Sonat Suer's thesis

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We are looking at differential algebraic subgroups of $(U, +)$ where U is a Δ-closed field for $\Delta = \{\partial_1, \ldots, \partial_m\}$.

Motivation: "Zilber conjecture" for regular types in $DCF_{0,m}$. That is, a regular type in $DCF_{0,m}$ is either $\not\perp$ to generic of a field of constants or is locally modular.

True in finite dimension.

In [Moosa, Pillay, Scanlon], this problem is reduced to the case where the regular type is the generic type of a Δ-subgroup of the additive group. (Proved using differential arc spaces.)

Y is an irreducible affine Δ-variety over $K \subseteq U$. Let $a \in Y(U)$ be a generic of Y.

- The type associated to Y is $p_Y := tp(a/K)$.
- The Δ-type of Y is denoted τ_Y and we define the Δ-type of p_Y to be τ_Y.
- The Kolchin polynomial of Y is $\omega_Y(x) = \sum_{i=0}^{\tau a_i}((X + i)$ choose $i)$ Then $a_\tau \neq 0$ and is called the *typical* Δ*-dimension*.

Fact (McGrail, Pong): $U(p_Y) \leq RM(p_Y) \leq \Delta{-}\dim(p_Y) \leq RD(p_Y) < \omega^\tau(a_\tau + 1)$.

Note that there are no nontrivial lower bounds. It need not be true that $\omega^\tau a_\tau \leq U(p_Y)$. Suer gives counter-examples with $\tau = 1$ coming from subgroups of the additive group.

Proposition $(m = 2)$ Let $G \leq (U, +)$ be a subgroup defined by $\delta_1(y) = f(y)$ with $f \in U[\delta_2]$ of order k. Then p_G has Δ-type 1 and typical dimension k but U-rank ω.

N.B.: Call Y Δ*-type minimal* if for every proper Δ-subvariety have a smaller Δ-type.

Theorem: Let $G \leq (U, +)$ have Δ-type one and be Δ-type minimal. ($\Longrightarrow$ $U(p_G) = \omega$ and p_G is regular) Then p_G is nonorthogonal to the generic of a definable field iff the typical Δ-dimension of G is one.

Fact: Every definable subfield of U is the field of constants of finitely many definable derivations of the form $\sum a_i \delta_i$. Moreover, nonorthogonality of G to the field F is witnessed by a definable, surjective homomorphism $G \to F$.

Theorem: $(m = 2)$ The group $\{x \in U : \delta_1(x) = \delta_2^2(x)\}$ is modular.

Modularity of the group G means (assuming that G is finite dimensional) that every definable subset of $G \times \cdots \times G$ is a finite Boolean combination of a definable subgroup. In the infinite dimensional case, this definition works "modulo small." Technically, one needs to discuss local p-weight, semiregular types, *et cetera*.

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