, for some small A, it is A-invariant and for all $B \supseteq A$ and $\bar{a}$ satisfying $\phi(\bar{x})$, either $\bar{a} \models p|B$ or $p|B \vdash p|B\bar{a}$.

(iii) Likewise we have the notions *definable -(strongly) regular* and *generically stable - (strongly) regular.*

Remark 2. If p is invariant-regular then as a witness-set A in the definition we can take any small A such that p is A-invariant. Likewise for strongly regular types. Also note that we have the obvious implications between properties of global types in Definition 3, such as x-strongly regular implies x-regular for $x =$ invariant, definable, or generically stable.

Definition 4. Let N be a submodel of $\bar{M}$ (possibly $N = \bar{M}$), and let $p(x) \in S_1(N)$. The operator cl_p is defined on (all) subsets of N by:

$$\mathrm{cl}_p(A) = \{a \in N \mid a \not\models p|A\}.$$

Remark 3. (i) Let $p \in S_1(\bar{M})$ be $\emptyset$-invariant and $\phi(x) \in p_0(x) = p(x)|\emptyset$. Then regularity of p can be expressed in terms of cl_p:

- $(p(x), x = x)$ is invariant-strongly regular iff $p|A \vdash p|\mathrm{cl}_p(A)$ for any small A.

- $(p(x), \phi(x))$ is invariant-strongly regular iff $p|A \vdash p|A \cup (\mathrm{cl}_p(A) \cap \phi(\bar{M}))$ for any small A .

- p is invariant-regular iff $p|A \vdash p|A \cup (\mathrm{cl}_p(A) \cap p_0(\bar{M}))$ for any small A. If we consider cl_p only as an operation on subsets of $p_0(\bar{M})$ we have that p is invariant-regular just if $p|A \vdash p|\mathrm{cl}_p(A)$ for all $A \subset p_0(\bar{M})$.

(ii) cl_p satisfies the monotonicity and has finite character, but in general it is not a closure operator. $\mathrm{cl}_p(\mathrm{cl}_p(\emptyset)) = \bar{M}$ can easily happen with $RM(p) = 2$ (and T is ω-stable).

Lemma 2. *Suppose $p \in S_1(\bar{M})$ is $\emptyset$-invariant and let $p_0 = p|\emptyset$.*

(i) If p is invariant-regular then:

- *(the restriction of)* cl_p *is a closure operator on $p_0(\bar{M})$.*

- cl_p *is a pregeometry operator on $p_0(\bar{M})$ iff every Morley sequence in p over $\emptyset$ is totally indiscernible.*

(ii) $(p(x), x = x)$ is strongly regular iff cl_p *is a closure operator on $\bar{M}$.*

(iii) Suppose that $(p(x), x = x)$ is strongly regular. Then cl_p *is a pregeometry operator on $\bar{M}$ iff every Morley sequence in p over $\emptyset$ is totally indiscernible.*

Proof. (i) Suppose that p is invariant-regular and consider cl_p only as an operation on subsets of $p_0(\bar{M})$. Let $A \subset p_0(\bar{M})$ and let $a \models p_0$. Then:

$$a \notin \mathrm{cl}_p(A) \text{ iff } a \models p|A \text{ iff } a \models p|\mathrm{cl}_p(A) \text{ iff } a \notin \mathrm{cl}_p(\mathrm{cl}_p(A));$$

The first and the last equivalence follow from the definition of cl_p, and the middle one is by Remark 3(i). Thus $\mathrm{cl}_p(A) = \mathrm{cl}_p(\mathrm{cl}_p(A))$. The other clause is proved as in (iii) below.

(ii) As in (i), for any $A \subset \bar{M}$ we have:

$$a \notin \mathrm{cl}_p(A) \text{ iff } a \models p|A \text{ iff } a \models p|\mathrm{cl}_p(A) \text{ iff } a \notin \mathrm{cl}_p(\mathrm{cl}_p(A)),$$

and cl_p is a closure operator on $\bar{M}$.

(iii) Suppose that every Morley sequence in p over $\emptyset$ is symmetric. To show that cl_p is a pregeometry operator, by part (ii), it suffices to verify the exchange property over finite $A \subset M$. Since, by (ii), cl_p is a closure operator on M there is a Morley sequence in p (over $\emptyset$) $(a_1, ..., a_n) \in A^n$ such that $\mathrm{cl}_p(A) = \mathrm{cl}_p(a_1, .., a_n)$. Now, let $a \models p|A$ and let $b \in \bar{M}$. Note that $(a_1, ..., a_n, a)$ is a Morley sequence and that $\mathrm{cl}_p(aA) = \mathrm{cl}_p(a_1, ..., a_n, a)$. We have:

$$b \notin \mathrm{cl}_p(aA) \text{ iff } b \notin \mathrm{cl}_p(a_1, ..., a_n, a) \text{ iff } (a_1, ..., a_n, a, b) \text{ is a Morley sequence}$$
$$\text{iff } (a_1, ..., a_n, b, a) \text{ is a Morley sequence iff } a \notin \mathrm{cl}_p(Ab).$$

The exchange follows. The other direction is similar. □

Theorem 1. *Suppose that $p(x) \in S_1(\bar{M})$ is $\emptyset$-invariant and regular. Then exactly one of the following cases holds:*

(1) p is symmetric, in which case cl_p is a pregeometry operator on $(p|\emptyset)(\bar{M})$, and if $(p(x), \phi(x))$ is strongly regular then cl_p is a pregeometry operator on $\phi(\bar{M})$.

(2) p is asymmetric, in which case there exists a finite A and a A-definable partial order $\leq$ such that every Morley sequence in p over A is strictly increasing.

Proof. If p is symmetric then every Morley sequence in p over $\emptyset$ is totally indiscernible and (1) follows from Lemma 2(i). So suppose that p is asymmetric. then there exists an asymmetric Morley sequence in p over some finite A', and let $(c_1, c_2, ..., c_n, a, b)$ be the shortest possible and let $A = A'\bar{c}$. By $\emptyset$-invariance we have $\mathrm{tp}(a, b/A) \neq \mathrm{tp}(b, a/A)$. Then (a, b) is an asymmetric Morley sequence over A so let $\phi(x, y) \in \mathrm{tp}(a, b/A)$ be asymmetric: $\models \phi(x, y) \rightarrow \neg\phi(y, x)$. Then from $a \in \mathrm{cl}_p(bA)$ and $b \notin \mathrm{cl}_p(aA)$ it follows that $\phi(a, x)$ is large while $\phi(x, b)$ is small. By invariance, $\phi(x, a)$ is small, too. We claim that

$$(p|A)(t) \cup \{\phi(t, a)\} \cup \{\neg\phi(t, b)\}$$

is inconsistent. Otherwise, there is d realizing $p|A$ such that $\models \phi(d, a) \wedge \neg\phi(d, b)$. Then d does not realize $p|(A, a)$ (witnessed by $\phi(x, a)$) so, by regularity, $p|(A, a) \vdash p|(A, a, d)$ and thus $b \models p|(A, a, d)$. In particular $b \models p|(A, d)$ and since, by invariance, $\phi(d, x)$ is large we conclude $\models \phi(d, b)$. A contradiction.

From the claim, by compactness, we find $\theta(t) \in p|A$ such that

$$\models (\forall t)(\phi(t,a) \wedge \theta(t) \rightarrow \phi(t,b)).$$

Let $x \preccurlyeq y$ be $(\forall t)(\phi(t,x) \wedge \theta(t) \rightarrow \phi(t,y))$. Clearly, $\preccurlyeq$ defines a quasi order and $a \preccurlyeq b$. Also:

$$\models \phi(a,b) \wedge \theta(a) \wedge \neg\phi(a,a);$$

The first conjunct follows by our choice of ϕ, the second from $a \models p|A$, and the third from the asymmetry of ϕ. Altogether they imply $b \not\preccurlyeq a$. Thus if $x < y$ is $x \preccurlyeq y \wedge y \not\preccurlyeq x$ we have $a < b$. $\square$

The next examples concern issues of whether symmetric regular types are definable or even generically stable. But we first give a case there this is true (although it depends formally on Theorem 3 of the next section).

Corollary 1. *Suppose that $(p(x), x = x)$ is invariant-strongly regular and symmetric. Then $p(x)$ is generically stable.*

Proof. If $(p(x), x = x)$ is invariant-strongly regular then, by Theorem 1(1) cl_p is a pregeometry operator on $\bar{M}$, and then $p(x)$ is generically stable by Theorem 3(ii). $\square$

Example 1. A symmetric, definable - strongly regular type which is not generically stable.

Let $L = \{U, V, E\}$ where U, V are unary and E is a binary predicate. Consider the bipartite graph (M, U^M, V^M, E^M) where $U^M = \omega$, V^M is the set of all finite subsets of ω, $M = U^M \cup V^M$, and $E^M = \{(u,v) : u \in U^M,\ v \in V^M, \text{ and } u \in v\}$. Let $A \subset M$ be finite. Then:

If $(c_1, ..., c_n), (d_1, ..., d_n) \in (U^M)^n$ have the same quantifier-free type over A then $\mathrm{tp}(c_1, ..., c_n/A) = \mathrm{tp}(d_1, ..., d_n/A)$,

since the involution of ω mapping c_i's to d_i's respectively, and fixing all the other elements of ω is an A-automorphism of M. Note that this is expressible by a set of first-order sentences, so is true in the monster.

Further, if $e_1, ..., e_n \in U^M$ are distinct and have the same type over A then, since every permutation of ω which permutes $\{e_1, ..., e_n\}$ and fixes all the other elements of ω is an A-automorphism of M, $(e_1, ..., e_n)$ is totally indiscernible over A. This is also expressible by a set of first-order sentences.

Let $p(x) \in S_1(M)$ be the type of a "new" element of U which does not belong to any element of V^M. Then, by the above, p is definable, its global heir $\bar{p}$ is symmetric, and $(\bar{p}(x), U(x))$ is strongly regular.

Example 2. A symmetric invariant - strongly regular type which is not definable.
Consider the bipartite graph (M, U^M, V^M, E^M) where $U^M = \omega$, V^M consists of all finite and co-finite subsets of ω, $M = U^M \cup V^M$, and E^M is $\in$.

Let $\bar{M}$ be the monster and let $p(x) \in S_1(\bar{M})$ be the type of a new element of $U^{\bar{M}}$, which belongs to all co-finite members of $V^{\bar{M}}$ (and no others). Arguing as in the previous example $(p(x), U(x))$ is strongly regular and symmetric. Since 'being a co-finite subsets of U^M' is not definable, p is not definable.

Definition 5. Let G be a definable (or even type-definable) group in $\bar{M}$. G is called an *invariant-regular group* if for some global type $p(x) \in S_G(\bar{M})$, $(p(x)$, "$x \in G$") is invariant-strongly regular.

We will see in Example 3 that asymmetric invariant-regular groups, and even fields, exist.

Theorem 2. *Suppose that G is a definable, invariant-regular group, witnessed by $p(x) \in S_G(\bar{M})$. Then:*

(i) $p(x)$ is both left and right translation invariant (and in fact invariant under definable bijections).

(ii) A formula $\phi(x)$ is in $p(x)$ iff two left (right) translates of $\phi(x)$ cover G iff finitely many left (right) translates of $\phi(x)$ cover G. (Hence $p(x)$ is the unique generic type of G.)

(iii) $p(x)$ is definable over $\emptyset$; in particular, G is a definable-regular group.

(iv) $G = G^0$ (i.e. G is connected).

Proof. (i) Suppose that $f : G \longrightarrow G$ is a B-definable bijection and $a \models p|B$. Since $p|B \vdash p|(B, f(a))$ is not possible, by strong regularity, we get $f(a) \models p|B$. Thus p is invariant under f.

(ii) Suppose that $D \subseteq G$ is defined by $\phi(x) \in p(x)$ which is over A. Let $g \models p|A$ and we show $G = D \cup g \cdot D$. If $b \in G \setminus D$ then b does not realize $p|A$ so, by strong regularity, $g \models p|(A, b)$. By (i) $g^{-1} \models p|(A, b)$, thus $g^{-1} \in D \cdot b^{-1}$ and $b \in g \cdot D$. This proves $G = D \cup g \cdot D$.

For the other direction, if finitely many translates of $\psi(x)$ cover G then at least one of them belongs to $p(x)$ and, by (i), $\psi(x) \in p(x)$.

(iii) and (iv) follow immediately from (ii). □

Question. Is every invariant-regular group commutative?

4 Homogeneous pregeometries

If (M, cl) is a pregeometry then, as usual, we obtain notions of independence and dimension: for $A, B \subset M$ we say that A is independent over B if $a \notin \mathrm{cl}(A \setminus \{a\} \cup B)$ for all $a \in A$. Given A and B, all subsets of A which are independent over B and maximal such, have the same cardinality, called $\dim(A/B)$. (M, cl) is infinite-dimensional if $\dim(M/\emptyset) \geq \aleph_0$.

Remark 4. (i) If $\bar{c}$ is a tuple of length n then $\dim(\bar{c}/B) \leq n$ for any B.

(ii) If $l(\bar{c}) = n$, $|A| \geq n+1$ and A is independent over B then there is $a \in A$ such that $a \notin \mathrm{cl}(B \cup \bar{c})$.

Definition 6. We call an infinite-dimensional pregeometry (M, cl) *homogeneous* if for any finite $B \subset M$, the set of all $a \in M$ such that $a \notin \mathrm{cl}(B)$ is the set of realizations in M of a complete type $p_B(x)$ over B.

Note that Definition 6 relates in some way the closure operation to the first-order structure. But note that it does not say anything about automorphisms, and nothing is being claimed about the homogeneity of M as a first-order structure.

Lemma 3. *Suppose* (M, cl) *is a homogeneous pregeometry.*

(i) $p_{\mathrm{cl}}(x) = \bigcup\{p_B(x) : B$ *finite subset of* $M\}$ *is a complete 1-type over* M*, which we call the generic type of the pregeometry* (M, cl).

(ii) $a \notin \mathrm{cl}(B)$ *iff* $a \models p_{\mathrm{cl}}(x)|B$. *In particular* $\mathrm{cl} = \mathrm{cl}_{p_{\mathrm{cl}}}$.

(iii) $I = (a_i : i < \omega)$ *is independent over* B *iff* $a_i \models p_{\mathrm{cl}}|(B, a_0, ..., a_{i-1})$ *for all* i*. In particular, if* $M = \bar{M}$ *and* p *is* B*-invariant then* I *is independent over* B *iff it is a Morley sequence in* p *over* B.

Proof. (i) Consistency is by compactness: given $A_1, ..., A_n$ finite subsets of M and $B = A_1 \cup ... \cup A_n$, clearly $p_{A_1}(x) \cup ... \cup p_{A_n}(x) \subseteq p_B(x)$ and the later is consistent. Completeness is clear.

(ii) and (iii) are easy. □

Lemma 4. *Suppose* (M, cl) *is a homogeneous pregeometry. Let* $(a_i : i \in \omega)$ *be an* $\emptyset$*-independent subset of* M*. Then for any* L*-formula* $\phi(x, \bar{y})$ *with* $l(\bar{y}) = n$*, and* n*-tuple* $\bar{b}$ *from* M:

$\phi(x, \bar{b}) \in p_{\mathrm{cl}}(x)$ *iff* $\models \wedge_{i \in w} \phi(a_i, b)$ *for some* $w \subset \{1, ...2n\}$, $|w| = n+1$.

In particular $p_{\mathrm{cl}}(x)$ *is definable.*

Proof. If $\phi(x,\bar{b})$ is large (namely in p_{cl}), then its negation is small, and thus if $M \models \neg\phi(a,\bar{b})$ then $a \in \mathrm{cl}(\bar{b})$. By Remark 4(ii), at most n many a_i's can satisfy $\neg\phi(x,\bar{b})$, hence at least $n+1$ among the first $2n+1$ a_i's satisfy $\phi(x,\bar{b})$. Conversely, if at least $n+1$ a_i's satisfy $\phi(x,\bar{b})$ then, again by Remark 4(ii), $\phi(x,\bar{b})$ can not be small, so it is large. $\square$

Proposition 2. *Suppose* (M,cl) *is a homogeneous pregeometry. Let* $p(x)$ *be the generic type and let* $\bar{p}(x)$ *be its (unique by definability) global heir.*

(i) $(\bar{M},\mathrm{cl}_{\bar{p}})$ *is a homogeneous pregeometry and* cl *is the restriction of* $\mathrm{cl}_{\bar{p}}$ *to* M.

(ii) If $(a_1,...a_n)$ *(from* $\bar{M}$*) is independent over* A *then:*

$$\mathrm{tp}(b_1,...,b_n/A) = \mathrm{tp}(a_1,...,a_n/A) \quad \textit{iff} \quad (b_1,...,b_n) \textit{ is independent over } A.$$

(iii) $\bar{p}(x)$ *is* $\emptyset$*-invariant and generically stable.*

(iv) $(\bar{p}(x), x=x)$ *is strongly regular*

Proof. (i) is an easy exercise, using the fact that $\bar{p}$ is defined by the same schema which defines p.

(ii) We prove it by induction on n. For $n=1$, by definition, we have $a_1, b_1 \models \bar{p}|A$. Now assume true for n and prove for $n+1$. Without loss of generality $A=\emptyset$. Suppose first that $\mathrm{tp}(b_1,...,b_{n+1}) = \mathrm{tp}(a_1,...,a_{n+1})$. Let a' realize $p|(a_1,...,a_{n+1},b_1,...,b_{n+1})$. So $\mathrm{tp}(a_1,...,a_n,a_{n+1}) = \mathrm{tp}(a_1,...,a_n,a')$. On the other hand, by the induction assumption (over $\emptyset$), $(b_1,...,b_n)$ is independent, so independent over a' (by symmetry). By induction assumption applied over a', $\mathrm{tp}(b_1,...,b_n,a') = \mathrm{tp}(a_1,...,a_n,a')$. Hence $\mathrm{tp}(b_1,...,b_n,a') = \mathrm{tp}(b_1,...,b_n,b_{n+1})$. As $a' \notin \mathrm{cl}_{\bar{p}}(b_1,...,b_n)$, also $b_{n+1} \notin \mathrm{cl}_{\bar{p}}(b_1,...,b_n)$. Thus $(b_1,...b_n,b_{n+1})$ is independent.

The converse (if $(b_1,...,b_{n+1})$ is independent then it realizes $\mathrm{tp}(a_1,...,a_{n+1})$) is proved in a similar fashion and left to the reader.

(iii) By part (i), Lemma 4 also applies to the pregeometry $\mathrm{cl}_{\bar{p}}$. Let $(a_i : i \in \omega)$ be $\mathrm{cl}_{\bar{p}}$-independent. Then $\bar{p}(x)$ is defined over $(a_i : i \in \omega)$ as in Lemma 4. But if $(b_i : i \in \omega)$ has the same type as $(a_i : i \in \omega)$ then, by (ii), it is also $\mathrm{cl}_{\bar{p}}$-independent, hence $\bar{p}(x)$ is defined over $(b_i : i \in \omega)$ in the same way. this implies that $\bar{p}$ is $\emptyset$-invariant. Thus, by Lemma 3(iii) a Morley sequence in $\bar{p}$ is the same thing as an infinite $\mathrm{cl}_{\bar{p}}$-independent set. By Lemma 4 and Definition 1, $\bar{p}$ is generically stable.

(iv) By part (i) $(\bar{M}, \mathrm{cl}_{\bar{p}})$ is a pregeometry so, by Lemma 2(ii), $(\bar{p}(x), x = x)$ is strongly regular. $\square$

We now drop (for a moment) all earlier assumptions and summarize the situation:

Theorem 3. *Let T be an arbitrary theory.*

(i) Let $p(x)$ be a global $\emptyset$-invariant type such that $(p(x), x = x)$ is strongly regular. Then $(\bar{M}, \mathrm{cl}_p)$ is a homogeneous pregeometry.

(ii) On the other hand, suppose $M \models T$ and (M, cl) is a homogeneous pregeometry. Then there is a unique global $\emptyset$-invariant generically stable type $p(x)$ such that $(p(x), x = x)$ is strongly regular, and such that the restriction of cl_p to M is precisely cl.

We end this section by pointing out the connection to exponential fields, as mentioned to us by Kirby. In [3], Jonathan Kirby proved that $ecl(-)$, "exponential algebraic closure", as originally defined by Macintyre, gives a pregeometry on *any* exponential field, and this result extends those of Wilkie [7] for the complex exponential field. It is an open question whether for the complex exponential field, $ecl(-)$ is *homogeneous* in the sense of Definition 6 above. A positive answer would yield quasiminimality for the complex exponential field as well as generic stability and strong regularity of its (unique) exponentially transcendental type. On the other hand, a positive answer does exist for Zilber's pseudoexponentiation and some other exponential fields.

5 Quasiminimal structures

Recall that a 1-sorted structure M in a countable language is called quasiminimal if M is uncountable and every definable (with parameters) subset of M is countable or co-countable; the definition was given by Zilber in [8]. Here we investigate the general model theory of quasiminimality, continuing an earlier work by Itai, Tsuboi and Wakai [2].

Throughout this section fix a quasiminimal structure M and its monster model $\bar{M}$. The set of all formulas (with parameters) defining a co-countable subset of M forms a complete 1-type $p(x) \in S_1(M)$; we will call it the generic type of M. If $p(x)$ happens to be definable we will denote its (unique) global heir by $\bar{p}(x)$.

Definition 7. (i) $p(x)$ (or M) is *countably based* if there is a countable $A \subset M$ such that p does not split over A; i.e. whenever $\bar{b}_1 \equiv \bar{b}_2(A)$ we have $(\phi(x, \bar{b}_1) \leftrightarrow \phi(x, \bar{b}_2)) \in p(x)$ for all $\phi(x, \bar{y})$ over A.

(ii) If p does not split over A then we say that $(a_1, ..., a_n)$ is a weak Morley sequence in p over A if a_k realizes $p|(A, a_1, ..., a_{k-1})$ for all relevant k.

As in the case of global invariant types weak Morley sequences are indiscernible over A.

Remark 5. (i) In quasiminimal structures Zilber's countable closure operator ccl is defined via cl_p :

$\mathrm{cl}_p^0(A) = \mathrm{cl}_p(A)$, $\mathrm{cl}_p^{n+1}(A) = \mathrm{cl}_p(\mathrm{cl}_p^n(A))$ and $\mathrm{ccl}(A) = \bigcup_{n \in \omega} \mathrm{cl}_p^n(A)$.

(ii) If A is countable then $\mathrm{ccl}(A)$ is countable, too. ccl is a closure operator on M.

(iii) cl_p is a closure operator iff $\mathrm{cl}_p = \mathrm{ccl}$ (which is in general not the case, see Example 3).

Lemma 5. *Suppose that $p(x)$ is countably based, witnessed by A. Then*

(i) cl_p *is a closure operator on M.*

(ii) (M, cl_p) *is a pregeometry iff every weak Morley sequence in p over A is totally indiscernible.*

Proof. (i) Without loss of generality suppose that $A = \emptyset$. Assuming that $\mathrm{cl}_p \neq \mathrm{ccl}$ we will find a non-indiscernible weak Morley sequence over some countable $C \subset M$, which is in contradiction with $\emptyset$-invariance. So suppose that $\mathrm{cl}_p \neq \mathrm{ccl}$. Then there are a (countable) $C \subset M$ and $a \in M$ such that $a \in \mathrm{cl}_p^2(C) \setminus \mathrm{cl}_p(C)$. Since $a \notin \mathrm{cl}_p(C)$ we have $a \models p|C$ so let $a_1, a_2 \notin \mathrm{ccl}(aC)$ be such that (a, a_1, a_2) is a weak Morley sequence over C. We will prove that it is not indiscernible.

Witness $a \in \mathrm{cl}_p^2(C)$ by a small formula $\varphi(x, \bar{b}) \in \mathrm{tp}(a/C\bar{b})$ such that $\varphi(x, y_1, ..., y_n)$ is over C and $(b_1, ..., b_n) = \bar{b} \in \mathrm{cl}_p(C)^n$. Choose $\theta_i(y_i) \in \mathrm{tp}(b_i/C)$ witnessing $b_i \in \mathrm{cl}_p(C)$ and let $x_1 \equiv_\varphi x_2$ denote the formula

$$(\forall \bar{y}) \left(\bigwedge_{1 \leq i \leq n} \theta_i(y_i) \rightarrow (\varphi(x_1, \bar{y}) \leftrightarrow \varphi(x_2, \bar{y})) \right).$$

It is, clearly, over C and we show $a \not\equiv_\varphi a_2$: from $\models \varphi(a, \bar{b})$ (witnessing $a \in \mathrm{cl}_p^2(C)$) and $a_2 \notin \mathrm{ccl}(C)$ we derive $\models \neg\varphi(a_2, \bar{b})$ and thus $a \not\equiv_\varphi a_2$. On the other hand, since all realizations of θ_i's are in $\mathrm{cl}_p(C)$, and since $\mathrm{tp}(a_1/\mathrm{cl}_p(C)) = \mathrm{tp}(a_2/\mathrm{cl}_p(C))$, we have $a_1 \equiv_\varphi a_2$. Therefore (a, a_1, a_2) is not indiscernible.

(ii) Having proved (i), the proof of Lemma 2(iii) goes through. $\square$

Let us note that if (M, cl_p) *is* a pregeometry, then as infinite-dimensionality and homogeneity are automatic (for quasiminimal M), we can apply Theorem 3(ii).

Theorem 4. *If $p(x)$ does not split over $\emptyset$ then exactly one of the following two holds:*

(1) Every weak Morley sequence in p over $\emptyset$ is totally indiscernible; in this case cl_p is a pregeometry operator on M, p is definable, $\bar{p}$ is generically stable and $(\bar{p}(x), x = x)$ is strongly regular.

(2) There exists an asymmetric weak Morley sequence in p (over some domain); then for some finite A there is an A-definable partial order $\leq$ such that every weak Morley sequence in p is strictly increasing.

Proof. First suppose that every weak Morley sequence in p over $\emptyset$ is symmetric. Then, by Lemma 5(i), cl_p is a closure operator and, by Lemma 2(ii), it is a pregeometry operator. Thus (M, cl_p) is a homogeneous pregeometry and the conclusion follows from Proposition 2.

Now suppose that there is an asymmetric weak Morley sequence in p over some finite A. By invariance it is indiscernible, so after adding an initial part to A we get a weak Morley sequence (a, b) over A which is not symmetric. Let $\phi(x, y) \in \mathrm{tp}(ab/A)$ be asymmetric ($\models \phi(x, y) \to \neg\phi(y, x)$). Then $\phi(a, x)$ is large and $\phi(x, b)$ is small; by invariance $\phi(x, a)$ is small, too. Every weak Morley sequence of length 2 satisfies this conditions, so (a, b) can be found such that each of a, b realizes $p|\mathrm{ccl}(A)$, and there is a countable, ccl-closed $M_0 \prec M$ containing A such that $a \in M_0$ and $b \notin M_0$ (i.e b realizes $p|M_0$). We claim:

$$\models (\forall t)(\phi(t, a) \to \phi(t, b)).$$

Let $d \in M$ be such that $\models \phi(d, a)$. Then $d \in \mathrm{ccl}(Aa)$, because $\phi(x, a)$ is small, and so $d \in M_0$ (M_0 is ccl-closed). Now, if $d \in \mathrm{ccl}(A)$ then $a \equiv b(\mathrm{ccl}(A))$ implies $\models \phi(d, b)$. Otherwise $d \models p|\mathrm{ccl}(A)$ so, since $p|M_0$ does not split over $\emptyset$, we have $(a, b) \equiv (d, b)(A)$. In particular $\models \phi(d, b)$. In both cases we have $\models \phi(d, b)$ proving the claim.

Let $x \preccurlyeq y$ be $(\forall t)(\phi(t, x) \wedge \theta(t) \to \phi(t, y))$. Clearly, $\preccurlyeq$ defines a quasi order and, as in the proof of Theorem 1, we have $\models \phi(a, b) \wedge \theta(b) \wedge \neg\phi(a, a)$ and $a < b$. ☐

Remark 6. If in the above Theorem, M is of cardinality at least $\aleph_2$, then case (1) holds. Because, using Lemma 5 (and the assumption that p is countably based), cl_p is a closure operator on M, and this suffices for the proof of Proposition 2.8 of [2] to go through.

Theorem 5. *Suppose that M is a quasiminimal group. Then $p(x)$ is definable over $\emptyset$, both left and right translation invariant, and $\bar{M}$ is a connected, definable-regular group witnessed by $\bar{p}(x)$.*

Proof. Let $X \subseteq M$ be definable. First we claim that X is uncountable iff $X \cdot X = M$. If X is uncountable, then X is co-countable, as is X^{-1}. So for any $a \in M$, $a \cdot X^{-1}$ is co-countable, so has nonempty intersection with X. If $d \in X \cap a \cdot X^{-1}$ then $a \in X \cdot X$, proving the claim.

It follows that $p(x)$ is definable over $\emptyset$. In particular, it is countably based and, by Lemma 5, cl_p is a closure operator on M. Then $\mathrm{cl}_{\bar{p}}$ is also a closure operator and $(p(x), x = x)$ is strongly regular by Lemma 2(ii). The rest follows from Theorem 6. $\square$

Example 3. An asymmetric quasiminimal field.
In fact, every strongly minimal structure of size $\aleph_1$ can be expanded to become asymmetric quasiminimal. Let $I = \omega_1 \times Q$ and let $\lhd$ be a (strict) lexicographic order on I. Further, let $B = \{b_i \,|\, i \in I\}$ be a maximal acl-independent subset of M. For each $a \in M$ let $i \in I$ be $\lhd$-maximal for which there are $i_1, ..., i_n \in I$ such that $a \in \mathrm{acl}(b_{i_1}, ..., b_{i_n}, b_i) \setminus \mathrm{acl}(b_{i_1}, ..., b_{i_n})$; Clearly, $i = i(a)$ is uniquely determined. Now, expand $(M, ..)$ to $(M, <, ...)$ where $b < c$ iff $i(b) \lhd i(c)$. We will prove that $(M, <, ...)$ is quasiminimal. Suppose that $M_0 \prec M$ is a countable, $<$-initial segment of M and that $B \setminus M_0$ does not have $\leq$-minimal elements, and let $a, a' \in M \setminus M_0$. Then there is an automorphism of $(B, <)$ fixing $B \cap M_0$ pointwise and moving $b_{i(a)}$ to $b_{i(a')}$. It easily extends to an M_0-automorphism of $(M, <, ...)$, so $b_{i(a)} \equiv b_{i(a')}(M_0)$ (in the expanded structure). Note that replacing $b_{i(a)}$ by a in B (in the definition of $<$) does not affect $<$, so $a \equiv a'(M_0)$ and there is a single 1-type over M_0 realized in $M \setminus M_0$. Since every countable set is contained in an M_0 as above, $(M, <, ...)$ is quasiminimal.

Question Is every quasiminimal field algebraically closed?

The following is an example of a quasiminimal structure, whose quasiminimality does not look like regularity at all: $\mathrm{cl}_p(A) \neq \mathrm{cl}_p(\mathrm{cl}_p(A))$ for arbitrarily large countable A's.

Example 4. (A quasiminimal structure where $\mathrm{cl}_p \neq \mathrm{ccl}$)
Peretyatkin in [4] constructed an $\aleph_0$-categorical theory of a 2-branching tree. Our quasiminimal structure will be its model.

The language consists of a single binary function symbol $L = \{\inf\}$. Let Σ be the class of all finite L-structures $(A, \inf)$ satisfying:

(i) $(A, \inf)$ is a semilattice;

(ii) $(A, <)$ is a tree (where $x < y$ iff $\inf(x, y) = x \neq y$);

(iii) (2-branching) No three distinct, pairwise $<$-incompatible elements satisfy: $\inf(x, y) = \inf(x, z) = \inf(y, z)$.

Then the Fraissé limit of Σ exists and its theory, call it T_2, is $\aleph_0$-categorical and has unique 1-type. If we extend the language to $\{\inf, <, \perp\}$, where $x \perp y$ stands for $x \nleq y \wedge y \nleq x$, then T_2 has elimination of quantifiers.

Let $(\bar{M}, <)$ be the monster model of T_2, let $\lhd$ be a lexicographic order on $I = \omega_1 \times Q$, and let $C = \{c_i \mid i \in \omega_1 \times Q\}$ be $<$-increasing. Then we can find a sequence of countable models $\{M_i \mid i \in \omega_1 \times Q\}$ satisfying:

(1) $M_i \prec M_j$ for all $i \lhd j$;

(2) $M_i \cap C = \{c_j \in C \mid j \unlhd i\}$ for all i;

(3) $M_i \cap C < M_j \setminus M_i$ for all $i \lhd j$.

Finally, let $M = \bigcup\{M_i \mid i \in I\}$. Clearly, C is an uncountable branch in M. Moreover, by (3), any other branch is completely contained in some M_i, and is so countable. This suffices to conclude that M is quasiminimal and that the generic type is determined by $C < x$.

Fix $c_i \in C$ and $a \in M \setminus C$ with $c_i < a$. Note that $x \nleq c_i$ is small, so $M_j \subset \mathrm{cl}_p(c_i)$ for all $j \lhd i$. Also, $x \perp a$ is large so $\mathrm{cl}_p(a)$ is the union of branches containing a. Since $c_i \in \mathrm{cl}_p(a)$ then $M_j \subseteq \mathrm{cl}_p(c_i) \subset \mathrm{cl}_p^2(a)$. Since $M_j \nsubseteq \mathrm{cl}_p(a)$ we conclude that $\mathrm{cl}_p(a) \neq \mathrm{cl}_p^2(a)$ and cl_p is not a closure operator. Similarly, for any countable $A \subset M$ we can find a, c_i as above much bigger than A, and thus both realizing $p|A$. Then $x \perp a \wedge \neg(x \perp c_i) \in p(x)$ witness that $p(x)$ is not countably invariant.

6 Local regularity

In this section we study conditions under which a type $p(x) \in S_1(M)$ (where M is not necessarily saturated) has a global, strongly regular, M-invariant extension. From the definition of strong regularity and Remark 2 $p(x)$ has to satisfy the following:

Definition 8. A non-isolated type $p(x) \in S_1(A)$ is *locally strongly regular via* $\phi(x) \in p(x)$ if $p(x)$ has a unique extension over $A\bar{b}$ whenever $\bar{b}$ is a finite tuple of realizations of $\phi(x)$ no element of which realizes p.

Proposition 3. *Suppose that* $p(x) \in S_1(M)$ *is definable and locally strongly regular via* $\phi(x) \in p(x)$*, and let* $\bar{p}(x)$ *be its global heir. Then* $(\bar{p}(x), \phi(x))$ *is definable-strongly regular.*

Proof. Suppose that $(\bar{p}(x), \phi(x))$ is not strongly regular and let $B \supset M$ be such that $\bar{p}|B \nvdash \bar{p}|\mathrm{cl}_{\bar{p}}(B)$. Then, without loss of generality, $B = M\bar{b}$ and there are $a \in \mathrm{cl}_{\bar{p}}(B) \cap \phi(\bar{M})$ and $c \models \bar{p}|B$ such that c does not realize $\bar{p}|Ba$. Witness $a \in \mathrm{cl}_{\bar{p}}(B)$ by $\theta(y, \bar{z})$ which is over M, implies $\phi(y)$, and $\models \theta(a, \bar{b}) \wedge \neg(d_p\theta)(\bar{b})$ (where d_p is the defining schema of p). Similarly, find $\varphi(x, y, \bar{z})$ over M such that $\models \varphi(c, a, \bar{b}) \wedge \neg(d_p\varphi)(a, \bar{b})$.

$$\models (\exists y)(\theta(y, \bar{b}) \wedge \neg(d_p\theta)(\bar{b}) \wedge \varphi(c, y, \bar{b}) \wedge \neg(d_p\varphi)(y, \bar{b})).$$

Since $\mathrm{tp}(c/M\bar{b})$ is an heir of $p(x)$ there is $\bar{m} \in M$ and a' such that

$$\models \theta(a', \bar{m}) \wedge \neg(d_p\theta)(\bar{m}) \wedge \varphi(c, a', \bar{m}) \wedge \neg(d_p\varphi)(a', \bar{m}).$$

The first two conjuncts witness $a' \in \phi(\bar{M}) \setminus p(\bar{M})$ while the last two witness that c is not a realization of $\bar{p}|Ma'$. A contradiction. □

For the next proposition, recall that if $p(x) \in S_1(M)$ then by a coheir sequence in p we mean a Morley sequence in p' over M for some global coheir of p.

Proposition 4. *Suppose that* $p(x) \in S_1(M)$ *is locally strongly regular via* $x = x$ *and that there exists an infinite, totally indiscernible (over* M*) coheir-sequence in* p*. Then* p *is definable, its global heir* $\bar{p}$ *is generically stable and* $(\bar{p}(x), x = x)$ *is strongly regular.*

Proof. Let $I = \{a_i : i \in \omega\}$ be a symmetric coheir sequence in p, let $p_n(x) = \mathrm{tp}(a_{n+1}/|Ma_1...a_n)$ and let $p_I(x) = \cup_{n\in\omega} p_n(x) \in S_1(MI)$. We will prove that $p_I(x)$ is locally strongly regular via $x = x$, then, by standard arguments, it follows that p has a global coheir $\bar{p}$ which extends p_I such that $(\bar{p}(x), x = x)$ is invariant-strongly regular and symmetric; the conclusion follows by Theorem 1.

Suppose, on the contrary, that p_I is not locally strongly regular. Then some p_n is not locally strongly regular, so there are $b_1...b_k = \bar{b}$, with none realizing $p_n(x)$, such that p_n has at least two extensions in $S_1(M\bar{a}\bar{b})$ (here $\bar{a} = a_1...a_n$). Let $\varphi(x, \bar{z}, \bar{y})$ be over M and such that both $\varphi(x, \bar{a}, \bar{b})$ and $\neg\varphi(x, \bar{a}, \bar{b})$ are consistent with $p_n(x)$.

Choose $\theta_i(y_i, \bar{a}) \in \mathrm{tp}(b_i/M\bar{a})$ witnessing that b_i does not realize p_n and let $\phi(x_1, x_2, \bar{a})$ be

$$(\exists \bar{y})\ \big(\bigwedge\nolimits_{1\le i\le n}(\theta_i(y_i, \bar{a}) \wedge \neg\theta_i(x_2, \bar{a})) \wedge \neg(\varphi(x_1, \bar{a}, \bar{y}) \leftrightarrow \varphi(x_2, \bar{a}, \bar{y}))\big).$$

It is, clearly, over M and we show $\models \phi(a_{n+2}, a_{n+1}, \bar{a})$. By our assumptions on φ and $\bar{b}$, there is $\bar{b}' \equiv \bar{b}(M\bar{a})$ such that $\models \neg(\varphi(a_{n+1}, \bar{a}, \bar{b}) \leftrightarrow \varphi(a_{n+1}, \bar{a}, \bar{b}'))$. Also $\models \bigwedge_{1\le i\le n}(\theta_i(b_i, \bar{a}) \wedge \theta_i(b_i', \bar{a}) \wedge \neg\theta_i(a_{n+1}, \bar{a}))$. Thus for any $c \in M$ either $\bar{b}$ or $\bar{b}'$ in place of $\bar{y}$ witnesses $\models \phi(c, a_{n+1}, \bar{a})$ and, since $\mathrm{tp}(a_{n+2}/M\bar{a}a_{n+1})$ is a coheir of p, we conclude $\models \phi(a_{n+2}, a_{n+1}, \bar{a})$.

By total indiscernibility, $\mathrm{tp}(\bar{a}/Ma_{n+1}a_{n+2})$ is a coheir of p, so there are $\bar{m} \in M$ and $\bar{d}$ such that

$$\bigwedge\nolimits_{1\le i\le n}(\theta_i(d_i, \bar{m}) \wedge \neg\theta_i(a_{n+1}, \bar{m})) \wedge \neg(\varphi(a_{n+2}, \bar{m}, \bar{d}) \leftrightarrow \varphi(a_{n+1}, \bar{m}, \bar{d})).$$

The first conjunct witnesses that each d_i does not realize p, and the second witnesses that p does not have a unique extension over $M\bar{d}$. A contradiction.

$\square$

Our next goal is to prove that the generic type of a quasiminimal structure is locally strongly regular via $x = x$. This we will do in a more general situation, for any (M and) $p \in S_1(M)$ for which the closure operator induced by cl_p (we will call it Cl_p) 'does not finitely generate M'. So, fix for now M and $p \in S_1(M)$.

$$\mathrm{Cl}_p(A) = \bigcup\{\mathrm{cl}_p^n(A) \mid n \in \omega\} \ \text{ where } \ \mathrm{cl}_p^0(A) = \mathrm{cl}_p(A),\ \ \mathrm{cl}_p^{n+1}(A) = \mathrm{cl}_p(\mathrm{cl}_p^n(A)).$$

Call $A \subseteq M$ *finitely* Cl_p*-generated* if there is finite $A_0 \subset A$ such that $A \subset \mathrm{Cl}_p(A_0)$. $\{a_i \mid i \in \alpha\}$ is a Cl_p*-free sequence over* $B \subset M$ if $a_i \notin \mathrm{Cl}_p(A_iB)$ for all $i \le \alpha$ ($A_i = (\{a_i \mid j < i\})$. Cl_p-free means Cl_p-free over $\emptyset$.

(1) $a \notin \mathrm{cl}_p(B)$ implies $a \models p \restriction B$; $a \notin \mathrm{Cl}_p(B)$ implies $a \models p \restriction \mathrm{Cl}_p(B)$.

(2) If $A = \{a_i \mid i \in \alpha\}$ is Cl_p-free then $p \restriction \mathrm{Cl}_p(A) = \bigcup_{i\in\alpha} p \restriction \mathrm{Cl}_p(A_i)$

(3) If $A = \{a_i \mid i \in \alpha\}$ is Cl_p-free and α is a limit ordinal then $p \restriction \mathrm{Cl}_p(A)$ is non-algebraic and finitely satisfiable in A.

(4) Maximal Cl_p-free sequences always exists. If $M_0 \subset M$ is not finitely Cl_p-generated and $\{a_i \,|\, i \in \alpha\} \subset M_0$ is a maximal Cl_p-free sequence such that α is minimal possible, then α is a limit ordinal (otherwise, take the last a_i, put it on the first place ...)

Proposition 5. *Suppose $p \in S_1(M)$ and M is not finitely Cl_p-generated. Then p is locally strongly regular via $x = x$.*

Proof. Suppose, on the contrary, that there are $d_1, d_2 \in p(\mathcal{M})$, a formula $\phi(x, \bar{y})$, and a tuple $\bar{b} = b_1 b_2 ... b_n \in \mathcal{M}^n$ such that none of b_i's realize p and:

$$\models \neg\phi(d_1, \bar{b}) \wedge \phi(d_2, \bar{b})).$$

Choose $\theta_i(y_i) \in \mathrm{tp}(b_i/M)$ such that $\theta_i(x) \notin p(x)$. Without loss of generality, after absorbing parameters into the language, we may assume that each $\theta_i(x)$ is over $\emptyset$. Let $I = \{a_i \,|\, i < \alpha\}$ be a maximal Cl_p-free sequence of minimal possible length. Then α is a limit ordinal and at least one of

$$\{i < \alpha \,|\, \models \phi(a_i, \bar{b})\} \ \text{ and } \ \{i < \alpha \,|\, \models \neg\phi(a_i, \bar{b})\}$$

is cofinal in α. Assume the first one is cofinal and let $I_0 = \{a_i \,|\, \models \phi(a_i, \bar{b})\}$. Then p is an I_0-type (that is, finitely satisfiable in I_0) and there is an I_0-type in $S_1(Md_1)$, it necessarily contains $\phi(x, \bar{b})$; wlog, let d_2 realizes it. Thus, both $\mathrm{tp}(d_1/M_0)$ and $\mathrm{tp}(d_2/Md_1)$ are I_0-types. We have:

$$\models (\exists \bar{y})(\textstyle\bigwedge_{1\le i\le n} \theta_i(y_i) \wedge \neg\phi(d_1, \bar{y}) \wedge \phi(d_2, \bar{y})).$$

Since $\mathrm{tp}(d_2/Md_1)$ is an I_0-type, there is $a_i \in I_0$ such that:

$$\models (\exists \bar{y})(\textstyle\bigwedge_{1\le i\le n} \theta_i(y_i) \wedge \neg\phi(d_1, \bar{y}) \wedge \phi(a_i, \bar{y})).$$

Since $\mathrm{tp}(d_1/M)$ is an I_0-type, there is $a_j \in I_0$ such that:

$$\models (\exists \bar{y})(\textstyle\bigwedge_{1\le i\le n} \theta_i(y_i) \wedge \neg\phi(a_j, \bar{y}) \wedge \phi(a_i, \bar{y})).$$

Finally, since $a_i, a_j \in M$ there is $\bar{b}' = b'_1 b'_2 ... b'_n \in M^n$ satisfying:

$$\models \textstyle\bigwedge_{1\le i\le n} (\theta_i(b'_i) \wedge \neg\phi(a_j, \bar{b}') \wedge \phi(a_i, \bar{b}')).$$

But $\bigwedge_{1\le i\le n} \theta_i(b'_i)$ implies $\bar{b}' \subset \mathrm{cl}_p(\emptyset)$ and thus $\mathrm{tp}(a_i/\mathrm{cl}_p(\emptyset)) \neq \mathrm{tp}(a_j/\mathrm{cl}_p(\emptyset))$. A contradiction. $\square$

Corollary 2. *If M is a quasiminimal structure then its generic type p is locally strongly regular via $x = x$. Moreover, whenever $M_0 \prec M_1 \prec ... \prec M$ are* ccl*-closed, then $p | \cup_{i\in\omega} M_i$ is locally strongly regular via $x = x$.*

Corollary 3. *Suppose that M is quasiminimal, p is its generic type, and that there exists an infinite* ccl*-free (or an uncountable* cl_p*-free), totally indiscernible sequence* $\subset M$*. Then p is definable, its global heir $\bar{p}$ is generically stable and $(\bar{p}(x), x = x)$ is strongly regular.*

Proof. Let $I = \{a_i : i \in \omega\}$ be a symmetric ccl-free sequence. Let $M_0 = \mathrm{ccl}(a_0)$ and $M_{i+1} = \mathrm{ccl}(M_i a_{i+1})$. Then $M_0 \prec M_1 \prec ... \prec M$ are ccl-closed and $p|M_\omega$ is almost strongly regular via $x = x$ (where $M_\omega = \cup_{i\in\omega} M_i$). Let J be an infinite indiscernible sequence over M_ω extending I. Then J is symmetric and the conclusion follows by Proposition 4 applied to $p|M_\omega$. □

Theorem 6. *Suppose that $G \subseteq M$ is a definable group and $p(x) \in S_G(M)$ is locally strongly regular via "$x \in G$". Then:*

(i) $p(x)$ is both left and right translation invariant (and in fact invariant under definable bijections).

(ii) A formula $\phi(x)$ is in $p(x)$ iff two left (right) translates of $\phi(x)$ cover G iff finitely many left (right) translates of $\phi(x)$ cover G. (Hence $p(x)$ is the unique generic type of G.)

(iii) $p(x)$ is definable over $\emptyset$ and G is connected.

(iv) $(\bar{p}(x),$ "$x \in G$"$)$ is strongly regular and $\bar{G}$ is a definable-regular group. (Here $\bar{p}$ is the unique heir of $p(x)$ and $\bar{G} \subseteq \bar{M}$ is defined by "$x \in G$").

Proof. (i) Suppose that $f : \bar{G} \longrightarrow \bar{G}$ is an M-definable bijection and $a \models p$. Since $p \vdash p|(M, f(a))$ is not possible, by local strong regularity we get $f(a) \models p$. Thus p is invariant under f.

(ii) The local strong regularity of $p(x)$ implies that whenever $g, g' \in \bar{G}$ do not realize p then $g \cdot g'$ does not realize p either. It follows that $a \cdot g \models p$ whenever $a \models p$ and $g \in \bar{G}$ does not realize p. Thus:

$$\phi(x) \in p(x) \quad \text{iff} \quad (\forall y \in \bar{G})(\neg\phi(y) \rightarrow \phi(y \cdot x)) \in p(x),$$

and $\phi(x) \in p(x)$ iff $\phi(\bar{G}) \cup a^{-1} \cdot \phi(\bar{G}) = \bar{G}$.

(iii) follows immediately from (ii), and then (iv) follows from Proposition 3. □

References

[1] E.Hrushovski, A.Pillay, On NIP and invariant measures, preprint 2009 (revised version).

[2] M.Itai, A.Tsuboi, K.Wakai, Constructions of saturated quasi-minimal structure, J.Symbolic Log. vol.69, pp. 9-22 (2004).

[3] J. Kirby, Exponential algebraicity in exponential fields, preprint 2008,

[4] M.G.Peretyatkin, On complete theories with a finite number of countable models (in Russian), Algebra i Logika vol.12(1973), pp.550-576.

[5] S. Shelah, Dependent first order theories, continued, to appear in Israel J. Math.

[6] P. Tanović, Some questions concerning minimal structures, Publications de l'Institut Mathematique, 82(96), (2007), p. 79-83.

[7] A. J. Wilkie, Some local definability theory for holomorphic functions. In *Model Theory with Applications to Algebra and Analysis, vol 1* (edited by Chatzidakis, Macpherson, Pillay, Wilkie), LMS Lecture Notes Series 349, CUP, 2008.

[8] B.Zilber, Dimensions and homogeneity in mathematical structures, in *Connections between Model Theory and Algebraic and Analytic Geometry* (edited by A. Macintyre), Quaderni di matematica, vol.6, Seconda Univ. di Napoli, 2000.