Generic stability, regularity, and quasiminimality

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September 12th, 2010

Abstract

We study (and sometimes introduce) the notions generic stability, regularity, homogeneous pregeometries, quasiminimality, and their mutual relations, in arbitrary first order theories. We prove that "infinite-dimensional homogeneous pregeometries" coincide with generically stable strongly regular types $(p(x), x = x)$. We prove that in a theory without the strict order property, regular types are generically stable. We give conditions under which a quasiminimal structure M is naturally a pregeometry, an example being when $|M| \geq \aleph_2$.

1 Introduction

The first author was motivated partly by hearing Wilkie's talks on his program for proving Zilber's conjecture that the complex exponential field is quasiminimal (definable subsets are countable or co-countable), and wondering about the first order (rather than infinitary) consequences of the approach. The second author was partly motivated by his interest in adapting

*Supported by EPSRC grant EP/F009712/1

†Supported by Ministry of Science and Technology of Serbia

his study of minimal structures (definable subsets are finite or cofinite) and his dichotomy theorems ([7]), to the quasiminimal context.

Zilber's conjecture (and the approach to it outlined by Wilkie) is closely related to the existence and properties of a canonical pregeometry on the complex exponential field. See the end of section 4 for a more detailed discussion. Also in section 5 of this paper we discuss to what extent a pregeometry can be recovered just from quasiminimality, sometimes assuming the presence of a definable group structure. This continues in a sense an earlier study of the general model theory of quasiminimality by Itai, Tsuboi, and Wakai [2].

A pregeometry is a closure relation on subsets of a not necessarily saturated structure M satisfying usual properties (including exchange). We will also assume "homogeneity" ($\mathrm{tp}(b/A)$ is unique for $b \notin \mathrm{cl}(A)$) and "infinite-dimensionality" ($\dim(M)$ is infinite). One of the points of this paper then is that the canonical "generic type" p of the pregeometry is "generically stable" and regular. This includes the statement that on realizations of p, the closure operation is precisely forking in the sense of Shelah. See Theorem 3 of section 4. Another main result of the paper is that a quasiminimal structure of cardinality at least $\aleph_2$ carries, in a canonical fashion, a homogeneous (in the above sense), infinite-dimensional pregeometry, improving on results in [2].

Generic stability, the stable-like behavior of a given complete type vis-a-vis forking, was studied in several papers including [6] and [1], but mainly in the context of theories with NIP (i.e. without the independence property). Here we take the opportunity, in section 2, to give appropriate definitions in an arbitrary ambient theory T, as well as discussing generically stable (strongly) regular types.

The notion of a regular type is central in stability theory and classification theory, where the counting of models of superstable theories is related to dimensions of regular types. Here (in section 3) we give appropriate generalizations of (strong) regularity for an arbitrary theory T (although it does not agree with the established definitions for simple theories). In section 3 a basic dichotomy theorem (Theorem 1) is proved for global regular types p; roughly speaking, either p is generically stable, or there is certain definable partial ordering on the set of realizations of p. This broad dichotomy, "symmetric" versus "asymmetric", in the context of regular-like behaviour, is explored and generalized in various ways in sections 6 and 7. Theorem 7 in section 6 provides our most general dichotomy theorem which, among other things, yields the results on quasiminimal structures of cardinality $\geq \aleph_2$. In section 7 a local version of regularity is given and further results connected to quasiminimal structures are deduced (see Theorem 8 and Corollaries 3 and 4).

The current paper is a revised and expanded version of a preprint "Remarks on generic stability, pregeometries, and quasiminimality" by the first author, which was written and circulated in June 2009.

The first author would like to thank Clifton Ealy, Krzysztof Krupinski, and Alex Usvyatsov for various helpful conversations and comments. After seeing the first author's preliminary results (and talk at a meeting in Lyon in July 2009), Ealy pointed out that the commutativity of regular groups is problematic (see our Question at the end of section 3) and suggested possible directions towards a counterexample. Krupinski independently came up with examples such as Example 3 of the current paper. And after a talk by the first author on the same subject in Bedlewo in August 2009, Usvyatsov pointed out examples such as Example 1 and 2 of the current paper. Both authors would like to thank Jonathan Kirby for helpful comments and questions, and for clarifying the connection to exponential algebraicity (see the end of section 4).

We now give our conventions and give a few basic definitions relevant to the paper. As the referee mentioned to us, this paper maybe of interest to readers who are not so familiar with stability-style model theory, and so following his/her suggestion, we give more details than we usually would concerning some standard constructions. A thorough treatment of the case of stable theories appears in [5] and adaptations of some of the ideas to unstable (mainly NIP) theories appears in [1].

T denotes an arbitrary complete 1-sorted theory in a language L and $\bar{M}$ denotes a saturated (monster) model of T. As a rule $a, b, c, ...$ denote elements of $\bar{M}$, and $\bar{a}, \bar{b}, \bar{c}$ denote finite tuples of elements. (But in some situations $a, b, ..$ may denote elements of $\bar{M}^{eq}$.) A, B, C denote small subsets, and $M, M_0, ...$ denote small elementary submodels of $\bar{M}$. By a "global type" we mean morally a complete type over a sufficiently saturated model. In practice we will mean a complete type $p(\bar{x}) \in S(\bar{M})$ over the "monster model". Such a type p is said to be A-invariant if p is $\mathrm{Aut}(\bar{M}/A)$-invariant; and p is said to be invariant if it is A-invariant for some small A. Notice that by saturation/homogeneity of $\bar{M}$, the A-invariance of p is equivalent to p *not splitting* over A, in the sense that for any L-formula $\phi(\bar{x}, \bar{y})$ and $\bar{b}$ from $\bar{M}$, whether or not $\phi(\bar{x},\bar{b}) \in p$ depends on $tp(\bar{b}/A)$. So an A-invariant type $p(\bar{x})$ comes with a kind of "infinitary" defining schema d_p over A. Namely for a given L-formula $\phi(\bar{x},\bar{y})$, $d_p(\phi(\bar{x},\bar{y}))$ is the set of $q(\bar{y}) \in S(A)$ such that for some (any) $\bar{b}$ realizing q, $\phi(\bar{x},\bar{b}) \in p$.

We now explain how to build the "nonforking iterates" $p^{(n)}(\bar{x}_1,..,\bar{x}_n) \in S(\bar{M})$ of p. Let $\phi(\bar{x}_1,\bar{x}_2,\bar{c})$ be a formula over $\bar{M}$ (with witnessed parameters $\bar{c}$). We will put such a formula in $p^{(2)}(\bar{x}_1,\bar{x}_2)$ if for some (any) $\bar{a}_1$ realizing $p|(A,\bar{c})$ (the restriction of p to $A,\bar{c}$), the formula $\phi(\bar{a}_1,\bar{x}_2,\bar{c})$ is in $p(\bar{x}_2)$. Having defined $p^{(n)}$, we put a formula $\phi(\bar{x}_1,..,\bar{x}_n,\bar{x}_{n+1},\bar{c})$ in $p^{(n+1)}$ if for for some (any) $(\bar{a}_1,...,\bar{a}_n)$ realizing $p^{(n)}|(A,\bar{c})$, the formula $\phi(\bar{a}_1,..,\bar{a}_n,\bar{x}_{n+1},\bar{c})$ is in $p(\bar{x}_{n+1})$. This construction depends only on the defining schema d of p. It is clear that $p^{(n)}(\bar{x}_1,..,\bar{x}_n) \subseteq p^{(n+1)}(\bar{x}_1,...,\bar{x}_{n+1})$, and we let $p^{(\omega)}$ be the increasing union of the $p^{(n)}$'s. It is a very basic result, proved by an easy induction, that any realization $(\bar{a}_i : i = 1,2,....)$ of $p^{(\omega)}$ (in an elementary extension of the monster model!) is an indiscernible sequence over $\bar{M}$, and a sketch proof is given below.

Assuming the global type p to be A-invariant, by a *Morley sequence in p over A* we mean a realization, in $\bar{M}$, $(\bar{a}_i : i = 1,2,....)$ of $p^{(\omega)}|A$. Clearly this can also be obtained by choosing $\bar{a}_1$ to realize $p|A$, choosing $\bar{a}_2$ to realize $p|(A,\bar{a}_1)$ etc. So a Morley sequence in p over A is among other things an A-indiscernible sequence, and can be stretched to an A-indiscernible sequence of any ordinal length α (a Morley sequence in p over A of length α).

Let us give a quick sketch proof of the indiscernibility of $(a_1,a_2,....)$ over A where $p(x) \in S_1(M)$ does not split over A, and where all $a_n \in M$ and a_{n+1} realizes $p|(Aa_1..a_n)$. Assume inductively that $tp(a_1,..a_n/A) = tp(a_{i_1},..,a_{i_n}/A)$ for all n, and $i_1 < .. < i_n$. Fix $i_1 < ... < i_{n+1}$. Suppose $\phi(x_1,..,x_{n+1})$ is over A and suppose $\models \phi(a_1,..,a_n,a_{n+1})$. So $\phi(a_1,..,a_n,x) \in p(x)$. By inductive assumption, and the fact that p does not split over A, $\phi(a_{i_1},..,a_{i_n},x) \in p$, so is realized by $a_{i_{n+1}}$.

In any case clearly the type over A of any Morley sequence of length α in p over A depends only on p and A.

Let us note that if $p(x) \in S(M)$ is a not necessarily global type, which does not split over $A \subset M$, and $a_{n+1} \in M$ realizes $p|Aa_1...a_n$ then $(a_1,a_2,...)$ is also an indiscernible sequence

An invariant global type $p(\bar{x})$ is said to be *symmetric* if for any n, formula $\phi(\bar{x}_1,..,\bar{x}_n)$ over $\bar{M}$ and permutation π of $\{1,..,n\}$, $\phi(\bar{x}_1,...,\bar{x}_n) \in p^{(n)}(\bar{x}_1,..,\bar{x}_n)$ if and only if $\phi(\bar{x}_{\pi(1)},...,\bar{x}_{\pi(n)}) \in p^{(n)}(\bar{x}_1,..,\bar{x}_n)$.

It is not hard to see that the condition for $n = 2$ implies it for all n.

Also note that p is symmetric if and only if for any small A such that p is A-invariant, any Morley sequence in p over A is *totally indiscernible*.

In some parts of this paper we will be considering analogous notions for complete types over "large" but not necessarily saturated or homogeneous (in the sense of model theory) structures M, where we are no longer able to use the expression "invariant" type.

Some other basic notions used in this paper are *definable type*, *heir*, *coheir*, *almost finitely satisfiability*. A *definable type* (over A) refers usually to a complete type $p(x)$ say over a model M such that for any $\phi(x, \bar{y}) \in L$, $\{\bar{b} \in M : \phi(x, \bar{b}) \in p\}$ is a definable set (over A) in M. If $p(x)$ is a complete type over a model M and N is a larger model (elementary extension) then an *heir* of p over N is an extension $q(x) \in S(N)$ of p such that for any formula $\phi(x, \bar{y})$ with parameters from M, if $\phi(x, \bar{c}) \in q(x)$ for some $\bar{c}$ from N, then $\phi(x, \bar{b}) \in p$ for some $\bar{b} \in M$. A basic and easy fact is that if $p(x) \in S(M)$ is a definable type, then it has a unique heir over any larger model (which is precisely given by applying the defining schema for p to the larger model).

In the same context, with $p(x) \in S(M)$, and $q(x) \in S(N)$ an extension of p, q is said to be a *coheir* of p if q is finitely satisfiable in M (any formula $\phi(x)$ in $q(x)$ is realized by some element of M, even though the formula may have parameters outside M). Finally if $q(x)$ is a global complete type and A a small set of parameters, q is said to be *almost finitely satisfiable* in A if q is finitely satisfiable in any model (elementary substructure) of $\bar{M}$ which contains A.

Of course pervasive notions are *dividing* and *forking* in the sense of Shelah. A formula $\phi(x, b)$ (with witnessed parameters b) is said to *divide over* A if for some A-indiscernible sequence $(b_i : i < \omega)$ of realizations of $tp(b/A)$, $\{\phi(x, b_i : i < \omega\}$ is inconsistent. The formula is said to *fork over* A if it implies a finite disjunction of formulas $\psi_j(x, c_j)$ each of which divides over A. The notions extend naturally to (partial, or complete) types in place of formulas.

In stable theories, the above notions cohere in the following senses:
- Given a type $p(x) \in S(M)$ over a model and $N \supseteq M$, p has a unique nonforking extension over N which coincides with its unique heir over N as well as with its unique coheir over N.
- Any complete type over a model is definable (in particular any global type is invariant).
- A global type $p(x)$ does not fork over A iff it is definable over $acl^{eq}(A)$ iff it is almost finitely satisfiable in A.

- Moreover any global type is definable in a specific way over some (any) Morley sequence. (See Proposition 1 (i) below.)

Although stable theories were originally the main environment for studying forking, we saw in the 1990's the fundamental role of forking in simple theories, and more recently its importance for NIP theories, including o-minimal theories and suitable valued fields. In the current paper we are concerned to a large extent with the role of forking in "pregeometries" in general first order theories.

Recall that (P, cl), where cl is an operation on subsets of P, is a *pregeometry* (or cl is a pregeometry on P) if for all $A, B \subseteq P$ and $a, b \in P$ the following hold:

Monotonicity: $A \subseteq B$ implies $A \subseteq \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$;

Finite character $\mathrm{cl}(A) = \bigcup\{\mathrm{cl}(A_0) \mid A_0 \subseteq A \text{ finite}\}$;

Transitivity $\mathrm{cl}(A) = \mathrm{cl}(\mathrm{cl}(A))$, and

Exchange (symmetry) $a \in \mathrm{cl}(A \cup \{b\}) \setminus \mathrm{cl}(A)$ implies $b \in \mathrm{cl}(A \cup \{a\})$.

cl is called a *closure operator* if it satisfies the first three conditions.

2 Generic stability

Definition 1. A non-algebraic global type $p(\bar{x}) \in S(\bar{M})$ is generically stable if, for some small A, it is A-invariant and:

if $(\bar{a}_i : i < \alpha)$ (any α, not only ω) is a Morley sequence in p over A then for any formula $\phi(\bar{x})$ (with parameters from $\bar{M}$) $\{i : \models \phi(\bar{a}_i)\}$ is either finite or co-finite.

Remark 1. If p is generically stable then as a witness-set A in the definition we can take any small A such that P is A-invariant.

Proposition 1. *Let $p(\bar{x}) \in S(\bar{M})$ be generically stable and A-invariant. Then:*

(i) For any formula $\phi(\bar{x}, \bar{y}) \in L$ there is n_ϕ such that for any Morley sequence $(\bar{a}_i : i < \omega)$ of p over A, and any $\bar{b}$:

$$\phi(\bar{x}, \bar{b}) \in p \quad \textit{iff} \quad \models \bigvee_{w \subset \{0,1,\ldots 2n_\phi\},\, |w| = n_\phi + 1} \bigwedge_{i \in w} \phi(\bar{a}_i, \bar{b}).$$

(ii) p is definable over A and almost finitely satisfiable in A.

(iii) Any Morley sequence of p over A is totally indiscernible.

(iv) p is the unique global nonforking extension of $p \,|\, A$.

Proof. This is a slight elaboration of the proof of Proposition 3.2 from [1].

(i) First note that for any $\phi(\bar{x}, \bar{y}) \in L$ there is n_ϕ such that for any Morley sequence $(\bar{a}_i : i < \omega)$ in p over A, and any $\bar{b}$ either at most n_ϕ many $\bar{a}_i$'s satisfy $\phi(\bar{x}, \bar{b})$ or at most n_ϕ many $\bar{a}_i$'s satisfy $\neg\phi(\bar{x}, \bar{b})$. For if not, then by compactness we can find a Morley sequence in p over A of length $\omega + \omega$ which violates Definition 1. Note that if at most n_ϕ $\bar{a}_i$'s satisfy $\neg\phi(\bar{x}, \bar{b})$ then $\phi(\bar{x}, \bar{b}) \in p$: for otherwise we could let $(\bar{a}_{\omega+j} : j < \omega)$ realize $p^{(\omega)}$ restricted to $A \cup \{\bar{a}_i : i < \omega\} \cup \{\bar{b}\}$, in which case $(\bar{a}_i : i < \omega + \omega)$ is a Morley sequence in p over A such that all but finitely many $\bar{a}_i$ for $i < \omega$ satisfy $\phi(\bar{x}, \bar{b})$ and all $\bar{a}_{\omega+j}$ satisfy $\neg\phi(\bar{x}, \bar{b})$, again a contradiction.

(ii) By (i) p is definable (over a Morley sequence), so by A-invariance p is definable over A. (If $\psi(\bar{y}, \bar{d})$ is the $\phi(\bar{x}, \bar{y})$ definition of p, then by A-invariance of p, this formula is preserved, up to equivalence, by automorphisms which fix A pointwise, hence is equivalent to a formula over A.) Let M be a model containing A, $\phi(\bar{x}, \bar{c}) \in p$ and let $I = (\bar{a}_i : i < \omega)$ be a Morley sequence in p over A such that $\mathrm{tp}(I/M\bar{c})$ is finitely satisfiable in M. Then, by (i), $\phi(\bar{x}, \bar{c})$ is satisfied by some $\bar{a}_i$ hence also by an element of M.

(iii) This follows from (i) exactly as in the proof of the Proposition 3.2 of [1] (where NIP was not used).

(iv) Let $q(\bar{x})$ be a global nonforking extension of $p \,|\, A$. We will prove that $q = p$.

Claim. Suppose $(\bar{a}_0, ..., \bar{a}_n, \bar{b})$ are such that $\bar{a}_0 \models q \,|\, A$, $\bar{a}_{i+1} \models q \,|\, (A, \bar{a}_0, ..., \bar{a}_i)$ and $b \models q \,|\, (A, \bar{a}_0, ..., \bar{a}_n)$. Then $(\bar{a}_0, ..., \bar{a}_n, \bar{b})$ is a Morley sequence in p over A.

Proof. We prove it by induction. Suppose we have chosen $\bar{a}_0, ..., \bar{a}_n$ as in the claim and we know (induction hypothesis) that $(\bar{a}_0, ..., \bar{a}_n)$ begins a Morley sequence $I = (\bar{a}_i \,|\, i < \omega)$ in p over A. Suppose that $\phi(\bar{a}_0, ..., \bar{a}_n, \bar{x}) \in q$. Then we claim that $\phi(\bar{a}_0, ..., \bar{a}_{n-1}, \bar{a}_i, x) \in q$ for all $i > n$. For otherwise, without loss of generality $\neg\phi(\bar{a}_0, ..., \bar{a}_{n-1}, \bar{a}_{n+1}, \bar{x}) \in q$. But then, by indiscernibility of $\{\bar{a}_i\bar{a}_{i+1} : i = n, n+2, n+4, ...\}$ over $(A, \bar{a}_0, ..., \bar{a}_{n-1})$, and the nondividing of q over A, we have that

$$\{\phi(\bar{a}_0, ..., \bar{a}_{n-1}, \bar{a}_i, x) \wedge \neg\phi(\bar{a}_0, ..., \bar{a}_{n-1}, \bar{a}_{i+1}, \bar{x}) \,:\, i = n, n+2, n+4, ...\}$$

is consistent, which contradicts Definition 1. Hence $\models \phi(\bar{a}_0, ..., \bar{a}_{n-1}, \bar{a}_i, \bar{b})$ for all $i > n$. So by part (i), $\phi(\bar{a}_0, ..., \bar{a}_{n-1}, \bar{x}, \bar{b}) \in p(\bar{x})$. The inductive assumption gives that $\mathrm{tp}(\bar{a}_0, ..., \bar{a}_{n-1}, \bar{a}_n/A) = \mathrm{tp}(\bar{a}_0, ..., \bar{a}_{n-1}, \bar{b}/A)$, so by A-invariance of p, $\phi(\bar{a}_0, ..., \bar{a}_{n-1}, \bar{x}, \bar{a}_n) \in p(\bar{x})$. Thus $\models \phi(\bar{a}_0, ..., \bar{a}_{n-1}, \bar{a}_{n+1}, \bar{a}_n)$ and, by total indiscernibility of I over A (part (iii)), $\models \phi(\bar{a}_0, ..., \bar{a}_{n-1}, \bar{a}_n, \bar{a}_{n+1})$. We have shown that $q \,|\, (A, \bar{a}_0, ..., \bar{a}_n) = p \,|\, (A, \bar{a}_0, ..., \bar{a}_n)$, which allows the induction process to continue. The claim is proved.

Now suppose that $\phi(\bar{x}, \bar{c}) \in q(\bar{x})$. Let $\bar{a}_i$ for $i < \omega$ be such that $\bar{a}_i$ realizes $q \,|\, (A, \bar{c}, \bar{a}_0, ..., \bar{a}_{i-1})$ for all i. By the claim $(\bar{a}_i\,;\, i < \omega)$ is a Morley sequence of p over A. But $\phi(\bar{a}_i, \bar{c})$ for all i, hence by the proof of (i), $\phi(\bar{x}, \bar{c}) \in p$. So $q = p$. □

Let us note for the record that for a global type p, generically stable implies definable implies invariant, and these are strict implications.

Let us restate the notion of generic stability for groups from [1] in the context of connected groups. Recall that a definable (or even type-definable) group G is connected if it has no relatively definable subgroup of finite index. A type $p(x) \in S_G(\bar{M})$ is left G-invariant iff for all $g \in G$:

$$(g \cdot p)(x) =^{def} \{\phi(g^{-1} \cdot x) \,:\, \phi(x) \in p(x)\} = p(x);$$

likewise for right G-invariant.

Definition 2. Let G be a definable (or even type-definable) connected group in $\bar{M}$. G is *generically stable* if there is a global complete 1-type $p(x)$ extending '$x \in G$' such that p is generically stable and left G-invariant.

As in the NIP context, we show that a generically stable left invariant type is also right invariant and is unique such (and we will call it the generic type):

Lemma 1. *Suppose that G is generically stable, witnessed by $p(x)$. Then $p(x)$ is the unique left-invariant and also the unique right-invariant type.*

Proof. First we prove that p is right invariant. Let (a, b) be a Morley sequence in p over (any small) A. By left invariance $g = a^{-1} \cdot b \models p \,|\, A$. By total indiscernibility we have $\mathrm{tp}(a, b) = \mathrm{tp}(b, a)$, so $g^{-1} = b^{-1} \cdot a \models p \,|\, A$. This proves that $p = p^{-1}$. Now, for any $g \in G$ we have

$$p = g^{-1} \cdot p = (g^{-1} \cdot p)^{-1} = p^{-1} \cdot g = p \cdot g\ ,$$

so p is also right-invariant.

For the uniqueness, suppose that q is a left invariant global type and we prove that $p = q$. Let $\phi(x) \in q$ be over A, let $I = (a_i : i \in \omega)$ be a Morley sequence in p over A, and let $b \models q \,|\, (A, I)$. Then, by left invariance of q, for all $i \in \omega$ we have $\models \phi(a_i \cdot b)$. By Proposition 1(i) we get $\phi(x \cdot b) \in p(x)$ and, by right invariance, $\phi(x) \in p(x)$. Thus $p = q$. □

3 Global regularity

For stable T, a stationary type $p(x) \in S(A)$ is said to be regular if for any $B \supseteq A$, and b realizing a forking extension of p over B, $p \,|\, B$ (the unique nonforking extension of p over B) has a unique complete extension over (B, b). Appropriate versions of regularity have been given for simple theories where we do not always have stationarity. Here we give somewhat different versions for global "invariant" types in arbitrary theories. So the reader should be aware that our definitions below are not consistent with usual terminology for *simple theories*, although they are of course consistent with the stable case.

Definition 3. Let $p(\bar{x})$ be a global non-algebraic type.

(i) $p(\bar{x})$ is said to be *regular* if, for some small A, it is A-invariant and for any $B \supseteq A$ and $\bar{a} \models p \,|\, A$: either $\bar{a} \models p \,|\, B$ or $p \,|\, B \vdash p \,|\, B\bar{a}$.

(ii) Suppose $\phi(\bar{x}) \in p$. We say that $(p(\bar{x}), \phi(\bar{x}))$ is *strongly regular* if, for some small A over which ϕ is defined, p is A-invariant and for all $B \supseteq A$ and $\bar{a}$ satisfying $\phi(\bar{x})$, either $\bar{a} \models p \,|\, B$ or $p \,|\, B \vdash p \,|\, B\bar{a}$.

Remark 2. If p is regular then as a witness-set A in the definition we can take any small A such that p is A-invariant. Similarly for strongly regular types. Also note that strongly regular implies regular.

Definition 4. Let N be any submodel of $\bar{M}$ (possibly $N = \bar{M}$), and let $p(x) \in S_1(N)$. The operator cl_p is defined on (all) subsets of N by:

$$\mathrm{cl}_p(X) = \{a \in N \,|\, a \not\models p \,|\, X\}.$$

Also define $\mathrm{cl}_p^A(X) = \mathrm{cl}_p(X \cup A)$ for any $A \subset N$ (and $X \subseteq N$.

Intuitively, having fixed $p(x) \in S(N)$ and $B \subset N$ then realizations of $p|B$ are considered as 'generic over B' so formulas in $p(x)$ should be considered as defining 'large' subsets of N and their negations as defining 'small' subsets. In this way, $\mathrm{cl}_p(B)$ is the union of all 'small' definable subsets. In the rest of this section we will be interested in the case where $N = \bar{M}$. But later in the paper will consider other N.

Remark 3. Let $p(x) \in S_1(\bar{M})$ be A-invariant, where A is small.

(i) Strong regularity of $(p(x), x = x)$ translates into a simpler expression using cl_p:

$$(p(x), x = x) \text{ is strongly regular iff } p \,|\, B \vdash p \,|\, \mathrm{cl}_p(B) \text{ for any small } B \supseteq A.$$

Since $a \models p \,|\, B$ is the same as $a \notin \mathrm{cl}_p(B)$, another equivalent way of expressing strong regularity of $(p(x), x = x)$ is:

$$a \models p|B \text{ iff } a \models p \,|\, \mathrm{cl}_p(B) \quad (\text{for all } a \in \bar{M} \text{ and small } B \supseteq A).$$

(ii) The corresponding expression is not so concise when $(p(x), \phi(x))$ is strongly regular or when p is just regular. For example:

$$p \text{ is regular iff } p \,|\, AB \vdash p \,|\, A \cup B \cup (\mathrm{cl}_p^A(B) \cap (p|A)(\bar{M})) \text{ for any small } B.$$

We can also consider the restriction of cl_p^A to $A \cup (p|A)(\bar{M})$. Formally we define (for B a small subset of $A \cup (p|A)(\bar{M})$) $\mathrm{cl}_{p_A}(B) = A \cup (\mathrm{cl}_p^A(B) \cap (p|A)(\bar{M}))$. Then we have a simple consequence of regularity; p is regular *implies*

$$p \,|\, AB \vdash p \,|\, \mathrm{cl}_{p_A}(B) \text{ for any small } B \subset (p|A)(\bar{M}).$$

As in (i) this is translated to: if p is regular then

$$a \models p \,|\, AB \text{ iff } a \models p \,|\, \mathrm{cl}_{p_A}(B) \quad (\text{for any } a \in \bar{M} \text{ and small } B \subset (p|A)(\bar{M})).$$

(iii) Note that in (i) and (ii) we took into account only small B's, while Definition 3 mentions all B's. But this does not matter, since in all of them the statements with and without 'small' are easily seen to be equivalent.

Lemma 2. *Suppose that A is small and $p \in S_1(\bar{M})$ is A-invariant.*

(i) $(p(x), x = x)$ is strongly regular iff cl_p^A is a closure operator on $\bar{M}$.

(ii) Suppose that $(p(x), x = x)$ is strongly regular. Then cl_p^A is a pregeometry operator on $\bar{M}$ iff every Morley sequence in p over A is totally indiscernible.

(iii) Suppose that p is regular. Then cl_{p_A} is a closure operator on $A \cup (p|A)(\bar{M})$; it is a pregeometry operator iff every Morley sequence in p over A is totally indiscernible.

Proof. (i) Assume that $(p(x), x = x)$ is strongly regular. Then for any small $B \supseteq A$ we have:

$$a \notin \mathrm{cl}_p(B) \ \text{ iff } \ a \models p \,|\, B \ \text{ iff } \ a \models p \,|\, \mathrm{cl}_p(B) \ \text{ iff } \ a \notin \mathrm{cl}_p(\mathrm{cl}_p(B)).$$

The first and the last equivalence follow from the definition of cl_p and the middle one is by Remark 3(i). Thus $\mathrm{cl}_p(B) = \mathrm{cl}_p(\mathrm{cl}_p(B)$ so cl_p^A is a closure operator on $\bar{M}$. The other direction is similar.

(ii) Suppose that $(p(x), x = x)$ is strongly regular and that every Morley sequence in p over A is symmetric. To show that cl_p^A is a pregeometry operator, by part (i), it suffices to verify the exchange property over finite extensions $B \supset A$. Let $(a_1, ..., a_n) \in B^n$ be a maximal Morley sequence in p over A; note that it is finite since it is in $B \backslash A$, which is finite. Then $B \subseteq \mathrm{cl}_p^A(a_1, .., a_n)$ (otherwise any element in the difference would contradict the maximality) and, since cl_p^A is a closure operator, we get $\mathrm{cl}_p^A(B) \subseteq \mathrm{cl}_p^A(a_1, .., a_n)$ and thus $\mathrm{cl}_p^A(B) = \mathrm{cl}_p^A(a_1, .., a_n)$.

To verify exchange, let $a \models p \,|\, B$ (so $a \notin cl_p^A(B)$) and let $b \in \bar{M}$. Note that $(a_1, ..., a_n, a)$ is a Morley sequence in p over A and $\mathrm{cl}_p^A(Ba) = \mathrm{cl}_p^A(a_1, ..., a_n, a)$. We have:

$b \notin \mathrm{cl}_p^A(Ba)$ iff $b \notin \mathrm{cl}_p^A(a_1, ..., a_n, a)$ iff $(a_1, ..., a_n, a, b)$ is a Morley sequence iff $(a_1, ..., a_n, b, a)$ is a Morley sequence iff $(b \notin \mathrm{cl}_p^A(B)$ and $a \notin \mathrm{cl}_p^A(Bb))$.

In particular $b \notin \mathrm{cl}_p^A(Ba)$ implies $a \notin \mathrm{cl}_p^A(Bb)$. This proves exchange.

The converse is immediate.

(iii) This is similar to (i) and (ii). Suppose that p is regular, let $B \subset (p|A)(\bar{M})$ be small and let $a \models p|\, A$. Then:

$$a \notin \mathrm{cl}_{p_A}(B) \ \text{ iff } \ a \models p \,|\, AB \ \text{ iff } \ a \models p \,|\, \mathrm{cl}_{p_A}(B) \ \text{ iff } \ a \notin \mathrm{cl}_{p_A}(\mathrm{cl}_{p_A}(B));$$

the first equivalence and the last equivalence follow from the definition of cl_{p_A} and the middle one is by Remark 3(ii). Thus $\mathrm{cl}_p(B) = \mathrm{cl}_p(\mathrm{cl}_p(B))$. The other clause is proved as in part (ii). □

Theorem 1. *Suppose that A is small and $p(x) \in S_1(\bar{M})$ is A-invariant and regular.*

(1) If p is symmetric, then cl_{p_A} is a pregeometry operator on $A \cup (p \,|\, A)(\bar{M})$, In the case where $(p(x), x = x)$ is strongly regular, cl_p^A is a pregeometry operator on $\bar{M}$.

(2) If p is not symmetric, then there exists a finite extension A_0 of A and a A_0-definable partial order $\leq$ such that every Morley sequence in p over A_0 is strictly increasing.

Proof. If p is symmetric then every Morley sequence in p over A is totally indiscernible and (1) follows from Lemma 2.

Now suppose that p is not symmetric. Then as mentioned in the introduction there is a finite extension $A_0 \supset A$ and a Morley sequence (a, b) of length 2 in p over A_0 such that $\mathrm{tp}(a, b/A_0) \neq \mathrm{tp}(b, a/A_0)$. Choose $\phi(x, y) \in \mathrm{tp}(a, b/A_0)$ witnessing the "asymmetry": $\models \phi(x, y) \rightarrow \neg\phi(y, x)$. Then $\phi(a, x) \wedge \neg\phi(x, a) \in \mathrm{tp}(b/aA_0) \subset p(x))$ so $\phi(x, a) \notin p(x)$. We claim that

$$(p \,|\, A_0)(t) \cup \{\phi(t, a)\} \cup \{\neg\phi(t, b)\}$$

is inconsistent. Otherwise, there is d realizing $p \,|\, A_0$ such that $\models \phi(d, a) \wedge \neg\phi(d, b)$. Then d does not realize $p \,|\, (A_0, a)$ (witnessed by $\phi(x, a)$) so, by regularity, $p \,|\, (A_0, a) \vdash p \,|\, (A_0, a, d)$ and thus $b \models p \,|\, (A_0, a, d)$. In particular $b \models p \,|\, (A_0, d)$ and since, by invariance, $\phi(d, x) \in p(x)$ we conclude $\models \phi(d, b)$. A contradiction.

By this claim, and compactness, we find $\theta(t) \in p \,|\, A_0$ such that

$$\models (\forall t)(\phi(t, a) \wedge \theta(t) \rightarrow \phi(t, b)).$$

Let $x \preccurlyeq y$ be $(\forall t)(\phi(t, x) \wedge \theta(t) \rightarrow \phi(t, y))$. Clearly, $\preccurlyeq$ defines a quasi order and $a \preccurlyeq b$. Also:

$$\models \phi(a, b) \wedge \theta(a) \wedge \neg\phi(a, a);$$

The first conjunct follows by our choice of ϕ, the second from $a \models p \,|\, A_0$, and the third from the asymmetry of ϕ. Altogether they imply $b \not\preccurlyeq a$. Thus if $x < y$ is $x \preccurlyeq y \wedge y \not\preccurlyeq x$ we have $a < b$. □

The next examples concern issues of whether symmetric regular types are definable or even generically stable. But we first give a case where this is true (although it depends formally on Theorem 3 of the next section).

Corollary 1. *Suppose that $(p(x), x = x)$ is strongly regular and symmetric. Then $p(x)$ is generically stable.*

Proof. If $(p(x), x = x)$ is invariant-strongly regular then, by Theorem 1(1) cl_p is a pregeometry operator on $\bar{M}$, and then $p(x)$ is generically stable by Theorem 3(ii). □

Example 1. A symmetric, definable, strongly regular type which is not generically stable.
Let $L = \{U, V, E\}$ where U, V are unary and E is a binary predicate. Consider the bipartite graph (M, U^M, V^M, E^M) where $U^M = \omega$, V^M is the set of all finite subsets of ω, $M = U^M \cup V^M$, and $E^M = \{(u, v) : u \in U^M,\ v \in V^M,\ \text{and } u \in v\}$. Let $A \subset M$ be finite. Then:

$$\text{If } \ (c_1, ..., c_n), (d_1, ..., d_n) \in (U^M)^n \ \text{ have the same quantifier-free type over } A \text{ then } \operatorname{tp}(c_1, ..., c_n/A) = \operatorname{tp}(d_1, ..., d_n/A),$$

since the involution of ω mapping c_i's to d_i's respectively, and fixing all the other elements of ω is an A-automorphism of M. Note that this is expressible by a set of first-order sentences, so is true in the monster.

Further, if $e_1, ..., e_n \in U^M$ are distinct and have the same type over A then, since every permutation of ω which permutes $\{e_1, ..., e_n\}$ and fixes all the other elements of ω is an A-automorphism of M, $(e_1, ..., e_n)$ is totally indiscernible over A. This is also expressible by a set of first-order sentences.

Let $p(x) \in S_1(M)$ be the type of a "new" element of U which does not belong to any element of V^M. Then, by the above, p is definable, its global heir $\bar{p}$ is symmetric, and $(\bar{p}(x), U(x))$ is strongly regular.

Example 2. A symmetric, strongly regular type which is not definable.
Consider the bipartite graph (M, U^M, V^M, E^M) where $U^M = \omega$, V^M consists of all finite and co-finite subsets of ω, $M = U^M \cup V^M$, and E^M is $\in$.

Let $\bar{M}$ be the monster and let $p(x) \in S_1(\bar{M})$ be the type of a new element of $U^{\bar{M}}$, which belongs to all co-finite members of $V^{\bar{M}}$ (and no others). Arguing as in the previous example $(p(x), U(x))$ is strongly regular and symmetric. Since "being a co-finite subset of U^M" is not definable, p is not definable.

Definition 5. Let G be a definable group in $\bar{M}$. G is called an *regular group* if for some global type $p(x) \in S_G(\bar{M})$, $(p(x)$, "$x \in G$") is strongly regular (in particular invariant over some small set).

We will see in Example 3 that non symmetric regular groups, and even fields, exist.

Theorem 2. *Suppose that G is a group definable over $\emptyset$, which is regular, witnessed by $p(x) \in S_G(\bar{M})$. Then:*

(i) $p(x)$ *is both left and right translation invariant (and in fact invariant under definable bijections).*

(ii) A formula $\phi(x)$ *is in* $p(x)$ *iff two left (right) translates of* $\phi(x)$ *cover* G *iff finitely many left (right) translates of* $\phi(x)$ *cover* G*. (Hence* $p(x)$ *is the unique "generic type" of* G*.)*

(iii) $p(x)$ *is definable over* $\emptyset$*.*

(iv) $G = G^0$ *(i.e.* G *is connected).*

Proof. (i) Suppose that $f : G \longrightarrow G$ is a B-definable bijection and $a \models p \mid B$. Since $p \mid B \vdash p \mid (B, f(a))$ is not possible, by strong regularity, we get $f(a) \models p \mid B$. Thus p is invariant under f.

(ii) Suppose that $D \subseteq G$ is defined by $\phi(x) \in p(x)$ which is over A. Let $g \models p \mid A$ and we show $G = D \cup g \cdot D$. If $b \in G \setminus D$ then b does not realize $p \mid A$ so, by strong regularity, $g \models p \mid (A, b)$. By (i) $g^{-1} \models p \mid (A, b)$, thus $g^{-1} \in D \cdot b^{-1}$ and $b \in g \cdot D$. This proves $G = D \cup g \cdot D$.

For the other direction, if finitely many translates of $\psi(x)$ cover G then at least one of them belongs to $p(x)$ and, by (i), $\psi(x) \in p(x)$.

(iii) and (iv) follow immediately from (ii). □

Question. Is every regular group commutative?

4 Homogeneous pregeometries

If (M, cl) is a pregeometry then, as usual, we obtain notions of independence and dimension: for $A, B \subset M$ we say that A is independent over B if $a \notin \mathrm{cl}(A \setminus \{a\} \cup B)$ for all $a \in A$. Given A and B, all subsets of A which are independent over B and maximal such, have the same cardinality, called $\dim(A/B)$. (M, cl) is infinite-dimensional if $\dim(M/\emptyset) \geq \aleph_0$.

Remark 4. (i) If $\bar{c}$ is a tuple of length n then $\dim(\bar{c}/B) \leq n$ for any B.

(ii) If $l(\bar{c}) = n$, $|A| \geq n + 1$ and A is independent over B then there is $a \in A$ such that $a \notin \mathrm{cl}(B \cup \bar{c})$.

Definition 6. We call an infinite-dimensional pregeometry (M, cl) *homogeneous* if for any finite $B \subset M$, the set of all $a \in M$ such that $a \notin \mathrm{cl}(B)$ is the set of realizations in M of a complete type $p_B(x)$ over B.

Note that Definition 6 relates in some way the closure operation to the first-order structure. But it does not say anything about automorphisms, and nothing is being claimed about the homogeneity or strong homogeneity of M as a first-order structure. In particular we do not want to assume M to be a saturated structure.

Lemma 3. *Suppose* (M, cl) *is a homogeneous pregeometry.*

(i) $\mathrm{p_{cl}}(x) = \bigcup\{p_B(x) : B$ *finite subset of* $M\}$ *is a complete 1-type over* M*, which we call the generic type of the pregeometry* (M, cl).

(ii) $a \notin \mathrm{cl}(B)$ *iff* $a \models \mathrm{p_{cl}}(x) \,|\, B$. *In particular* $\mathrm{cl} = \mathrm{cl}_{\mathrm{p_{cl}}}$.

(iii) $I = (a_i : i < \omega)$ *is independent over* B *iff* $a_i \models \mathrm{p_{cl}} \,|\, (B, a_0, ..., a_{i-1})$ *for all* i. *In particular, if* $M = \bar{M}$ *and* $\mathrm{p_{cl}}$ *is* B*-invariant then* I *is independent over* B *iff it is a Morley sequence in* $\mathrm{p_{cl}}$ *over* B.

Proof. (i) Consistency is by compactness: given $A_1, ..., A_n$ finite subsets of M and $B = A_1 \cup ... \cup A_n$, clearly $p_{A_1}(x) \cup ... \cup p_{A_n}(x) \subseteq p_B(x)$ and the later is consistent. Completeness is clear.

(ii) and (iii) are easy. □

Lemma 4. *Suppose* (M, cl) *is a homogeneous pregeometry. Let* $(a_i : i \in \omega)$ *be an* $\emptyset$*-independent subset of* M. *Then for any* L*-formula* $\phi(x, \bar{y})$ *with* $l(\bar{y}) = n$*, and* n*-tuple* $\bar{b}$ *from* M:

$$\phi(x, \bar{b}) \in \mathrm{p_{cl}}(x) \quad \text{iff} \quad \models \wedge_{i \in w} \phi(a_i, b) \ \text{for some} \ w \subset \{1, ...2n\},\ |w| = n + 1.$$

In particular $\mathrm{p_{cl}}(x)$ *is definable.*

Proof. If $\phi(x, \bar{b})$ is large (namely in p_{cl}), then its negation is small, and thus if $M \models \neg\phi(a, \bar{b})$ then $a \in \mathrm{cl}(\bar{b})$. By Remark 4(ii), at most n many a_i's can satisfy $\neg\phi(x, \bar{b})$, hence at least $n+1$ among the first $2n+1$ a_i's satisfy $\phi(x, \bar{b})$. Conversely, if at least $n + 1$ a_i's satisfy $\phi(x, \bar{b})$ then, again by Remark 4(ii), $\phi(x, \bar{b})$ can not be small, so it is large. □

In the next Proposition we make use of Definition 4 from the previous section.

Proposition 2. *Suppose* (M, cl) *is a homogeneous pregeometry. Let* $p(x)$ *be the generic type and let* $\bar{p}(x)$ *be its (unique by definability) global heir.*

(i) $(\bar{M}, \mathrm{cl}_{\bar{p}})$ *is a homogeneous pregeometry and* cl *is the restriction of* $\mathrm{cl}_{\bar{p}}$ *to* M.

(ii) If $(a_1, ...a_n)$ *(from* $\bar{M}$*) is independent over* A *then:*

$\mathrm{tp}(b_1,...,b_n/A) = \mathrm{tp}(a_1,...,a_n/A)$ *iff* $(b_1,...,b_n)$ *is independent over* A.

(iii) $\bar{p}(x)$ *is* $\emptyset$*-invariant and generically stable.*

(iv) $(\bar{p}(x), x = x)$ *is strongly regular*

Proof. (i) is a reasonably routine exercise, using the fact that $\bar{p}$ is defined by the same schema which defines p, and is left to the reader. But we will briefly point out why exchange holds: Note first that $\bar{p}^{(2)}(x_1, x_2)$ is also definable, over the same parameters as p. Let $\phi(x_1, x_2, \bar{b})$ be an arbitrary formula over $\bar{M}$, with parameters $\bar{b}$ witnessed. To prove exchange it suffices to see that $\phi(x_1, x_2, \bar{b}) \in \bar{p}^{(2)}(x_1, x_2)$ iff $\phi(x_2, x_1, \bar{b}) \in \bar{p}^{(2)}(x_1, x_2)$. By definability of $\bar{p}^{(2)}$, there are formulas $\delta_1(\bar{z})$ and $\delta_2(\bar{z})$ over M such that for any $\bar{b}'$ from $\bar{M}$, we have
(a) $\phi(x_1, x_2, \bar{b}') \in \bar{p}^{(2)}(x_1, x_2)$ iff $\models \delta_1(\bar{b}')$ and
(b) $\phi(x_2, x_1, \bar{b}') \in \bar{p}^{(2)}(x_1, x_2)$ iff $\models \delta_2(\bar{b}')$.
As exchange holds inside M, we have $M \models \forall \bar{z}(\delta_1(\bar{z}) \leftrightarrow \delta_2(\bar{z}))$. Hence this also holds in $\bar{M}$.

(ii) We prove it by induction on n. For $n = 1$, by definition, we have $a_1, b_1 \models \bar{p} \,|\, A$. Now assume true for n and prove for $n + 1$. Without loss of generality $A = \emptyset$. Suppose first that $\mathrm{tp}(b_1, ..., b_{n+1}) = \mathrm{tp}(a_1, ..., a_{n+1})$. Let a' realize $p \,|\, (a_1, ..., a_{n+1}, b_1, ..., b_{n+1})$. So $\mathrm{tp}(a_1, ..., a_n, a_{n+1}) = \mathrm{tp}(a_1, ..., a_n, a')$. On the other hand, by the induction assumption (over $\emptyset$), $(b_1, ..., b_n)$ is independent, so independent over a' (by symmetry). By induction assumption applied over a', $\mathrm{tp}(b_1, ..., b_n, a') = \mathrm{tp}(a_1, ..., a_n, a')$. Hence $\mathrm{tp}(b_1, ..., b_n, a') = \mathrm{tp}(b_1, ..., b_n, b_{n+1})$. As $a' \notin \mathrm{cl}_{\bar{p}}(b_1, ..., b_n)$, also $b_{n+1} \notin \mathrm{cl}_{\bar{p}}(b_1, ..., b_n)$. Thus $(b_1, ...b_n, b_{n+1})$ is independent.

The converse (if $(b_1, ..., b_{n+1})$ is independent then it realizes $\mathrm{tp}(a_1, ..., a_{n+1})$) is proved in a similar fashion and left to the reader.

(iii) By part (i), Lemma 4 also applies to the pregeometry $\mathrm{cl}_{\bar{p}}$. Let $(a_i : i \in \omega)$ be $\mathrm{cl}_{\bar{p}}$-independent. Then $\bar{p}(x)$ is defined over $(a_i : i \in \omega)$ as in Lemma 4. But if $(b_i : i \in \omega)$ has the same type as $(a_i : i \in \omega)$ then, by (ii), it is also $\mathrm{cl}_{\bar{p}}$-independent, hence $\bar{p}(x)$ is defined over $(b_i : i \in \omega)$ in the same way. This implies that $\bar{p}$ is $\emptyset$-invariant. Thus, by Lemma 3(iii) a Morley sequence in $\bar{p}$ is the same thing as an infinite $\mathrm{cl}_{\bar{p}}$-independent set. By Lemma 4 and Definition 1, $\bar{p}$ is generically stable.

(iv) By part (i) $(\bar{M}, \mathrm{cl}_{\bar{p}})$ is a pregeometry so, by Lemma 2(i), $(\bar{p}(x), x = x)$ is strongly regular. □

We now drop (for a moment) all earlier assumptions and summarize the situation:

Theorem 3. *Let T be an arbitrary theory.*

(i) Let $p(x)$ be a global $\emptyset$-invariant type such that $(p(x), x = x)$ is strongly regular. Then $(\bar{M}, \mathrm{cl}_p)$ is a homogeneous pregeometry.

(ii) On the other hand, suppose $M \models T$ and (M, cl) is a homogeneous pregeometry. Then there is a unique global $\emptyset$-invariant generically stable type $p(x)$ such that $(p(x), x = x)$ is strongly regular, and such that the restriction of cl_p to M is precisely cl.

We end this section by pointing out the connection to exponential fields, as mentioned to us by Kirby. In [3], Jonathan Kirby proved that ecl$(-)$, "exponential algebraic closure", as originally defined by Macintyre, gives a pregeometry on *any* exponential field, and this result extends those of Wilkie [8] for the complex exponential field. It is an open question whether for the complex exponential field, ecl$(-)$ is *homogeneous* in the sense of Definition 6 above. A positive answer would yield quasiminimality for the complex exponential field as well as generic stability and strong regularity of its (unique) exponentially transcendental type. On the other hand, a positive answer does exist for Zilber's pseudoexponentiation and some other exponential fields.

5 Quasiminimal structures

Recall that a 1-sorted structure M in a countable language is called quasiminimal if M is uncountable and every definable (with parameters) subset of M is countable or co-countable; the definition was given by Zilber in [9]. Here we investigate the general model theory of quasiminimality, continuing an earlier work by Itai, Tsuboi and Wakai [2].

Throughout this section fix a quasiminimal structure M and let $\bar{M}$ be a saturated elementary extension. The set of all formulas (with parameters) defining a co-countable subset of M forms a complete 1-type $p(x) \in S_1(M)$; we will call it the generic type of M. If $p(x)$ happens to be definable we will denote its (unique) global heir by $\bar{p}(x)$.

Remark 5. (i) In quasiminimal structures Zilber's countable closure operator ccl is defined via cl_p :

$$\mathrm{cl}_p^0(A) = \mathrm{cl}_p(A),\ \ \mathrm{cl}_p^{n+1}(A) = \mathrm{cl}_p(\mathrm{cl}_p^n(A)) \ \text{ and } \ \mathrm{ccl}(A) = \bigcup_{n\in\omega} \mathrm{cl}_p^n(A).$$

(ii) If A is countable then $\mathrm{ccl}(A)$ is countable, too. ccl is a closure operator on M.

(iii) cl_p is a closure operator iff $\mathrm{cl}_p = \mathrm{ccl}$ (which is in general not the case, see Example 4).

In Section 2 of [2] Itai, Tsuboi and Wakai, studied the case when M is strongly $\aleph_1$-homogeneous, in the model theoretic sense, namely, any partial elementary map between countable subsets extends to an automorphism of M. They proved in Proposition 2.10 there that if ccl is *not* a pregeometry operator then some uncountable subset of M is totally ordered by a formula, and they also remark that Maesono has strengthened in one direction the conclusion to: $\mathrm{Th}(M)$ has the strict order property (Remark 2.16 there). They also proved (assuming model-theoretic strong $\aleph_1$ homogeneity)that ccl is a pregeometry operator whenever $|M| \geq \aleph_2$.

In Theorem 4 below we will use a weaker assumption than strong $\aleph_1$-homogeneity, which we call countable basedness, and derive a stronger dichotomy similar to the one for strongly regular types in Theorem 1. Then, in Corollary 2 we prove that any quasiminimal structure of size at least $\aleph_2$ must be of symmetric type; in particular, (M, ccl) is a homogeneous pregeometry (in the sense of the previous section) after possibly countably many parameters from M into the language. In Theorem 5 we will prove that the failure of countable basedness implies the strict order property.

Recall from the introduction that p does not split over A, if $(\phi(x, \bar{b}_1) \leftrightarrow \phi(x, \bar{b}_2)) \in p(x)$ for all $\phi(x, \bar{y})$ over A and all $\bar{b}_1, \bar{b}_2 \in M$ realizing the same type over A. We talk explicitly about splitting rather than invariance, because M need not be saturated/homogeneous. Remember that in this section we are taking $p(x) \in S_1(M)$ to be the unique "generic" type of M consisting of all formulas defining co-countable sets.

Definition 7. (i) $p(x)$ is *based on* A if: p does not split over A and $\mathrm{ccl}(AC) \subsetneq M$ for all finite $C \subset M$

(ii) $p(x)$ (or M) is *countably based* if there is a countable $A \subset M$ such that it is based on A.

The technical condition $\mathrm{ccl}(AC) \subsetneq M$ is satisfied by any countable A, so countable baseness is equivalent to the existence of a countable subset A over which p does not split. In the proof of Corollary 2 we will use an uncountable base set, and this is why it is added: it describes a relative smallness of A in M.

If p does not split over A then we say that $(a_1, ..., a_n)$ is a weak Morley sequence in p over A if a_k realizes $p \,|\, (A, a_1, ..., a_{k-1})$ for all relevant k. As in the case of global invariant types weak Morley sequences are indiscernible over A. Now we can one of our main theorems, whose proof will be presented in a somewhat more general context in the next section, and then draw an important Corollary.

Theorem 4. *Suppose that $p(x)$ is based on $A \subset M$. Then cl_p^A is a closure operator and exactly one of the following two holds:*

(1) Every (some) weak Morley sequence in p over A is totally indiscernible; in this case cl_p^A is a pregeometry operator on M, p is definable, $\bar{p}$ (its unique global heir) is generically stable and $(\bar{p}(x), x = x)$ is strongly regular. Moreover (M, cl_p^A) is an infinite-dimensional homogeneous pregeometry (after adding constants for elements of A).

(2) There exists a weak Morley sequence in p over A which is not totally indiscernible; in this case there is a finite extension A_0 of A and an A_0-definable partial order $\leq$ such that every weak Morley sequence in p over A_0 is strictly increasing.

Corollary 2. *If $|M| \geq \aleph_2$ then p is definable (hence countably based) and Case (1) of Theorem 4 holds.*

Proof. Suppose $|M| \geq \aleph_2$ and let $M_0 \subset M$ be ccl-closed of size $\aleph_1$. First we show that $p(x)$ is finitely satisfiable in M_0 (actually it is finitely satisfiable in any uncountable subset of M): for any $\phi(x) \in p(x)$ $\phi(M)$ is co-countable, so $\phi(M)$ intersects M_0. Thus p does not split over M_0 and, since $|\mathrm{ccl}(M_0C)| = \aleph_1$ for any countable C, p is based on M_0. Theorem 4 applies; we will show that Case (1) holds. Otherwise, there is a definable $<$ and a strictly increasing sequence $\{a_i \,|\, i < \omega_2\} \subset M$. Then $x < a_{\omega_1}$ and $a_{\omega_1} < x$ define uncountable mutually disjoint subsets of M. A contradiction. Thus Case (1) holds and the conclusion follows. □

Theorem 5. *If p is not countably based then there exists a definable partial order (on some M^n) which has uncountable strictly increasing chains.*

Proof. Suppose that p is not countably based. Then p is finitely satisfiable in no countable subset of M and, inductively, we can find a sequence $(M_i \mid i < \omega_1)$ of countable ccl-closed submodels and a sequence $(\bar{a}_i \mid i < \omega_1)$ of tuples such that for all $i < \omega_1$:

(1) $M_i \prec M_{i+1}$ and $M_\alpha = \cup_{j<\alpha} M_j$ for $\alpha < \omega_1$ a limit ordinal;

(2) $\bar{a}_i \in M_{i+1}$;

(3) $p \mid M_i\bar{a}_i$ is not finitely satisfiable in M_i.

For each $i < \omega_1$ witness (3) by an L-formula $\phi_i(x, \bar{y}, \bar{z})$ and $\bar{m}_i \in M_i$ such that $\phi_i(x, \bar{m}_i, \bar{a}_i) \in p(x)$ is not satisfied in M_i. Thus $\neg\phi_i(M, \bar{m}_i, \bar{a}_i)$ is countable and contains M_i. Since $M_{i+1} \supset M_i\bar{a}_i$ is ccl-closed we have:

$$M_i \subseteq \neg\phi_i(M, \bar{m}_i, \bar{a}_i) \subseteq M_{i+1} \ .$$

We will now find uncountable $S \subset \omega_1$, and $\phi(x, \bar{y}, \bar{z}) \in L$ and $\bar{m}$ from M, such that for all $i \in S$ such that for all $i \in S$ $\phi_i(x, \bar{y}, \bar{z}) = \phi(x, \bar{y}, \bar{z})$ and $\bar{m}_i = \bar{m}$: Without loss of generality assume that the universe of M is ω_1. Then $C = \{\alpha \in \omega_1 \mid M_\alpha = \alpha\}$ is a club subset of ω_1. By the Pressing Down Lemma the function $\alpha \mapsto \bar{m}_\alpha$ is constant on a stationary $S_1 \subset C$, so the $\bar{m}_i$'s are the same for all $i \in S_1$ (say $\bar{m}$). Since there are only countably many possibilities for the ϕ_i there is uncountable $S \subset S_1$ such that ϕ_i's are the same for all $i \in S$ (say ϕ). The family $(\neg\phi(M, \bar{m}, \bar{a}_i) \mid i \in S)$ is a strictly increasing chain of definable subsets of M, yielding the Theorem. □

As we see now the existence of a definable group operation on a quasiminimal structure has strong (but easy) consequences.

Theorem 6. *Suppose that M is a quasiminimal group. Then $p(x)$ is definable over $\emptyset$, both left and right translation invariant, and $\bar{M}$ is a regular group, in the sense of Definition 5, witnessed by $\bar{p}(x)$, where $\bar{p}$ is the unique global heir of p.*

Proof. Let $X \subseteq M$ be definable. First we claim that X is uncountable iff $X \cdot X = M$. If X is uncountable, then X is co-countable, as is X^{-1}. So for any $a \in M$, $a \cdot X^{-1}$ is co-countable, so has nonempty intersection with X. If $d \in X \cap a \cdot X^{-1}$ then $a \in X \cdot X$, proving the claim.

It follows that $p(x)$ is definable over $\emptyset$. In particular it is based on $\emptyset$ and, by Theorem 4, cl_p is a closure operator on M. The definability of p implies that $\mathrm{cl}_{\bar{p}}$ is also a closure operator and $(\bar{p}(x), x = x)$ is strongly regular by Lemma 2(i). The rest follows from Theorem 2. □

Several examples of quasiminimal structures with "bad" properties are given in [2]. For example $\omega_1 \times \mathbb{Q}$ equipped with the lexicographic order is quasiminimal, its generic type is definable over $\emptyset$, but which is non symmetric in the sense that Case (2) of Theorem 4 holds. We give here some other examples, including algebraic ones.

Example 3. A quasiminimal field with countably based "generic type", which is "asymmetric"

In fact, every strongly minimal structure M of size $\aleph_1$ can be expanded to such a quasiminimal structure. Let again $I = \omega_1 \times \mathbb{Q}$ and let $\lhd$ be the strict lexicographic order on I. Further, let $B = \{b_i \mid i \in I\}$ be a basis of the strongly minimal structure M. For each $a \in M$ let $i \in I$ be $\lhd$-maximal for which there are $i_1, ..., i_n \in I$ such that $a \in \mathrm{acl}(b_{i_1}, ..., b_{i_n}, b_i) \setminus \mathrm{acl}(b_{i_1}, ..., b_{i_n})$; Clearly, $i = i(a)$ is uniquely determined. Now, expand $(M, ..)$ to $(M, <, ...)$ where $b < c$ iff $i(b) \lhd i(c)$. We will prove that $(M, <, ...)$ is quasiminimal. Suppose that $M_0 \prec M$ is a countable, $<$-initial segment of M and that $B \setminus M_0$ does not have $\leq$-minimal elements, and let $a, a' \in M \setminus M_0$. Then there is an automorphism of $(B, \lhd)$ fixing $B \cap M_0$ pointwise and moving $b_{i(a)}$ to $b_{i(a')}$. It easily extends to an M_0-automorphism of $(M, <, ...)$, so $b_{i(a)} \equiv b_{i(a')}(M_0)$ (in the expanded structure). Note that replacing $b_{i(a)}$ by a in B (in the definition of $<$) does not affect $<$, so $a \equiv a'(M_0)$ and there is a single 1-type over M_0 realized in $M \setminus M_0$. Since every countable set is contained in an M_0 as above, $(M, <, ...)$ is quasiminimal. The remaining details are left to the reader.

Question Is every quasiminimal field algebraically closed?

The following is an example of a quasiminimal structure, which is very far from being regular: $\mathrm{cl}_p(A) \neq \mathrm{cl}_p(\mathrm{cl}_p(A))$ for arbitrarily large countable A's. In particular the generic type is not countably based.

Example 4. (A quasiminimal structure where $\mathrm{cl}_p \neq \mathrm{ccl}$)

Peretyatkin in [4] constructed an $\aleph_0$-categorical theory of a 2-branching tree. Our quasiminimal structure will a model of this theory.

The language consists of a single binary function symbol $L = \{\inf\}$. Let Σ be the class of all finite L-structures $(A, \inf)$ satisfying:

(i) $(A, \inf)$ is a semilattice;

(ii) $(A, <)$ is a tree (where $x < y$ iff $\inf(x, y) = x \neq y$);

(iii) (2-branching) No three distinct, pairwise $<$-incompatible elements satisfy: $\inf(x,y) = \inf(x,z) = \inf(y,z)$.

Then the Fraissé limit of Σ exists and its theory, call it T_2, is $\aleph_0$-categorical and has unique 1-type. If we extend the language to $\{\inf, <, \perp\}$, where $x \perp y$ stands for $x \not\leq y \wedge y \not\leq x$, then T_2 has elimination of quantifiers.

Let $(\bar{M}, <)$ be the monster model of T_2, let $\triangleleft$ be a lexicographic order on $I = \omega_1 \times Q$, and let $C = \{c_i \,|\, i \in \omega_1 \times Q\}$ be $<$-increasing. Then we can find a sequence of countable models $\{M_i \,|\, i \in \omega_1 \times Q\}$ satisfying:

(1) $M_i \prec M_j$ for all $i \triangleleft j$;

(2) $M_i \cap C = \{c_j \in C \,|\, j \trianglelefteq i\}$ for all i;

(3) $M_i \cap C < M_j \setminus M_i$ for all $i \triangleleft j$.

Finally, let $M = \bigcup\{M_i \,|\, i \in I\}$. Clearly, C is an uncountable branch in M. Moreover, by (3), any other branch is completely contained in some M_i, and is so countable. This suffices to conclude that M is quasiminimal and that the generic type is determined by $C < x$.

Fix $c_i \in C$ and $a \in M \setminus C$ with $c_i < a$. Note that $x \not\leq c_i$ is small, so $M_j \subset \mathrm{cl}_p(c_i)$ for all $j \triangleleft i$. Also, $x \perp a$ is large so $\mathrm{cl}_p(a)$ is the union of branches containing a. Since $c_i \in \mathrm{cl}_p(a)$ we have $M_j \subseteq \mathrm{cl}_p(c_i) \subset \mathrm{cl}_p^2(a)$; since $M_j \not\subseteq \mathrm{cl}_p(a)$ we conclude that $\mathrm{cl}_p(a) \neq \mathrm{cl}_p^2(a)$ and cl_p is not a closure operator. Similarly, for any countable $A \subset M$ we can find a, c_i as above much bigger than A, and thus both realizing $p\,|\,A$. Then $x \perp a \wedge \neg(x \perp c_i) \in p(x)$ witness that $p(x)$ splits over A.

6 A general dichotomy theorem

In this section we prove a dichotomy theorem, Theorem 7, which immediately yields Theorem 4 (the countably based quasiminimal case), and also subsumes the global strongly regular case (Theorem 1). Our set up will also apply to κ-quasiminimal structures, for κ regular (defined in the obvious way). We fix an arbitrary model N (allowing the possibility that $N = \bar{M}$), and a complete 1-type $p(x)$ over N. The operator cl_p defined on subsets of N may not be a closure operator, but generates one:

Definition 8. For $X \subset N$ define: $\mathrm{Cl}_p(X) = \bigcup\{\mathrm{cl}_p^n(X) \,|\, n \in \omega\}$ where $\mathrm{cl}_p^0(X) = X$, $\mathrm{cl}_p^{n+1}(X) = \mathrm{cl}_p(\mathrm{cl}_p^n(X))$.

Note that Cl_p is a closure operator and that cl_p is a closure operator if and only if $\mathrm{cl}_p = \mathrm{Cl}_p$. If N is quasiminimal and p is the generic type, then Cl_p agrees with ccl. However, in the general case we can easily have $\mathrm{Cl}_p(\emptyset) = N$ which is not interesting at all; the interesting case is when N is 'infinite dimensional'.

Definition 9. $A \subseteq N$ is *finitely* Cl_p*-generated over* $B \subset N$ if there is a finite $\bar{a} \subset A$ such that $A \subseteq \mathrm{Cl}_p(B\bar{a})$; if $B = \emptyset$ then we simply say that A is finitely Cl_p-generated.

The interesting case for us is when N itself is not finitely Cl_p-generated, which *will be assumed from now on.* This is already a weak regularity assumption on p, as we will see in the next section where it is proved that p is 'locally strongly regular'. The 'relative smallness' of $A \subset N$ is expressed by: N is not finitely Cl_p-generated over A.

Definition 10. A sequence $\{a_i \mid i \in \alpha\}$ is cl_p*-free over* $B \subset N$ if for all $i \leq \alpha$ $a_i \notin \mathrm{cl}_p(B \cup \{a_i \mid j < i\})$; cl_p-free means cl_p-free over $\emptyset$. Similarly Cl_p*-free* sequences are defined.

Definition 11. p is *based on* $A \subset N$ if p does not split over A, and N is not finitely Cl_p-generated over A.

Note that Cl_p-free sequences are also cl_p-free. Moreover, if $A \subset N$ and p does not split over A then every cl_p-free sequence over (any domain containing) A is indiscernible, by the standard argument.

Lemma 5. *Suppose that $p(x)$ is based on A. Then*

(i) cl_p^A *is a closure operator on N.*

(ii) (N, cl_p^A) *is a pregeometry iff every* cl_p*-free sequence over A is totally indiscernible.*

Proof. Without loss of generality assume $A = \emptyset$.

(i) Assuming that cl_p is not a closure operator we will find a non-indiscernible cl_p-free sequence over C for some finite $C \subset N$, which is in contradiction with non-splitting over $\emptyset$. So suppose that cl_p is not a closure operator. Then there are a finite $C \subset N$ and $a \in N$ such that $a \in \mathrm{cl}_p^2(C) \setminus \mathrm{cl}_p(C)$. Since $a \notin \mathrm{cl}_p(C)$ we have $a \models p \mid C$ so, since N is not finitely Cl_p-generated, there are $a_1, a_2 \in N$ such that (a_1, a_2) is a Cl_p-free sequence over Ca. We will prove that (a, a_1, a_2) is not indiscernible over C; since it is cl_p-free over C we have a contradiction as p does not split over C.

Witness $a \in \mathrm{cl}_p^2(C)$ by a formula $\varphi(x, \bar{b}) \in \mathrm{tp}(a/C\bar{b})$ which is not in $p(x)$, where $\varphi(x, y_1, ..., y_n)$ is over C and $(b_1, ..., b_n) = \bar{b} \in \mathrm{cl}_p(C)^n$. Choose $\theta_i(y_i) \in \mathrm{tp}(b_i/CA)$ witnessing $b_i \in \mathrm{cl}_p(C)$ (i.e. $\theta_i(y_i) \notin p(y_i)$) and let $x_1 \equiv_\varphi x_2$ denote the formula

$$(\forall \bar{y}) \left(\bigwedge_{1 \le i \le n} \theta_i(y_i) \to (\varphi(x_1, \bar{y}) \leftrightarrow \varphi(x_2, \bar{y})) \right).$$

It is, clearly, over C and we show $a \not\equiv_\varphi a_2$: from $\models \varphi(a, \bar{b})$ (witnessing $a \in \mathrm{cl}_p^2(C)$) and $a_2 \notin \mathrm{Cl}_p(C)$ we derive $\models \neg\varphi(a_2, \bar{b})$ and thus $\bar{b}$ witnesses $a \not\equiv_\varphi a_2$. On the other hand, since all realizations of θ_i's are in $\mathrm{cl}_p(C)$, and since $\mathrm{tp}(a_1/\mathrm{cl}_p(C)) = \mathrm{tp}(a_2/\mathrm{cl}_p(C))$, we have $a_1 \equiv_\varphi a_2$. Therefore (a, a_1, a_2) is not indiscernible over C.

(ii) Having proved (i), the proof of Lemma 2(ii) goes through: Let A_0 be finite, let $(a_1, ..., a_n) \in A_0^n$ be cl_p-free over $\emptyset$ such that $\mathrm{cl}_p(A_0) = \mathrm{cl}_p(a_1, .., a_n)$, let $a \models p \,|\, A_0$ and let $b \in N$. Then $(a_1, ..., a_n, a)$ is cl_p-free over $\emptyset$ and $\mathrm{cl}_p(aA_0) = \mathrm{cl}_p(a_1, ..., a_n, a)$. We have:

$b \notin \mathrm{cl}_p(aA_0)$ iff $b \notin \mathrm{cl}_p(a_1, ..., a_n, a)$ iff $(a_1, ..., a_n, a, b)$ is cl_p-free over $\emptyset$ iff $(a_1, ..., a_n, b, a)$ is cl_p-free over $\emptyset$ iff $(b \notin \mathrm{cl}_p(A_0)$ and $a \notin \mathrm{cl}_p(A_0 b))$.

□

Theorem 7. *Suppose that $p(x)$ is based on $A \subset N$. Then cl_p^A is a closure operator and exactly one of the following holds:*

(1) Every cl_p-free sequence over A is totally indiscernible; in this case cl_p^A is a pregeometry operator on N, p is definable, $\bar{p}$ (its unique global heir) is generically stable and $(\bar{p}(x), x = x)$ is strongly regular.

(2) Otherwise. In which case there is a finite extension A_0 of A and an A_0-definable partial order $\le$ such that every cl_p-free sequence over A_0 is strictly increasing.

Proof. To simplify the notation assume $A = \emptyset$, it will not affect the generality. First suppose that every cl_p-free sequence over $\emptyset$ is symmetric. Then, by Lemma 5(i), cl_p is a closure operator and, by Lemma 5(ii), it is a pregeometry operator. Since N is not finitely cl_p-generated it is infinite-dimensional so (N, cl_p) is a homogeneous pregeometry and the conclusion follows from Proposition 2.

Now suppose otherwise. Then over some finite A_0 there is a cl_p-free sequence (a, b) over A_0 such that $tp(a, b/A_0) \neq tp(b, a)/A_0$. So for some $\phi(x, y)$ over A_0, we have:

(1) $a \models p|\,\mathrm{cl}_p(A_0)$, $b \models p|\,\mathrm{cl}_p(A_0, a)$ and $\models \phi(a, b)$;

(2) $\models \phi(x, y) \to \neg\phi(y, x)$.

We *claim* $\phi(N, a) \subsetneq \phi(N, b)$. To prove it, first note that $\models \phi(a, b) \wedge \neg\phi(b, a)$ and $b \notin \mathrm{cl}_p(Aa)$ imply $\neg\phi(x, a) \in p(x)$, so $\phi(N, a) \subseteq \mathrm{cl}_p(A_0 a)$. Now suppose $d \in \phi(N, a)$ and we will show $d \in \phi(N, b)$. By the above $d \in \mathrm{cl}_p(A_0 a)$. We have two possibilities for d. The first is $d \in \mathrm{cl}_p(A_0)$, where $a \equiv b\,(\mathrm{cl}_p(A_0))$ and $\models \phi(d, a)$ imply $\models \phi(d, b)$ and we're done. The second is $d \notin \mathrm{cl}_p(A_0)$. Then a and d realize $p|\,\mathrm{cl}_p(A_0)$ and, since p does not split over A_0, we have $(\phi(a, x) \leftrightarrow \phi(d, x)) \in p(x)$; since $d \in \mathrm{cl}_p(A_0 a)$ we have $(\phi(a, x) \leftrightarrow \phi(d, x)) \in p\,|\,\mathrm{cl}_p(aA_0)$ and, since $b \models p|\,\mathrm{cl}_p(A_0 a)$, we get $\models \phi(a, b) \leftrightarrow \phi(d, b)$. Thus $\models \phi(d, b)$. This proves $\phi(N, a) \subseteq \phi(N, b)$.

Finally, the asymmetry of $\phi(x, y)$ implies $\models \neg\phi(a, a)$ so $a \notin \phi(N, a)$ and $a \in \phi(N, b) \setminus \phi(N, a)$; this proves that the inclusion is proper.

'$\phi(N, x) \subsetneq \phi(N, y)$' is an A_0-definable strict-ordering relation $x < y$ on N and we have $a < b$. By non-splitting, $a' < b'$ is true whenever (a', b') is cl_p-free over A_0, so every cl_p-free sequence over A_0 is increasing. □

7 Local regularity

Here we introduce and study "locally strongly regular types" and give applications to quasiminimal structures (see Corollary 4).

Given an arbitrary model M, $p(x) \in S_1(M)$ and $\phi(x) \in p$ we can ask when p extends to a global M-invariant $\bar{p}$, such that $(\bar{p}, \phi)$ is strongly regular. When T is stable the situation is well understood, but we are interested in the general case. Our definition of local strong regularity is actually a necessary condition for such an extension to exist. It is convenient to work with types over arbitrary sets (not just models).

Definition 12. A non-isolated type $p(x) \in S_1(A)$ is *locally strongly regular via* $\phi(x) \in p(x)$ if $p(x)$ has a unique extension over $A\bar{b}$ whenever $\bar{b} \in \bar{M}$ is a finite tuple of realizations of $\phi(x)$ no element of which realizes p.

Proposition 3. *Suppose that $p(x) \in S_1(M)$ is definable and locally strongly regular via $\phi(x) \in p(x)$, and let $\bar{p}(x)$ be its global heir. Then $(\bar{p}(x), \phi(x))$ is strongly regular (and of course definable).*

Proof. Suppose that $(\bar{p}(x), \phi(x))$ is not strongly regular. Then there are $B = M\bar{b}$ and $a \in \mathrm{cl}_{\bar{p}}(B) \cap \phi(\bar{M})$ and $c \models \bar{p}|B$ such that c does not realize $\bar{p}\,|\,Ba$. Witness $a \in \mathrm{cl}_{\bar{p}}(B)$ by $\theta(y, \bar{z})$ which is over M, implies $\phi(y)$, and $\models \theta(a, \bar{b}) \wedge \neg(d_p\theta)(\bar{b})$ (where d_p is the defining schema of p). Similarly, find $\varphi(x, y, \bar{z})$ over M such that $\models \varphi(c, a, \bar{b}) \wedge \neg(d_p\varphi)(a, \bar{b})$.

$$\models (\exists y)(\theta(y, \bar{b}) \wedge \neg(d_p\theta)(\bar{b}) \wedge \varphi(c, y, \bar{b}) \wedge \neg(d_p\varphi)(y, \bar{b})).$$

Since $\mathrm{tp}(c/M\bar{b})$ is an heir of $p(x)$ there is $\bar{m} \in M$ and a' such that

$$\models \theta(a', \bar{m}) \wedge \neg(d_p\theta)(\bar{m}) \wedge \varphi(c, a', \bar{m}) \wedge \neg(d_p\varphi)(a', \bar{m}).$$

The first two conjuncts witness $a' \in \phi(\bar{M}) \setminus p(\bar{M})$ while the last two witness that c is not a realization of $\bar{p}\,|\,Ma'$. A contradiction. □

For the sake of this section we will call a sequence $(a_i \,|\, i \in \alpha)$ a *coheir sequence over* C if $\mathrm{tp}(a_j/M(a_i \,|\, i < j))$ is finitely satisfiable in C for any $j < \alpha$. In particular $tp(a_0/C)$ is finitely satisfiable in C.

Proposition 4. *Suppose that $p(x) \in S_1(C)$ is locally strongly regular via $x = x$ and that there exists an infinite, totally indiscernible (over C) sequence of realizations of p which is a coheir sequence over C. Then p has a global C-invariant extension $\bar{p}$ such that $(\bar{p}(x), x = x)$ is strongly regular and generically stable.*

Proof. Let $I = \{a_i : i \in \omega\}$ be a totally indiscernible (over C) sequence of realizations of p which is a coheir sequence in C and let $p_n(x) = \mathrm{tp}(a_{n+1}/Ca_1...a_n)$. We will first prove that each $p_n(x)$ is locally strongly regular via $x = x$. Suppose, on the contrary, that $p_n(x)$ is not locally strongly regular. Then there are $b_1...b_k = \bar{b}$, with none realizing $p_n(x)$, such that p_n has at least two extensions in $S_1(C\bar{a}\bar{b})$ (here $\bar{a} = a_1...a_n$). Let $\varphi(x, \bar{z}, \bar{y})$ be over C and such that both $\varphi(x, \bar{a}, \bar{b})$ and $\neg\varphi(x, \bar{a}, \bar{b})$ are consistent with $p_n(x)$.

Choose $\theta_i(y_i, \bar{a}) \in \mathrm{tp}(b_i/C\bar{a})$ witnessing that b_i does not realize $p_n(y_i)$ and let $\phi(x_1, x_2, \bar{a})$ be

$$(\exists \bar{y}) \left(\textstyle\bigwedge_{1 \le i \le n}(\theta_i(y_i, \bar{a}) \wedge \neg\theta_i(x_2, \bar{a})) \wedge \neg(\varphi(x_1, \bar{a}, \bar{y}) \leftrightarrow \varphi(x_2, \bar{a}, \bar{y}))\right).$$

It is, clearly, over C and we show $\models \phi(a_{n+2}, a_{n+1}, \bar{a})$. By our assumptions on φ and $\bar{b}$, there is $\bar{b}' \equiv \bar{b}(C\bar{a})$ such that $\models \neg(\varphi(a_{n+1}, \bar{a}, \bar{b}) \leftrightarrow \varphi(a_{n+1}, \bar{a}, \bar{b}'))$. Also $\models \bigwedge_{1 \le i \le n}(\theta_i(b_i, \bar{a}) \wedge \theta_i(b'_i, \bar{a}) \wedge \neg\theta_i(a_{n+1}, \bar{a}))$. Thus for any $c \in C$ either $\bar{b}$ or $\bar{b}'$ in place of $\bar{y}$ witnesses $\models \phi(c, a_{n+1}, \bar{a})$ and, since $\mathrm{tp}(a_{n+2}/C\bar{a}a_{n+1})$ is f.s. in C, we conclude $\models \phi(a_{n+2}, a_{n+1}, \bar{a})$.

By total indiscernibility, $\mathrm{tp}(\bar{a}/Ca_{n+1}a_{n+2})$ is finitely satisfiable in C, so there are $\bar{c} \in C$ and $\bar{d}$ such that

$$\bigwedge_{1\le i\le n}(\theta_i(d_i,\bar{c}) \wedge \neg\theta_i(a_{n+1},\bar{c})) \wedge \neg(\varphi(a_{n+2},\bar{c},\bar{d}) \leftrightarrow \varphi(a_{n+1},\bar{c},\bar{d})).$$

$\bigwedge_{1\le i\le n}(\theta_i(d_i,\bar{c})\wedge\neg\theta_i(a_{n+1},\bar{c}))$ witnesses that no d_i realizes p, and $\neg(\varphi(a_{n+2},\bar{c},\bar{d}) \leftrightarrow \varphi(a_{n+1},\bar{c},\bar{d}))$ witnesses that p does not have a unique extension over $C\bar{d}$; a contradiction.

Therefore each $p_n(x)$ is locally strongly regular via $x = x$. It easily follows that $p_I(x) = \cup_{n\in\omega}p_n(x) \in S_1(CI)$ is locally strongly regular via $x = x$ as well. Moreover, the same is true whenever $I' \subset \bar{M}$ is an indiscernible (over C) extension of I; then I' is also totally indiscernible over C and $p_{I'}(x) \in S(CI')$, defined by:

$\phi(x,\bar{a}') \in p_{I'}$ (where $\phi(x,\bar{y})$ is over C and $\bar{a}' \in I'$) iff $\phi(x,\bar{a}) \in p_I(x)$ for some $\bar{a} \in I$ with $\bar{a} \equiv \bar{a}'\ (C)$

is locally strongly regular via $x = x$ and does not split over C.

Now let $J \subset \bar{M}$ be a maximal indiscernible (over C) extension of I. By total indiscernibility and maximality of J no element of $\bar{M}\setminus CJ$ realizes $p_J(x)$, so the local strong regularity implies that p_J has a unique global extension $\bar{p}$. Since $p_J(x)$ does not split over C, it is easy to conclude that $\bar{p}$ is C-invariant. To show that $(\bar{p}(x), x = x)$ is strongly regular let $A \supseteq CI$ be small and we will prove that $\bar{p}\,|\,A \vdash \bar{p}\,|\,Ab$ for any b which does not realize $\bar{p}\,|\,A$. So fix such a b and let $A_0 \subset A$ be maximal such that $I_0 = I \cup A_0$ is a Morley sequence in $\bar{p}$ over C. Then $\bar{p}\,|\,CI_0(x)$ is $p_{I_0}(x)$ (as defined above for $I' = I_0$), so is locally strongly regular via $x = x$. The maximality of A_0 implies that no $a \in A$ realizes $\bar{p}\,|\,CI_0$ so, by local strong regularity, $\bar{p}\,|\,CI_0 \vdash \bar{p}\,|\,A$. In particular, since b does not realize $\bar{p}\,|\,A$, we have that b does not realize $\bar{p}\,|\,CI_0$ either. Thus no element of Ab realizes $\bar{p}\,|\,CI_0(x)$ so $\bar{p}\,|\,CI_0(x) \vdash \bar{p}\,|\,Ab$; in particular $\bar{p}\,|\,A \vdash p\,|\,Ab$ and $(\bar{p}(x), x = x)$ is strongly regular.

Having proved that $(\bar{p}(x), x = x)$ is strongly regular we can apply Corollary 1 to $\bar{M}$ and $\bar{p}$. Note that I is a Morley sequence in $\bar{p}$ over C, and is totally indiscernible, so $\bar{p}$ is symmetric, and hence generically stable. □

Our next goal is to prove that the generic type of a quasiminimal structure is locally strongly regular via $x = x$. This we will do in a more general situation, for any M and $p \in S_1(M)$ for which M is not finitely Cl_p-generated (with notation as in Section 6).

Proposition 5. *Suppose $p \in S_1(M)$ and M is not finitely Cl_p-generated. Then:*

(i) Whenever $I \subseteq M$ is a maximal Cl_p-free sequence then $(p\,|\,I)(x)$ is locally strongly regular via $x = x$, and $(p\,|\,I)(x) \vdash p(x)$.

(ii) $p(x)$ is locally strongly regular via $x = x$.

Proof. (i) Let $I \subseteq M$ be a maximal Cl_p-free sequence. We will prove that $(p\,|\,I)(x)$ is locally strongly regular via $x = x$. After passing to a subset and rearranging I if necessary we may assume that $I = \{a_i\,|\,i < \alpha\}$ where α is a limit ordinal.

Suppose, on the contrary, that there are $d_1, d_2 \in \bar{M}$ realizing $p\,|\,I$, a formula $\phi(x, \bar{y})$, and a tuple $\bar{b} = b_1 b_2 ... b_n \in \bar{M}^n$ such that none of b_i's realize $p\,|\,I$ and:

$$\models \neg\phi(d_1, \bar{b}) \wedge \phi(d_2, \bar{b})).$$

Choose $\theta_i(y_i) \in \mathrm{tp}(b_i/I)$ such that $\theta_i(x) \notin (p|I)(x)$. To simplify notation we will assume that $\phi(x, \bar{y})$, as well as each $\theta_i(x)$, are over $\emptyset$ (absorbing a few parameters from I into the language won't hurt the generality). At least one of

$$\{i < \alpha\,|\, \models \phi(a_i, \bar{b})\} \text{ and } \{i < \alpha\,|\, \models \neg\phi(a_i, \bar{b})\}$$

is cofinal in α. Assume the first one is cofinal and let $I_0 = \{a_i\,|\, \models \phi(a_i, \bar{b})\}$. The cofinality implies: first that $p|I$ is finitely satisfiable in I_0 (so $\mathrm{tp}(d_1/I)$ is finitely satisfiable in I_0); second, that $p(x) \cup \{\phi(x, \bar{b}\}$ is finitely satisfiable in I_0, so there is a type containing it in $S_1(I d_1 \bar{b})$ which is finitely satisfiable in I_0; wlog, let d_2 realizes it. Thus, both $\mathrm{tp}(d_1/I)$ and $\mathrm{tp}(d_2/I d_1 \bar{b})$ are finitely satisfiable in I_0.

$$\models (\exists \bar{y})(\textstyle\bigwedge_{1 \le i \le n} \theta_i(y_i) \wedge \neg\phi(d_1, \bar{y}) \wedge \phi(d_2, \bar{y})).$$

Since $\mathrm{tp}(d_2/I d_1)$ is finitely satisfiable in I_0, there is $a_i \in I_0$ such that:

$$\models (\exists \bar{y})(\textstyle\bigwedge_{1 \le i \le n} \theta_i(y_i) \wedge \neg\phi(d_1, \bar{y}) \wedge \phi(a_i, \bar{y})).$$

Since $\mathrm{tp}(d_1/M)$ is finitely satisfiable in I_0, there is $a_j \in I_0$ such that:

$$\models (\exists \bar{y})(\textstyle\bigwedge_{1 \le i \le n} \theta_i(y_i) \wedge \neg\phi(a_j, \bar{y}) \wedge \phi(a_i, \bar{y})).$$

Finally, since $a_i, a_j \in M$ there is $\bar{b}' = b'_1 b'_2 ... b'_n \in M^n$ satisfying:

$$\models \textstyle\bigwedge_{1 \le i \le n} (\theta_i(b'_i) \wedge \neg\phi(a_j, \bar{b}') \wedge \phi(a_i, \bar{b}')).$$

But $\bigwedge_{1\le i\le n}\theta_i(b'_i)$ implies $\bar{b}'\subset \mathrm{cl}_p(\emptyset)$ and thus $\mathrm{tp}(a_i/\mathrm{cl}_p(\emptyset))\neq \mathrm{tp}(a_j/\mathrm{cl}_p(\emptyset))$. A contradiction. Therefore $(p\,|\,I)(x)$ is locally strongly regular via $x=x$. The maximality of I implies $M=\mathrm{cl}_p(I)$ so, by local strong regularity of $p\,|\,I$, we have $(p\,|\,I)(x)\vdash p(x)$.

(ii) $p\,|\,I$ is locally strongly regular and $(p\,|\,I)(x)\vdash p(x)$ implies that $p(x)$ is locally strongly regular via $x=x$. □

Corollary 3. *The generic type of a quasiminimal structure is locally strongly regular via $x=x$.*

Theorem 8. *Suppose that $p\in S_1(M)$ and that (M,Cl_p) is an infinite dimensional pregeometry. Then p is definable and $(\bar{p}(x),x=x)$ is strongly regular and generically stable (where $\bar{p}$ is the unique global heir of p).*

Proof. First note that, by Proposition 5(ii), p is locally strongly regular via $x=x$. Now we will find an infinite, totally indiscernible sequence of realizations of p which is a coheir sequence over M. Towards this aim we first prove:

Claim. If $A\subset M$ and $a,b\in M$ are Cl_p-independent over A then $\mathrm{tp}(a,b/A)=\mathrm{tp}(b,a/A)$.

It suffices to prove the claim for A finite. Suppose, on the contrary, that $\phi(x,y)$ is over A and $\models\phi(a,b)\wedge\neg\phi(b,a)$. Since $b\notin\mathrm{Cl}_p(Aa)$ we have $\phi(a,x)\in p(x)$ and, since $a\notin\mathrm{Cl}_p(Ab)$, we have $\neg\phi(b,x)\in p(x)$. By infinite dimensionality and since Aab is finite, there is $e\in M\setminus\mathrm{Cl}_p(Aab)$. Then e realizes $p\,|\,Aab$ so $\models\phi(a,e)\wedge\neg\phi(b,e)$. But $a\notin\mathrm{Cl}_p(Ae)$ implies $\phi(x,e)\in p(x)$ and $b\notin\mathrm{Cl}_p(Ae)$ implies $\neg\phi(x,e)\in p(x)$. A contradiction. The claim is proved.

Now, let $I\subseteq M$ be a maximal Cl_p-free sequence. We can find an infinite M-indiscernible sequence $J=(a_i:i<\omega)$ such that for all i any formula satisfied by a_i over $M\cup\{a_j:j<i\}$ is satisfied by some element of I. Using the claim, we conclude that J is totally indiscernible over M. Now apply Proposition 4.

□

Corollary 4. *Suppose that M is quasiminimal and let $p(x)\in S_1(M)$ be the "generic type" of M (consisting of formulas defining uncountable sets). Suppose that (M,ccl) is a pregeometry. Then p is based on some countable $A\subset M$ (p does not split over A) and Case (1) of Theorem 4 holds ((M,cl_p^A) is* *a homogeneous infinite-dimensional pregeometry, p has a global M-invariant generically stable extension etc.) Moreover we can choose A to be some ccl-free sequence.*

Proof. Clearly, (M, ccl) is infinite-dimensional, so Theorem 8 applies to give the first part of the Corollary. To prove the 'moreover' part, let $M_0 \prec M$ be countable, of infinite ccl-dimension, such that p is definable over M_0 and such that p is the heir of $p \,|\, M_0$. Further, let $I \subset M_0$ be a maximal ccl-free subset of M_0. Then we can apply Proposition 5 (i) to M_0 and I: $p \,|\, I \vdash p \,|\, M_0$. In particular, p is I-invariant, so I is a base for p. □

So if M is quasiminimal and (M, ccl) is a pregeometry, then the generic type is definable. The following example shows that parameters may be required.

Example 5. A quasiminimal structure where ccl is a pregeometry operator but the generic type is not based on $\emptyset$.

This is a slight variation of an example from [2]. We have two unary predicates U, V and binary relation symbols E and f. The domain M is the disjoint union $U(M) \cup V(M)$. $U(M)$ is uncountable and E is an equivalence relation on it having $\aleph_0$ many classes with all of them but one of size $\aleph_0$. $V(M)$ is countable and contains 'names' for E-classes. Namely $f : U(M) \longrightarrow V(M)$ is a surjection(and $f(a) = f(a')$ iff $E(a, a')$). So the structure M is quasiminimal. Let $p(x)$ be the "generic type": $p(x)$ says that $x \in U$ and is in the unique uncountable E-class. Then $\mathrm{cl}_p(\emptyset) = V(M)$, while $\mathrm{cl}^2_p(\emptyset) = \mathrm{Cl}_p(\emptyset)$ contains also all the countable E-classes. Also $\mathrm{Cl}_p(X) = \mathrm{Cl}_p(\emptyset) \cup X$ so (M, Cl_p) is an infinite-dimensional pregeometry. But p is not $\emptyset$-definable: all the elements of $U(M)$ have the same type while $E(x, a) \wedge \neg E(x, b) \in p(x)$ for a in the uncountable class and b in a countable one. Therefore p is not definable over $\emptyset$.

Theorem 9. *Let M be an arbitrary model, and G a $\emptyset$-definable group. Suppose $p(x) \in S_G(M)$ via "$x \in G$".*

(i) $p(x)$ is both left and right $G(M)$-translation invariant (and in fact invariant under definable bijections from M to itself).

(ii) A formula $\phi(x)$ is in $p(x)$ iff two left (right) translates of $\phi(x)(M)$ cover $G(M)$ iff finitely many left (right) translates of $\phi(x)(M)$ cover $G(M)$.

(iii) $p(x)$ is definable over $\emptyset$ and G is connected.

(iv) If $\bar{p}$ is the unique global heir of p, then ($\bar{p}(x)$, "$x \in G$") is strongly regular, so G is a regular group in the sense of Definition 5.

Proof. (i) Suppose that $f : G \longrightarrow G$ is an M-definable bijection and $a \models p$. Clearly $p(x)$ has at least two extensions to complete types over $(M, f(a))$ (one containing "$f(x) = f(a)$" and another containing "$f(x) \neq f(a)$". By local strong regularity, $f(a) \models p$.

(ii) The local strong regularity of $p(x)$ implies that whenever $g, g' \in \bar{G}$ do not realize p then $g \cdot g'$ does not realize p either. It follows that $a \cdot g \models p$ whenever $a \models p$ and $g \in \bar{G}$ does not realize p. Thus:

$$\phi(x) \in p(x) \quad \text{iff} \quad (\forall y \in \bar{G})(\neg\phi(y) \rightarrow \phi(y \cdot x)) \in p(x),$$

and $\phi(x) \in p(x)$ iff $\phi(\bar{G}) \cup a^{-1} \cdot \phi(\bar{G}) = \bar{G}$.

(iii) follows immediately from (ii), and then (iv) follows from Proposition 3. □

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