# Ax's Theorem for iterative Hasse fields

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June 9, 2007

# Examples of Ax-Schanuel inequalities

$(K, \delta)$ is a differential field, $C$ – constants, $\mathbb{Q} \subseteq C$. Let $x = (x_1, \ldots, x_n) \in K^n, y = (y_1, \ldots, y_n), z = (z_1, \ldots, z_n) \in (K^*)^n$.

**Theorem (Ax)**

*If* $(\partial x_i = \frac{\partial y_i}{y_i})_{i \leqslant n}$ *and* $(\partial x_i)_{i \leqslant n}$ *are* $\mathbb{Q}$*-linearly independent, then* $\operatorname{trdeg}_C(x, y) \geqslant n + 1$.

**Theorem (Torus version of Ax Theorem)**

*If* $\alpha \in C \setminus \mathbb{Q}^{\mathrm{alg}}$, $(\frac{\partial z_i}{z_i} = \alpha \frac{\partial y_i}{y_i})_{i \leqslant n}$ *and* $(\frac{\partial z_i}{z_i})_{i \leqslant n}$ *are* $\mathbb{Q}$*-linearly independent, then* $\operatorname{trdeg}_C(y, z) \geqslant n + 1$.

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# Algebraic non-example

**The scheme of Ax-Schanuel inequalities**

Good differential equation $+$ linear independence $\Longrightarrow$ large trdeg.

**Example**

For any $y \in \mathbb{G}_m^n(K)$, $\frac{\partial y_i^2}{y_i^2} = 2\frac{\partial y_i}{y_i}$ and $\mathrm{trdeg}_C(y, y^2) \leqslant n$. But for a generic $y$, $(\frac{\partial y_i}{y_i})_{i \leqslant n}$ is $\mathbb{Q}$-linearly independent.

**Bad differential equation**

For any $\alpha \in \mathbb{Q}$, $\frac{\partial z_i}{z_i} = \alpha\frac{\partial y_i}{y_i}$ $(i \leqslant n)$ is not a differential equation giving Ax-Schanuel inequalities.

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# Non-algebraic non-example

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For the differential equation $\frac{\partial z_i}{z_i} = \sqrt{2}\frac{\partial y_i}{y_i}$ take:

- $y_1, y_2$ such that $\frac{\partial y_1}{y_1} = 1, \frac{\partial y_2}{y_2} = \sqrt{2} + 1$
- $z_1 = y_1^{-1} y_2, \; z_2 = y_1 y_2$.

Then $\frac{\partial y_1}{y_1}, \frac{\partial y_2}{y_2}$ are $\mathbb{Q}$-linearly independent, $\mathrm{trdeg}_C(x, y) \leqslant 2$ and:

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## Remark

Here "$z = y^{\sqrt{2}}$", $z$ is not of the form $f(y)$ for an algebraic morphism $f$.

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# Towards general formulation of Ax

Assume for a while $C = \mathbb{C}$.

## Analytic maps

- Ax is related with $\exp : \mathbb{G}_a^n(\mathbb{C}) \to \mathbb{G}_m^n(\mathbb{C})$ – analytic homomorphism.
- "Torus Ax" is related with $\mathbb{G}_m^n(\mathbb{C}) \ni x \mapsto x^\alpha \in \mathbb{G}_m^n(\mathbb{C})$ – local analytic homomorphism.

## Relating analytic maps with differential equations

For $A, B$ commutative algebraic groups over $\mathbb{C}$ and $\phi : A(\mathbb{C}) \to B(\mathbb{C})$ local analytic, we need to define the differential equation of $\phi$.

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# Logarithmic derivative

## Definition

For a commutative algebraic group $A$ over $\mathbb{C}$, there is a canonical $\partial$-definable homomorphism $l_A\partial : A(K) \to T_0A(K)$ called logarithmic derivative.

## Logarithmic derivative and homomorphisms

For a homomorphism of algebraic groups $\phi : A \to B$, the following diagram is commutative:

$$\begin{array}{ccc} T_0A(K) & \xrightarrow{\phi'} & T_0B(K) \\ \uparrow{\scriptstyle l_A\partial} & & \uparrow{\scriptstyle l_B\partial} \\ A(K) & \xrightarrow{\phi} & B(K) \end{array}$$

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# Differential equation

## Analytic maps and tangent spaces

- Any local analytic homomorphism $\phi : A(\mathbb{C}) \to B(\mathbb{C})$ induces a linear map $\phi' : T_0 A(K) \to T_0 B(K)$.
- But there is no function $\phi : A(K) \to B(K)$.

## Definition

The differential equation of $\phi$ is $l_B \partial(y) = \phi'(l_A \partial(x))$.

## Remark

The solution set of this equation corresponds (up to constants) to the graph of the non-existing function $\phi : A(K) \to B(K)$.

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# Examples of differential equations

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$\partial(x_i) = \frac{\partial(y_i)}{y_i}, i \leqslant n$, since $\exp'(0) = 1$.

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# Good analytic maps

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## Good analytic maps?

- Since our differential equations come from analytic maps, we need to specify which analytic maps are good.
- If $\phi$ is algebraic, then $(x, \phi(x))$ is always a solution of the differential equation of $\phi$, hence algebraic ones are not good.

## Definition

$\phi$ is nowhere algebraic if there are no algebraic subgroups $A_0 \leqslant A, B_0 < B$ and no algebraic epimorphism $\psi : A_0 \twoheadrightarrow B/B^0$ such that $\psi' - \phi' : T_0 A_0 \to T_0(B/B_0)$ has a non-trivial kernel.

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# Ax's theorem for nowhere algebraic $\phi$

**Theorem (K.)**

*Assume $A, B$ are commutative algebraic groups of dimension $n$, $\phi : A(\mathbb{C}) \to B(\mathbb{C})$ is local analytic and nowhere algebraic and $(x, y)$ satisfies the differential equation of $\phi$. If $\mathrm{trdeg}_{\mathbb{C}}(x, y) \leqslant n$, then there is a proper algebraic subgroup $B_0 < B$ such that $y \in c + B_0(K)$ for some $c \in B(\mathbb{C})$.*

**Remark**

For $B = \mathbb{G}_m^n$, $y$ is in a constant coset of a subtorus if and only if $(\frac{\partial y_i}{y_i})_{i \leqslant n}$ are $\mathbb{Q}$-linearly dependent.

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# Essentially different algebraic groups

**Definition**

$A, B$ are essentially different, if for any algebraic subgroup $H \leqslant A \times B$, $H$ is of the form $A_0 \times B_0$.

**Fact**

If $A, B$ are essentially different, then any non-zero local analytic $\phi : A(\mathbb{C}) \to B(\mathbb{C})$ is nowhere algebraic.

**Example (essentially different algebraic groups)**

- $A = \mathbb{G}_a^n$, $B$ – semi-abelian (Jonathan's talk).
- $A = \mathbb{G}_m^n$, $B$ – abelian.

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If $A, B$ are essentially different and $B$ has no vectorial quotients, then any non-zero local analytic $\phi : A(\mathbb{C}) \to B(\mathbb{C})$ is nowhere algebraic.

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## Other constants

What if constants do not coincide with $\mathbb{C}$? Two equivalent solutions.

## Solution 1

Talk only about linear maps on tangent spaces.

## Solution 2

Replace "local analytic" with "formal" (stop worrying about convergence). Over $\mathbb{C}$ these two notions are equivalent.

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# Hasse derivations needed

## Problem

Assume $\mathrm{char}(K) = p > 0$. Then for each $x \in K^p$, $\partial(x) = 0$, so $K$ is algebraic over $C$ and the statement of Ax is meaningless.

## Solution

Replace derivation $\partial$ with a Hasse derivation $D = (D_i)_{i<\omega}$, i.e.: each $D_i : K \longrightarrow K$ is additive, $D_0$ is the identity map,

$$D_i(xy) = \sum_{n+m=i} D_n(x)D_m(y),$$

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Let $C$ be the constant field of all $D_i$. Then $K$ is usually not algebraic over $C$.

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# Logarithmic derivative in the Hasse case

## Arcs

- Replace the tangent space $TV(K) = V(K[X]/X^2)$ with $\operatorname{Arc} V(K) = V(K[[X]])$.
- If $A$ is a commutative algebraic group, then $\operatorname{Arc} A \longrightarrow A$ is a group homomorphism which splits.
- $U_A$ denotes $\operatorname{Arc}_0 A$, the fiber over 0.

## Definition

$lD_A : A(K) \to U_A(K)$ is the HS-logarithmic derivative – the composition of $D : A(K) \to \operatorname{Arc} A(K)$ and $\operatorname{Arc} A(K) \to U_A(K)$.

## Example

$$lD_{\mathbb{G}_a}(x) = (D_i(x))_{i\geqslant 1}, \quad lD_{\mathbb{G}_m}(x) = (\frac{D_i(x)}{x})_{i\geqslant 1}.$$

# Logarithmic derivative in the Hasse case

## Arcs

- Replace the tangent space $TV(K) = V(K[X]/X^2)$ with $\operatorname{Arc} V(K) = V(K[[X]])$.
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# HS-differential equations of formal maps

**Remark**

Any formal map $\phi : A \longrightarrow B$ induces an algebraic homomorphism $U_\phi : U_A \longrightarrow U_B$.

**Definition**

The HS-differential equation for $\phi$ is $U_\phi(I_D A(x)) = I_D B(y)$.

**Remark**

As before, the solution set of the HS-differential equation for $\phi$ may be thought of the graph (up to $C$) of the non-existing function $\phi : A(K) \longrightarrow B(K)$.

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# Formal maps in positive characteristic

## Example (of formal maps)

- There is a formal isomorphism between $\mathbb{G}_m$ and an ordinary elliptic curve.
- There are no non-zero formal maps between $\mathbb{G}_a$ and $\mathbb{G}_m$ (no exp in positive characteristic)
- $\sum_{n=0}^{\infty} X^{p^n}$ is a formal endomorphism of $\mathbb{G}_a$.

## Definition

$\phi$ is nowhere algebraic if there are no algebraic subgroups $A_0 \leqslant A, B_0 < B$ and no algebraic epimorphism $\psi : A_0 \twoheadrightarrow B/B^0$ such that $U_{\psi-\phi} : U_{A_0} \to U_{B/B_0}$ has a non-trivial kernel.

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# Iterative Ax???

## Ax in arbitrary characteristic (in progress again)

Assume $A, B$ are commutative algebraic groups of dimension $n$, $\phi : A \to B$ is formal, nowhere algebraic and $(x, y)$ satisfies the differential equation of $\phi$. If $\mathrm{trdeg}_{\mathbb{C}}(x, y) \leqslant n$, then there is a proper algebraic subgroup $B_0 < B$ such that $y \in c + B_0(K)$ for some $c \in B(\mathbb{C})$.

## Problems

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